King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2016–2017
Academic Year (H) 1437–1438
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First midterm QMF: Actu. 468 (25%)

Sunday, November 6, 2016 / Rabi I 6, 1438 (1 – 2:30) pm

Exercise 1. (6 marks)

We specify below the basic elements of a financial market with T periods:

- A finite probability space $\Omega = \{\omega_1, \ldots, \omega_k\}$ with k elements.
- A probability measure P on Ω, such that $P(\omega) > 0$ for all $\omega \in \Omega$.
- A riskless asset (a saving account) $S_t^0, t \in \{0, 1, 2, \ldots T\}$ such that $S_0^0 = 1$ with a constant interest rate r.
- A d–dimensional price process S_t, $t \in \{0, 1, 2, \ldots T\}$ where $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$ and S_t^i stands for the price of the asset i at time t.

1. **(1 mark)** Give the definition of a portfolio in this market
2. **(1 mark)** Recall the self–financing property for this model
3. **(1 mark)** Give the definition of attainable payoffs for this model
4. **(1 mark)** Give the definition of a RNPM (risk neutral probability measure) in this setting.
5. **(1 mark)** Give the definition of a complete market
6. **(1 mark)** Give the definition of an incomplete market

Exercise 2. (6 marks)

Assume that $T = 1$ and $k = 2$, $r = \frac{1}{4}$. Let $(S_t^1)_{t \in \{0,1\}}$ be the price of a stock with initial price $S_0^1 = 100$ SAR and has two possible values a time $T = 1$:

$$S_1^1(\omega) = \begin{cases} 200 \text{ SAR} & \text{if } \omega = \omega_1 \\ 75 \text{ SAR} & \text{if } \omega = \omega_2. \end{cases}$$

Denote by F the payoff of an European put option with strike price $K = 150\ SAR$.

1. **(1 mark)** Give the value of F at time $T = 1$.
2. **(1 mark)** Is the market $(S_t^0, S_t^1)_{t \in \{0,1\}}$ arbitrage free.
3. **(1 mark)** Is the market $(S_t^0, S_t^1)_{t \in \{0,1\}}$ complete.
4. **(1 mark)** Compute the price of the put option at time 0 using the RNPM.
5. **(1 mark)** Is the option F attainable ?
6. **(1 mark)** If yes find its replicating portfolio.

Solution:

Exercise 3. (7 marks)

Now assume that $k = 3$, $r = 0$, $S_0^1 = 100$ SAR and assume that the price of the stock S_1^1 is given by

$$S_1^1(\omega) = \begin{cases} 200 \text{ SAR} & \text{if } \omega = \omega_1 \\ 150 \text{ SAR} & \text{if } \omega = \omega_2 \\ 75 \text{ SAR} & \text{if } \omega = \omega_3. \end{cases}$$

1. **(2 mark)** Find RNPM if any for the model $(S_t^0, S_t^1)_{t\in\{0,1\}}$?
2. **(1 mark)** Is the market $(S_t^0, S_t^1)_{t\in\{0,1\}}$ arbitrage free ?
3. **(1 mark)** Is the market $(S_t^0, S_t^1)_{t\in\{0,1\}}$ complete ?
4. **(1 mark)** Find the set of attainable contingent claims.
5. **(1 mark)** Show that the value at time zero of an attainable claim is the same for all RNPM.
6. **(1 mark)** Give an example of non attainable asset.

Exercise 4. (6 marks)

Assume that $k = 3$, $r = 0$ and consider now a financial market on which are negotiated two stocks with prices $(S_t^1)_{t\in\{0,1\}}$ and $(S_t^2)_{t\in\{0,1\}}$, their values at time 1 are given by:

$$S_0^1 = 10 \quad \text{and} \quad S_1^1(\omega) = \begin{cases} 20 \text{ SAR} & \text{if } \omega = \omega_1 \\ 15 \text{ SAR} & \text{if } \omega = \omega_2 \\ 7.5 \text{ SAR} & \text{if } \omega = \omega_3. \end{cases}$$

and

$$S_0^2 = 4 \quad \text{and} \quad S_1^2(\omega) = \begin{cases} 5 \text{ SAR} & \text{if } \omega = \omega_1 \\ 3 \text{ SAR} & \text{if } \omega = \omega_2 \\ 4 \text{ SAR} & \text{if } \omega = \omega_3. \end{cases}$$

1. **(2 mark)** Find a RNPM for the model $(S_t^0, S_t^1, S_t^2)_{t\in\{0,1\}}$.
2. **(1 mark)** Is the model $(S_t^0, S_t^1, S_t^2)_{t\in\{0,1\}}$ arbitrage free and complete ?
3. **(1 mark)** Give an example of an attainable contingent claim for this model ?
4. **(1 mark)** Give its price at time zero
5. **(1 mark)** Find its replicating portfolio.

Exercise 5. (8 marks)

Consider the following probability space $(\Omega = \{\omega_1,\ \omega_2,\ \ldots,\ \omega_5\}$, on which is defined two period market model consisting of a riskless asset (bond or saving account) with price $S_t^0 = 1$ for $t = 0,\ 1,\ 2$ (for simplicity assume that $r = 0$) and two risky assets (stocks) with prices S^1 and S^2 given by:

t	S_t^0	S_t^1					S_t^2				
		ω_1	ω_2	ω_3	ω_4	ω_5	ω_1	ω_2	ω_3	ω_4	ω_5
0	1	6	6	6	6	6	3.75	3.75	3.75	3.75	3.75
1	1	5	5	5	7	7	3	3	3	4.5	4.5
2	1	3	4	8	6	8	2	3	4	4	5

We denote for $t \in \{0,1\}$ $\mathcal{F}_t = \sigma(\{S_k^0,\ S_k^1,\ S_k^2\},\ k \leq t)$ a set which describes all the in formations available in the market up to time t and $\mathcal{F}_2 = \mathcal{P}(\Omega)$ (the power set of $\mathcal{P}(\Omega)$).

1. **(1 mark)** Find $\mathcal{F}_0$ and $\mathcal{F}_1$. (remember that you need this objects to compute conditional expectations)
2. **(2 marks)** Find RNPM for this model if any ?
3. **(1 mark)** Is this market model arbitrage free and complete?
   Assume that F is random variable on Ω given by:

t	F				
	ω_1	ω_2	ω_3	ω_4	ω_5
2	1	$\frac{1}{4}$	3	2	0

4. **(2 marks)** Calculate the values of F at times 0 and 1.
5. **(2 marks)** Find a replicating portfolio of F for the period 1 and the period 2.