Chapter 1

The theory of interest: It is well that 100$ to be received after 1 year is worth less than the same amount today.

The way in which money changes it is value in time is a complex issue of fundamental importance in mathematical finance.

In this chapter we discuss the following 2 questions :

- What is the future value of an amount invested or borrowed today?
- What is the present value of an amount to be paint or received at a certain time in the future?

1) simple interest:

Suppose that an amount is paid in to a bank account where it is to earn interest, the future value of the investment after t time :

$V(t) = p+I$ p: principal I:intrest

We now consider the case when interest is attracted only by the principal → simple interest in this case :

$I=p\times r\times t$ r:rate of return t: time in year.

Example : someone interesting 4000$ for 2 years with the interest rate 5.5% how much interest will earn after 2 years?

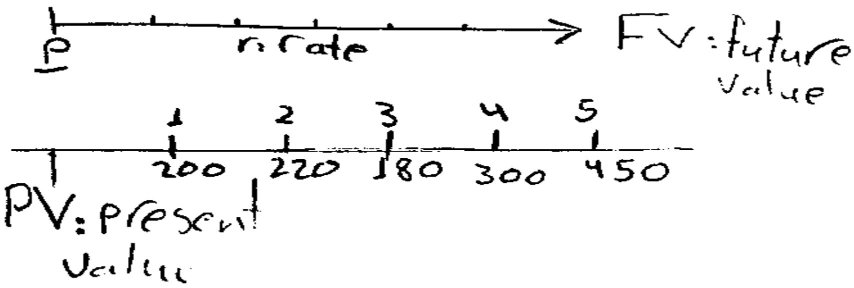
Answer : $I=4000\times.055\times2=440\$$

Ex (homework):

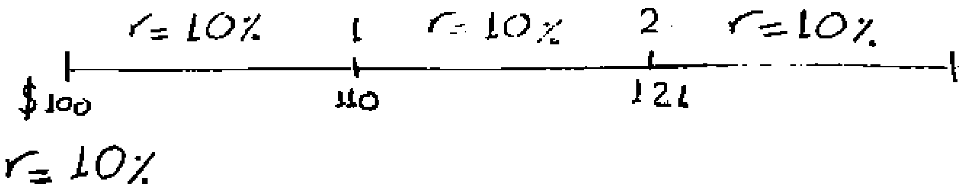
Consider a deposit of 150$ held for 20 days . How much interest will earn after 20 days with 8% as rate of interest.

$v(t)=p(1+rt)= 150(1+.08(20/365))=150.66\$$

Note : time value of money



2)compound interest:



Suppose now that an amount P is deposited in a bank and assume that the interest earned will be added to the principal periodically (annually - monthly - quarterly - semi-annually).

In this situation, will be attracted not just by the original deposit but also by all interest earned so far . We say that we have a periodic compound interest.

after 1year

$$\$100 \longrightarrow 100 + 10\%(100)$$

$$100\underbrace{\left(1+\frac{10}{100}\right)}_{1.1} = 110$$

after 2 years

$$\longrightarrow \underbrace{110}_{(1+0.1)} + 10\% \text{ of } 110$$

$$= 100(1.1) + 100(1.1)(1.1)$$

$$= 100\left(1+\frac{10}{100}\right)^2$$

after x years

$$= 100\left(1+\frac{10}{100}\right)^x$$

Question: how long will it take to double the capital?

Answer :

$$100(1.1)^x = 200$$

$$(1.1)^x = 2$$

$$x = \log_{1.1} 2$$

Formula of compound interest :

P:deposit.

r:rate of compound monthly interest.

$$\text{After 1 month} = P + \frac{r}{12}P = \left(1 + \frac{r}{12}\right)P$$

$$\text{2 month} = \left(1 + \frac{r}{12}\right)P + \frac{r}{12}\left(1 + \frac{r}{12}\right)P$$

$$= P\left(1 + \frac{r}{12}\right)^2$$

$$\text{After 1 year} = P\left(1 + \frac{r}{12}\right)^{12}$$

$$\text{After } t \text{ year} = P\left(1 + \frac{r}{12}\right)^{12t}$$

In general

$$V(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

m=1 compounded yearly.

m=12 compounded monthly.

m= 365 compounded daily.

EX :How long will it take to double a capital attracting interest 6%compounded daily?

Proposition : the future value **v(t)** increase if anyone of the parameters **m,t,r** or **p** increase.

Proof: $v(t) = P(1+\frac{r}{m})^{tm}$

- The proof is clear for t, p, r
- Let $m<k$

$$(1+\frac{r}{m})^m < (1+\frac{r}{k})^k$$

$$(a+b)^n = \sum_{i=0}^{n} C_n^i a^i b^{i-1} \quad \text{(Formla of Binomial of Newton)}$$

$$\binom{i}{n}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \cdots$$

$$(1+\frac{r}{m})^m = 1 + m\frac{r}{m} + \frac{m(m-1)r^2}{2m^2} + \cdots$$

$$\leq 1 + r + \frac{k(k-1)r^2}{2k} + \cdots$$

$$\frac{m(m-1)}{2m^2} = \frac{1-\frac{1}{m}}{2}$$

$$m<k \longrightarrow \frac{1}{m} > \frac{1}{k}$$

$$-\frac{1}{m} < -\frac{1}{k}$$

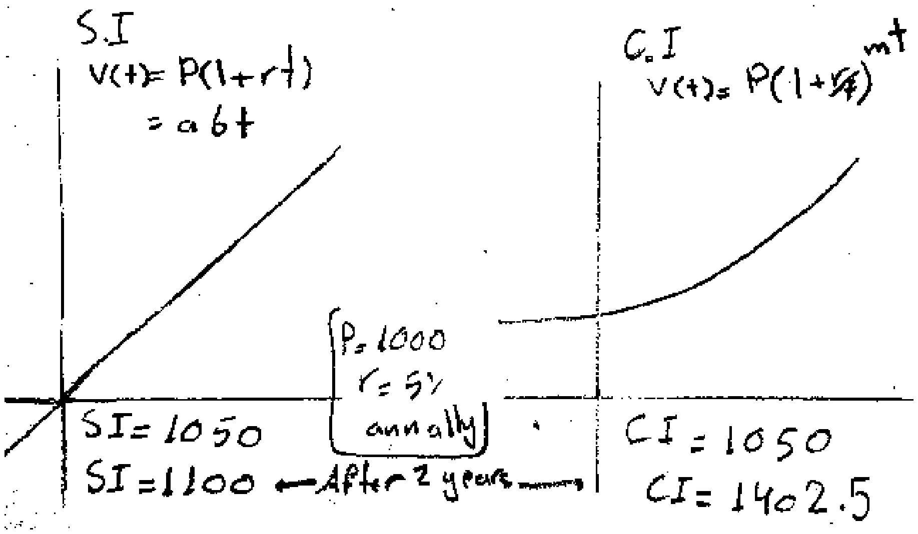
$$\therefore \frac{1-\frac{1}{m}}{2} < \frac{1-\frac{1}{k}}{2}$$

we deduce that

$$(1+\frac{r}{m})^m \leq (1+\frac{r}{k})^k$$

_ this proof in page 31.

Note :comparison between compounded and simple interest.



Effective interest rate :

re : The simple annual interest rate which wanted generate the same amount as the annual compounded rate.

Example : Annual interest rate of 5.75% compounded monthly is equivalent to an effective annual rate of 5.90%.

In general

$$re = \left(1 + \frac{r}{m}\right)^m - 1$$

Homework : which will deliver a higher future value after 1 year. A deposit of 1000$ attracting interest at 15% compounded daily or at 15.5 compounded semi-annually.

Example : what initial investment subject to annual compounding at 12% is needed to produce 1000$ after 2 years.

3) continuous compounding :

If a future value v(t) of a principal P attracting interest at a rate r compounded m times with m very large.

$$v(t) = \lim_{m\to\infty} \underline{P}\left(1 + \frac{r}{m}\right)^{mt}$$

Note:

$$\lim_{m\to\infty} \underline{P}\left(1 + \frac{r}{m}\right)^{mt} = 1^{\infty} = (IF)$$

L´Hôpital rule

$$\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{m} ?$$

$$\lim_{m\to\infty} m \ln\left(1 + \frac{r}{m}\right) = \lim_{m\to\infty} \frac{\ln(1 + r/m)}{1/m} \quad \frac{0}{0}$$

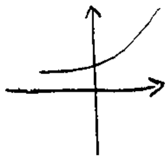
$$= \lim_{m\to\infty} \frac{-r/m^2}{(1 + r/m)} \cdot (m^2)$$

$$\Rightarrow \lim_{m\to\infty} (t) = r$$

$$\Rightarrow \lim_{m\to\infty} v(t) = \underline{P}\, e^{rt}$$

Continuous compounding :

$V(t) = \underline{P}\, e^{rt}$



Proposition :

Continues produces higher future value than periodic compounding with any frequency m .

Proof:

$Pe^{rt} \geqslant P\left(1+\frac{r}{m}\right)^{mt}$

It suffice to show that

$e^{r} \geqslant \left(1+\frac{r}{m}\right)^{m}$

$\{a_m\} = \left\{1+\frac{r}{m}\right\}^{m}$ is an increasing sequence and

$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m} = e^{r}$

$\left(1+\frac{r}{m}\right)^{m} < e^{r}$

$\Rightarrow$ conclusion .

Note : for continuous compounding

$re = e^{r} - 1$

Example : Suppose P is deposited with annual rate of 6.15 compounded continuous. Then find re .

$re = e^{0.0615} - 1 \simeq 6.34\,\%$

Note :

$$V(t) = Pe^{rt}$$
$$V'(t) = rPe^{rt} = rV(t)$$

*rate of change of the future value is proportional to the value itself.

Ex: Suppose that an interest will receive payments given in the following rate

Years	1	2	3	4	5	6
payments	465	233	632	365	324	248

If the interest rate is 3.99% compounded monthly. What is the present value of the investment?

1 2 3 4 5 6

$$465\left(1+\frac{0.0399}{12}\right)^{-12}$$
$$+$$
$$233\left(1+\frac{0.0399}{12}\right)^{-24}$$
$$+$$
$$632\left(1+\frac{0.0399}{12}\right)^{-36}$$
$$+$$
$$365\left(1+\frac{0.0399}{12}\right)^{-48}$$
$$+$$
$$334\left(1+\frac{0.0399}{12}\right)^{-60}$$
$$+$$
$$248\left(1+\frac{0.0399}{12}\right)^{-72}$$

$\Rightarrow PV = \$ 2003.31$ *Note: $PV \neq$ sum of payments $\$2277$

4) Annuity : an annuity is a sequence of faintly many payments of a fixed amount due at equal time intervals.

Suppose that someone borrows P from the bank with interest (r) compounded in time per years.

The monthly payment can be calculated as following:

X: the fixed monthly payments.

1 2 3

$$P = \begin{array}{l} x\left(1+\frac{r}{m}\right)^{-1} \\ x\left(1+\frac{r}{m}\right)^{-2} \\ \vdots \end{array}$$

$$P_V = \sum_{i=1}^{mt} x\left(1+\frac{r}{m}\right)^{-i}$$

$$S = 1 + a + a^2 + a^3 + \cdots + a^n \quad [illegible]$$

$$aS = a + a^2 + a^3 + \cdots + a^{n+1}$$

$$(1-a)S = 1 - a^{n+1}$$

$$S = \frac{1-a^{n+1}}{1-a}$$

$$S = a + a^2 + a^3 + \cdots + a^n$$

$$\Rightarrow S = a\frac{(1-a^n)}{1-a}$$

$$P_V = \sum_{i=1}^{mt} x\left(1+\frac{r}{m}\right)^{-i} = x\left(1+\frac{r}{m}\right)^{-1}\frac{\left[1-\left(1+\frac{r}{m}\right)^{-mt}\right]}{1-\left(1+\frac{r}{m}\right)^{-1}}$$

$$1-\left(\frac{m+r}{m}\right)^{-1}$$

$$1-\frac{m}{m+r}$$

$$\frac{m+r-m}{m+r} = \frac{r}{m+r}$$

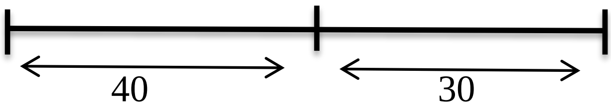
$$= \frac{r}{m}\left(\frac{1}{1+\frac{r}{m}}\right)$$

$$P_V = x\,\frac{m}{r}\left[1-\left(1+\frac{r}{m}\right)^{-mt}\right]$$

Annuity can be used to the concept of retirement .

Ex : Suppose a person is 25 years today plan to retire at at age 65 years .For the next 40 years he plan to invest a portion of his monthly incomes in securities which earn interest at the rate of 10% compounded monthly after retirement the person plan to receive 1500$ per month for 30 years.

What will be the monthly deposit of the person ?



x: deposit amount per month .

x ? , x we will be calculated by the equation :

$$P_v(40 \text{ years}) = P_v(30 \text{ years})$$

$$25 \rightarrow 26 \qquad\qquad 65 \rightarrow 95$$

$$\sum_{i=1}^{480} x\left(1+\frac{0.1}{12}\right)^{-i} = x\,\frac{12}{0.1}\left[1-\left(1+\frac{0.1}{12}\right)^{-480}\right]$$

$$\sum_{i=481}^{840} 1500\left(1+\frac{0.1}{12}\right)^{-i} = 1500\,\frac{12}{0.1}\left[1-\left(1+\frac{0.1}{12}\right)^{-360}\right]$$

$$\Rightarrow x = 27.03\$ \text{ per month.}$$

4) perpetuity :

A perpetuity is an infinite sequence of payment of a fixed amount **C** occurring at the end of each year.

The present value of a perpetuity can be obtained as follows :

$$P_v = \sum_{i=1}^{\infty} c \left(1+\frac{r}{1}\right)^{-i \cdot 1}$$

$$P_v = c \sum_{i=1}^{\infty} (1+r)^{-i} \quad : \text{Infinite geometric series}$$

We recall that :

if $\sum_{i=1}^{\infty} a^i$, then $S_n = \sum_{i=1}^{n} a^i$

$$= a \frac{(1-a^n)}{1-a}$$

and if $|a|<1$, then $a^n \xrightarrow[n\to\infty]{} 0$.

Hence $\sum_{i=1}^{\infty} a^i = \frac{a}{1-a}$

In our Case $a = \frac{1}{1+r} \Rightarrow \frac{a}{1-a} = \frac{1}{r}$

$$\Rightarrow \boxed{P_v = \frac{c}{r}}$$

Homework .

You want to endow a fund which pays out a scholarship of 1000$ every year in perpetuity .

The first scholarship will be paid out in the five years' time. Assume that the interest rate is 7% . How much do you need to pay into the fund.

5) Return - rate of return :

The return on an investment commencing at time S and terminating at T is defined by:

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = \frac{V(t)}{V(s)} - 1$$

Note $\quad V(t) = V(s) + K(s,t)\, V(s)$

Example :

1) for simple interest :

$$k(s,t) = \frac{(1+rt)}{(1+rs)} - 1 = \frac{r(t-s)}{1+rs}$$

2) In the case of compounding interest :

$$K(s,t) = \left(1 + r/m\right)^{(t-s)m} - 1$$

Note:

The return can be used to compute interest rate (compounded) :

K (0,1/m) = r/m

3) for continuous compounded interest :

$$k(s+t) = e^{r(t-s)} - 1$$

Remark: For continuous and discrete compounded interest, the return fails to be additive.

Example:

$$k(0,1) = e^r - 1 \qquad k(1,2) = e^r - 1$$

$$k(0,2) = (e^r)^2 - 1$$

We have $k(0,1) + k(1,2) \neq k(0,2)$

It will be more convenient to introduce the logarithmic return :

$$k(s,t) = \operatorname{Ln}\left[\frac{V(t)}{V(s)}\right]$$

Proposition: The logarithmic return is additive.

$$k(s,t) + k(t,u) = k(s,u)$$

Proof :

$$k(s,t) + k(t,u) = \operatorname{Ln}\frac{V(t)}{V(s)} + \operatorname{Ln}\frac{V(u)}{V(t)}$$

$$= \operatorname{Ln}\frac{V(u)}{V(s)}$$

$$= k(s,u)$$

The rate of return is a profit on an investment over a period of time.

If someone invest **P** now and receive a sequence of positive payoffs **{ A_1, A_2 ,.......,A_n}** at regular intervals.

In this case the rate of return per period is the interest rate such that the present value of the sequence of payoffs is equal to the amount invested.

In this case :

$$P = \sum_{i=1}^{n} A_i (1+r)^{-i}$$

r will be a solution of the equation

$$f(r) = 0 \text{ when } f(r) = -P + \sum_{i=1}^{n} A_i (1+r)^{-i}$$

Note : r is called zero of **f** .

We can see that :

- $f(r)$ is continuous on $(-1, \infty)$
- $\lim_{r \to -1^+} f(r) = \infty$, $\lim_{r \to \infty} f(r) = -P < 0$

By the intermediate value theorem, there exist **r*** , with **-1 < r* <** ∞ such that **f(r*) = 0**

Note : r* is unique.

Homework : Suppose you have the choice of investing 1000$ in just on of two ways.

Each investment will pay you an amount given by the following table :

year	1	2	3	5	6
Investment A	225	215	250	225	205
Investment B	220	225	250	250	210

1) using the present value of the investment to make the decision, which investment would you choose? We assume that the annual interest rate is 4.33%

2) using the rate of return per year of the investment to make the decision, which investment would you choose?