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Solution of the final exam ACTU. 464 fall 2018 (40%) two pages

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Problem 1. (8 marks)

1. Determine whether the following model is individual or collective: The number of claims per day N has a negative binomial distribution with mean 15. The size of each claim has a Pareto distribution with mean 5000. The number of losses and loss sizes are **not** mutually independent. Justify your answer.

2. Let X denotes a loss of reinsurance company such that the c.d.f. F of this loss is given as follows

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 20 \\ \dfrac{x+20}{80} & \text{for} \quad 20 \leq x < 40 \\ 1 & \text{for} \quad x \geq 40 \end{cases}$$

2 (a) Using $P(X = k) = P(X \leq k) - \lim_{\varepsilon \longrightarrow 0} P(X \leq k - \varepsilon)$, find $P(X = 20)$ and $P(X = 40)$.

(b) Find the probability mass and density function for this mixed distribution

(c) Calculate the expected loss and the standard deviation of this loss.

Solution:

1. The model is collective since the number of claims is random.

2. We have a mixed distribution.

   (a) Notice first that

$$P(X = 20) = P(X \leq 20) - \lim_{\varepsilon \longrightarrow 0} P(X \leq 20 - \varepsilon) = \frac{20+20}{80} - 0 = \frac{1}{2},$$

and

$$P(X = 40) = P(X \leq 40) - \lim_{\varepsilon \longrightarrow 0} P(X \leq 40 - \varepsilon) = 1 - \lim_{\varepsilon \longrightarrow 0} \frac{40 - \varepsilon + 20}{80} = 1 - \frac{60}{80} = 1 - \frac{6}{8} = \frac{1}{4}.$$

Moreover for $20 \leq x < 40$, $\left(\dfrac{x+20}{80}\right)' = \dfrac{1}{80}$.

   (b) Then the p.m.d.f. of X is given by

$$f(x) = \begin{cases} 0 & \text{for} \quad x < 20 \\ \frac{1}{2} & \text{for} \quad x = 20 \\ \left(\dfrac{x+20}{80}\right)' = \dfrac{1}{80} & \text{for} \quad 20 \leq x < 40 \\ \frac{1}{4} & \text{for} \quad x = 40 \\ 0 & \text{for} \quad x > 40 \end{cases}$$

(c) The expected loss is

$$
\begin{aligned}
E[X] &= 20P(X=20) + \int_{20}^{40} x\frac{1}{80}dx + 40P(X=40) = \frac{20}{2} + \frac{1}{160}\left[x^2\right]_{20}^{40} + \frac{40}{4} \\
&= 10 + \frac{1}{160}\left(40^2 - 20^2\right) + \frac{40}{4} = \frac{55}{2} = 27.5.
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[X^2\right] &= 20^2P(X=20)\int_{20}^{40} x^2\frac{1}{80}dx + 40^2P(X=40) = \frac{20^2}{2} + \frac{1}{240}\left[x^3\right]_{20}^{40} + \frac{40^2}{4} \\
&= \frac{20^2}{2} + \frac{1}{240}\left(40^3 - 20^3\right) + \frac{40^2}{4} = \frac{2500}{3} = 833.33
\end{aligned}
$$

and the $\mathrm{sd}(X) = \sqrt{833.33 - (27.5)^2} = 8.7795$.

Problem 2. (8 marks) A risk averse agent, whose utility $U(x) = \ln(x)$ and wealth \$50,000 is faced with a potential loss X of \$10,000 with a probability of $p = 0.25$ and \$1,000 with a probability of $p = 0.75$

The agent's expected wealth without insurance is $E[W - X]$ while their expected utility of wealth without insurance is $E[U(W-X)]$. The maximum premium $\Pi_{\max}$ that the insured accepts to pay with initial wealth W is calculated by the utility function U as $U(W - \Pi_{\max}) = E[U(W - X)]$.

The insurer, with wealth W, faces a similar problem. The minimum premium, $\Pi_{\min}$ that they would accept as $U(W) = E[U(W + \Pi_{\min} - X)]$.

1. Calculate the agent's expected wealth without insurance $V \to E(w-x) =$
2. Calculate the expected utility of wealth without insurance $\to E u(w-x))$ $\ln(w-x)$
3. What is the maximum premium they would be willing to pay to protect themselves against this loss?
4. What is the minimum premium that an insurer, with the same utility function and with wealth \$$10^6$, be willing to charge to cover this loss?
   The unique solution of the equation $10^6 = (999000 + x)^{0.75}(990000 + x)^{0.25}$ is: 3257.6

Solution:

1. The agent's expected wealth without insurance is

$$
\begin{aligned}
E[W - X] &= ((50000 - 1000) \times 0.75) + ((50000 - 10000) \times 0.25) \\
&= (49000 \times 0.75) + (40000 \times 0.25) = 46750.
\end{aligned}
$$

2. The expected utility of wealth without insurance is

$$
\begin{aligned}
u(w-p) = E[U(W - X)] &= E[\ln(W - X)] = (\ln(50000 - 1000) \times 0.75) + (\ln(50000 - 10000) \times 0.25) \\
&= (\ln(49000) \times 0.75) + (\ln(40000) \times 0.25) = 10.74884.
\end{aligned}
$$

3. To identify the maximum premium we need to solve the equation $U(W - \Pi_{\max}) = E[U(W - X)]$ that $\ln(50000 - \Pi_{\max}) = 10.74884$ hence $50000 - \Pi_{\max} = \exp(10.74884)$ which leads to $\Pi_{\max} = 50000 - \exp(10.74884) = 3424.0310$

   $u(w-p) = Eu(w-x)$

4. The insurer will set $\Pi_{\min}$ by equating $\ln(10^6) = E[\ln(10^6 + \Pi_{\min} - X)]$. Which gives

$$
\begin{aligned}
\ln(10^6) &= 0.75 \times \ln(10^6 + \Pi_{\min} - 1000) + 0.25 \times \ln(10^6 + \Pi_{\min} - 10^4) \\
&= \ln\left[(999000 + \Pi_{\min})^{0.75}\right] + \ln\left[(990000 + \Pi_{\min})^{0.25}\right] \\
&= \ln\left[(999000 + \Pi_{\min})^{0.75}(990000 + \Pi_{\min})^{0.25}\right]
\end{aligned}
$$

Therefore $10^6 = (999000 + x)^{0.75}(990000 + x)^{0.25}$, hence the solution is: $\Pi_{\min} = 3257.6$.

$\ln(w) = u(w) = E[u(w + P - x)]$

Problem 3. **(8 marks)** In order to simplify an actuarial analysis Actuary A uses an aggregate distribution $S = X_1 + X_2 + \cdots + X_N$, where N has a Poisson distribution with mean 10 and $\boxed{X_i = 1.5 \text{ for all } i.}$

Actuary A's work is criticized because the actual severity distribution (the distribution of the individual loss) is given by $P(Y_i = 1) = P(Y_i = 2) = \frac{1}{2}$, for all i, where Y_i's are independent and Y_i is independent from N

Set $S^* = Y_1 + Y_2 + \cdots + Y_N$.

1. Find S in terms of N and compare the two expected aggregate losses.
2. Compare the variances of the two aggregate losses.
3. Calculate $E[SS^*]$ using $E[E[SS^* \mid N]] = \sum_{n=0}^{\infty} E[SS^* \mid N = n] P(N = n)$ and the fact that $E[Y_j \mid N = n] = E[Y_1]$ for all j.
4. Calculate the correlation coefficient $\rho(S; S^*)$ between S and S^*, (where $\rho(S; S^*) = \frac{\mathrm{Cov}(S; S^*)}{\sigma_S \sigma_{S^*}}$)

Solution:

1. We have $S = 1.5N$, then $E[S] = 1.5E[N] = 1.5 \times 10 = 15$ and $E[S^*] = E[N]E[Y] = 10(1.5) = 15$, then $E[S] = E[S^*]$.
2. For the variance $\mathrm{Var}(S) = (1.5)^2 \mathrm{Var}(N) = (1.5)^2 \times 10 = 22.5$ and $\mathrm{Var}(S^*) = E[N]\mathrm{Var}(Y) + \mathrm{Var}(N)(E[Y])^2 = 10(0.25) + 10(1.5)^2 = 25$, then $\mathrm{Var}(S^*) > \mathrm{Var}(S)$
3. We write

$$
\begin{aligned}
E[SS^*] &= E[E[SS^* \mid N]] = \sum_{n=0}^{\infty} E[SS^* \mid N = n] P(N = n) \\
&= \sum_{n=0}^{\infty} E[1.5N(Y_1 + Y_2 + \cdots + Y_N) \mid N = n] P(N = n) \\
&= \sum_{n=0}^{\infty} E\left[1.5n\left(\sum_{j=1}^{n} Y_j\right) \mid N = n\right] P(N = n) \\
&= 1.5\sum_{n=0}^{\infty} n \sum_{j=1}^{n} E[Y_j \mid N = n] P(N = n) = 1.5\sum_{n=0}^{\infty} n^2 E[Y_1] P(N = n) \\
&= 1.5E[Y_1] \sum_{n=0}^{\infty} n^2 P(N = n) = 1.5E[Y_1] E[N^2].
\end{aligned}
$$

Now, we $E[Y_1] = \frac{1}{2} + 2 \times \frac{1}{2} = \frac{3}{2} = 1.5$ and $E[N^2] = \mathrm{Var}(N) + (E[N])^2 = 10 + 10^2 = 110$. Thus $E[SS^*] = 1.5 \times 1.5 \times 110 = 247.5$.

4. We have $\mathrm{Cov}(S; S^*) = E[SS^*] - E[S]E[S^*] = 247.5 - 15 \times 15 = 22.5$, therefore

$$
\rho(S; S^*) = \frac{\mathrm{Cov}(S; S^*)}{\sigma_S \sigma_{S^*}} = \frac{22.5}{\sqrt{22.5}\sqrt{25}} = 0.94868.
$$

Conclusion, the two aggregate loses S and S^* are positively correlated. And there is a linear dependence between them.

Problem 4. **(8 marks)**

$E(Z) + c\sqrt{Var(Z)}$

1. For an insurance portfolio:
(i) The **number of claims** has the probability distribution N

k	0	1	2	3
p_k	0.1	0.4	0.3	0.2

(ii) Each **claim amount** has a Poisson distribution with mean 3. X
(iii) The number of claims and claim amounts are mutually independent.
Calculate the premium $\Pi_{sd}(b)$ corresponding to S with standard deviation loading $b = 2.5\%$.

2. The **number of claims** N, in a period has a geometric distribution with mean 3. The **amount of each claim** X follows $P(X = x) = \frac{1}{4} = 0.25$, $x = 1, 2, 3, 4$. The number of claims and the claim amounts are independent. S is the aggregate claim amount in the period. The p.m.f. of a geometric distribution is $p_k = p(1-p)^k = \frac{\beta^k}{(1+\beta)^{k+1}}$, for $n = 0, 1, 2, 3, \ldots$ where $0 < p < 1$ that is its mean is $\frac{1-p}{p} = \beta$, its variance is $\beta(1+\beta)$ and its p.g.f. $P_N(t) = \frac{1}{1-\beta(t-1)}$.
Calculate the probability that the aggregate loss is less or equal than 3. (Hint use Panjer's recursion).

$$f_S(n) = \frac{(p_1 - (a+b)p_0)\, f_X(n) + \sum_{j=1}^{n} \left(a + \frac{b}{n} j\right) f_X(j) f_S(n-j)}{1 - a f_X(0)} \quad \text{and } f_S(0) = P_N(f_X(0)) = P_N(P(X=0))$$

Solution:

1. We have $\Pi_{sd}(b) = E[S] + b\sqrt{\mathrm{Var}(S)}$. Recall that $E[S] = E[N]E[X]$ and $Var(S) = E[N]Var(X) + (E[X])^2 Var(N)$ $1.6 \times 3 = 4.8$. We need $E[N] = 0.4 + 0.3 \times 2 + 0.2 \times 3 = 1.6$ and $E[N^2] = 0.4 + 0.3 \times 2^2 + 0.2 \times 3^2 = 3.4$, so $Var(N) = 3.4 - (1.6)^2 = 0.84$. Therefore

$$E[S] = 1.6 \times 3 = 4.8 \quad \text{and} \quad \mathrm{Var}(S) = 1.6 \times 3 + 3^2 \times 0.84 = 12.36$$

Finally $\Pi_{sd}(b) = 4.8 + 0.025\sqrt{12.36} = 4.8879$.

2. We want to calculate $P(S \le 3) = F_S(3) = f_S(0) + f_S(1) + f_S(2) + f_S(3)$.
First

$$f_S(0) = P(N=0) = P_N(f_X(0)) = P_N(P(X=0)) = P_N(0) = \frac{1}{1 - 3(0-1)} = \frac{1}{4} = 0.25.$$

Recall that the geometric distribution is in the $C(\frac{\beta}{1+\beta}, 0, 0)$ class where $E[N] = \beta = 3$, that is Then

$$f_S(n) = P(S=n) = \frac{\frac{3}{4}\sum_{j=1}^{n} f_X(j) f_S(n-j)}{1 - \frac{3}{4} f_X(0)} = \frac{3}{4}\sum_{j=1}^{n} f_X(j) f_S(n-j) \quad (\text{since } f_X(0) = 0).$$

Consequently in our case we have

$$f_S(n) = \frac{3}{4}\sum_{j=1}^{n} \frac{1}{4} f_S(n-j) = \frac{3}{4}\frac{1}{4}\sum_{j=1}^{n} f_S(n-j) \quad (\text{since } f_X(j) = \frac{1}{4}).$$

Hence $f_S(1) = \frac{3}{4}\frac{1}{4} f_S(0) = \frac{3}{4}\frac{1}{4}\frac{1}{4} = 0.046875$,

$$f_S(2) = \frac{3}{4}\frac{1}{4}\sum_{j=1}^{2} f_S(2-j) = \frac{3}{4}\frac{1}{4}(f_S(0) + f_S(1))) = \frac{3}{4}\frac{1}{4}(0.25 + 0.046875) = 0.055664$$

$$f_S(3) = \frac{3}{4}\frac{1}{5}\sum_{j=1}^{3} f_S(3-j) = \frac{3}{4}\frac{1}{4}(f_S(0) + f_S(1) + f_S(2)) = \frac{3}{4}\frac{1}{4}(0.25 + 0.046875 + 0.055664) = 0.066101.$$

So

$$F_S(3) = 0.25 + 0.046875 + 0.055664 + 0.066101 = 0.41864.$$

Problem 5. (8 marks)

1. A life insurance company issues 1–year term life contracts for benefit amounts of 1 and 2 units to individuals with probabilities of death of 0.02 or 0.10. The following table gives the number of individuals n_k in each of the four classes created by a benefit amount b_k and a probability of claim q_k.

k	q_k	b_k	n_k
1	0.02	1	500
2	0.02	2	500
3	0.10	1	300
4	0.10	2	500

The security loading total premium for this insurance is $\Pi_\theta = (1+\theta)E[S]$.

(a) What is the suitable risk model for this insurance. Justify your answer

(b) Use normal approximation to find θ such that the probability of the aggregate loss is less or equal to Π_θ equals 0.95. (95^{th} percentile of the standard normal distribution $\mathcal{N}(0,1)$ is 1.645).
Some useful notations $S = \sum_{i=1}^{n} X_i$ where X_i are independent. The random variable I_k indicates whether or not the k^{th} policy produced a payment. If the claim has occurred, then $I_k = 1$; if there has not been any claim, $I_k = 0$. Therefore we can write

$$X_k = I_k B_k,\ \ \mu_k = E\left[B_k\right],\ \ q_k = P(I_k = 1) \quad \text{and} \quad 1 - q_k = P(I_k = 0)$$

Then

$$E\left[X_k\right] = \mu_k q_k, \quad \text{and } \operatorname{Var}(X_k) = \mu_k^2 q_k(1-q_k) + \sigma_k^2 q_k$$

2. If N be a discrete non–negative random variable with p.m.f. p_k, a zero–modified distribution is of the form: $p_k^M = \frac{1-p_0^M}{1-p_0} p_k$ where $p_0^M \in [0,1)$.
Consider the zero–modified geometric distribution: $p_0^M = \frac{1}{2}$, $p_k^M = \frac{1}{6}\left(\frac{2}{3}\right)^{k-1}$, $k = 1, 2, 3, \ldots$

(a) Find the parameter $p = p_0$ of the initial geometric distribution p_k of N. (recall that $p_k = p(1-p)^k$, $k \geq 0$).

(b) Let N^M be a r.v. whose distribution is the zero–modified geometric distribution p_k^M given above. Find moment generating function of N^M.

(c) Find the exponential premium $\Pi_{\exp}(\alpha) = \dfrac{\ln\left(M_{N^M}(\alpha)\right)}{\alpha}$, for $\alpha = -1.5$.

Solution:

1. (a) The total number of insured individuals in this insurance is 1800. so the individual model is suitable for this type of insurance is the individual risk model.

(b) To find θ we solve the equation $P\left(S \leq \Pi_\theta\right) = 0.95$. Thence

$$P\left(S \leq (1+\theta)E[S]\right) = P\left(S - E[S] \leq \theta E[S]\right) = P\left(\frac{S - E[S]}{\sigma_S} \leq \theta\frac{E[S]}{\sigma_S}\right).$$

Set $T = \frac{S-E[S]}{\sigma_S}$, then $E[T] = 0$ and $\operatorname{Var}(T) = 1$, thus using normal approximation for $S = \sum\limits_{i=1}^{1800} X_i$, T follows a standard normal distribution $\mathcal{N}(0,1)$. Therefore $P(T \leq \theta\frac{E[S]}{\sigma_S}) = 0.95$ this means that

$\theta\dfrac{E[S]}{\sigma_S}$ is the 95^{th} percentile of $\mathcal{N}(0,1)$, consequently $\theta\dfrac{E[S]}{\sigma_S} = 1.645$, hence $\theta = 1.645\dfrac{\sigma_S}{E[S]}$. We need $E[S]$ and σ_S. Th

k	q_k	b_k	n_k	mean $q_k b_k$	Variance $q_k(1-q_k)b_k^2$
1	0.02	1	500	0.02	0.0196
2	0.02	2	500	0.04	0.0784
3	0.10	1	300	0.10	0.0900
4	0.10	2	500	0.20	0.3600

Therefore

$$E[S] = \sum_{i=1}^{1800} E[X_i] = \sum_{i=1}^{4} n_k b_k q_k = 160$$

and

$$\mathrm{Var}(S) = \sum_{i=1}^{1800} \mathrm{Var}(X_i) = \sum_{i=1}^{4} n_k q_k (1-q_k) b_k^2 = 256$$

and $\theta = 1.645\frac{\sqrt{256}}{160} = 1.645\frac{16}{160} = 0.1645$.

2. Recall that a zero–modified distribution is of the form: $p_k^M = \frac{1-p_0^M}{1-p_0}p_k$.

   (a) We know that for any $k \geq 1$, $p_k^M = \frac{1-p_0^M}{1-p_0}p_k$ then

$$p_k = \frac{1-p_0}{1-p_0^M}p_k^M = 2(1-p_0)\frac{1}{6}\left(\frac{2}{3}\right)^{k-1} = \frac{1-p_0}{3}\left(\frac{2}{3}\right)^{k-1} = p(1-p)^k \text{ for any } k \geq 1$$

   Notice that $p_0 = p$ thus in particular for $k=1$ we have $\frac{1-p}{3} = p(1-p)$ then $p = \frac{1}{3}$.

   (b) The m.g.f. of N^M is

$$\begin{aligned} M_{N^M}(\alpha) &= \sum_{k=0}^{\infty} e^{k\alpha}p_k^M = \frac{1}{2} + \sum_{k=1}^{\infty} e^{k\alpha}\frac{1}{6}\left(\frac{2}{3}\right)^{k-1} = \frac{1}{2} + \frac{1}{6}\times\frac{3}{2}\sum_{k=1}^{\infty}\left(\frac{2e^{\alpha}}{3}\right)^k \\ &= \frac{1}{2} + \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{2e^{\alpha}}{3}\right)^k = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{1-\frac{2e^{\alpha}}{3}} - 1\right) = \frac{1}{4}\left(1 - \frac{1}{\frac{2}{3}e^{\alpha} - 1}\right) \end{aligned}$$

   (c) So for $\alpha = -1.5$ we get

$$M_{N^M}(-1.5) = \frac{1}{4}\left(1 - \frac{1}{\frac{2}{3}e^{-1.5} - 1}\right) = 0.54369.$$

   Thus

$$\Pi_{\exp}(-1.5) = \frac{\ln(0.54369)}{-1.5} = 0.40625.$$