The Black-Scholes formula

This model is a continuous version of the binomial model.
The Black-Schols formula for a European Call option on a stock that pays dividends at the continuous rate δ is given by:

$$C_0 = C(S_0, K, \sigma, r, T, \delta) = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2).$$

where: N is the c.d.f of the standard normale distribution and

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

C_0 is the premium of the European Call option with strike price K and maturity T for an underlying asset modeled by the Black-schols model: That is:

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$; where $(W_t)_{t\geq 0}$ is a Brownian motion.

<u>Example</u>: Let $S_0 = 41$, $K = 40$, $\sigma = 30\%$, $r = 8\%$, $T = 3$ months and $\delta = 0$.

Compute the B-S. price of a call option: $C_0 = 3.399$.

From the call-put parity we deduce the price of a European put option with the same characteristics:

$$C_0 - P_0 = S_0 e^{-\delta T} - K e^{-rT} \Leftrightarrow P_0 = C_0 + K e^{-rT} - S_0 e^{-\delta T}.$$

$$\Rightarrow P_0 = 1.60703.$$

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The price of a call option can be computed using the Black-Scholes formula call spreadsheet, BSCALL:

$$BSCALL(S_0, K, \sigma, r, t, \delta) = BSCALL(41, 40, 0.3, 0.08, 0.25, 0).$$
$$= 3.399.$$

For the ~~p~~ european put option we have.

$$P(S_0, K, \sigma, r, T, \delta) = Ke^{-rT}(N(-d_2)) - S_0 e^{-\delta T} N(-d_1).$$

Indeed from the call-put parity we have.

$$P_0 = C_0 + Ke^{-rT} - S_0 e^{-\delta T}$$
$$= S_0 e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} - S_0 e^{-\delta T}$$
$$= Ke^{-rT}(1 - N(d_2)) - S_0 e^{-\delta T}(1 - N(d_1))$$

But we know that $N(x) + N(-x) = 1 \quad \forall x \in \mathbb{R}$.

Therefore $P_0 = Ke^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$.

Let C_t, P_t denote the prices of European call and put options given by the Black-Scholes formula: then

$$C_t = S_t e^{-\delta(T-t)} N(d_1(t)) - Ke^{-r(T-t)} N(d_2(t)).$$

where: $d_1(t) = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2(t) = d_1(t) - \sigma\sqrt{T-t}$,

Similarly $P_t = Ke^{-r(T-t)} N(-d_2(t)) - S_t e^{-\delta(T-t)} N(-d_1(t))$ for all $0 \le t \le T$

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for all $0 \le t \le T$

When is the Black-Scholes formula valid?

The derivation of the B-S formula makes a number of assumptions that can be sorted into two groups: Assumptions about how the stock price is distributed, and assumptions about the economic environment.

D.S.P
- continuously compounded returns on the stock are normally distributed and independent over time.
- The volatility of continuously compounded returns is known and constant.
- Future dividends are known, either as a dollar amount or as a fixed dividend yield.

A.E.E
- The ~~fi~~ risk-free rate is known and constant.
- There are no transaction costs or taxes.
- It is possible to short-sell costlessly and to borrow at the risk-free rate.

· Applying the B-S formula to other assets:

Remark first that we can rewrite d_1 in the B-S formula as follows:

$$d_1 = \frac{\ln(S_0 e^{-\delta T}/K e^{-rT}) + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}$$

because:

$$\ln\left(\frac{S_0 e^{-\delta T}}{K e^{-rT}}\right) = \ln\left(\frac{S_0}{K}\right) + \ln\left(e^{(r-\delta)T}\right)$$

$$= \ln\left(\frac{S_0}{K}\right) + (r-\delta)T.$$

The prepaid forward prices of the stock and strike asset are:

$$F^P_{0,T}(S) = S_0 e^{-\delta T}, \quad F^P_{0,T}(K) = K e^{-rT}$$

Now let us consider a call option (european type) on the prepaid forward price. This means that the underlying of the option is the prepaid forward price: The price of this call is:

$$C\left(F^P_{0,T}(S_0),\ F^P_{0,T}(K),\ \sigma, T\right) = F^P_{0,T}(S_0)\,N(d_1) - F^P_{0,T}(K)\,N(d_2) \quad (**)$$

where
$$d_1 = \frac{\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(K)}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Remark: This version of the B-S formula is interesting because the dividend yield and the interest rate ~~do~~ do not appear explicitly; they are implicitly incorporated into the prepaid forward prices.

To price options on underlying assets other than stocks, we can aply or use formula (**) in conjunction with the forward price formulas.

next page exercices (practice)

① $S_0 = 52$, $K = 50$; $r = 12\%$; $\sigma = 30\%$; $T = 3$ months.
Find the price of a call option with these characteristics.

$$C_0 = 52 \times 0.7041 - e^{-0.12 \times 0.25} \; 50 \times 0.6504 = 5.0543.$$

② $S_0 = 69$, $K = 70$, $r = 5\%$, $\sigma = 35\%$ and $T = 6$ Months.
Find the price of a put option. (european type).

$d_1 = 0.16662$ and $d_2 = -0.0809$.

$$P_0 = 70\, e^{-0.5 \times 0.05} N(0.0809) - 69\, N(-0.16662)$$
$$= 70\, e^{-0.5 \times 0.05} \times 0.5319 - 69 \times 0.4364 = 6.2 ..$$

« Excel: BSCALL & BSPUT »

③ : $S_0 = 58.96$; $K = 60.00$, Expected annual return on the stock is 10%; $\sigma = 20\%$; $\delta = 5\%$, $T = 3$ months; $r = 6\%$
Find C_0 of a call option and P_0 of a put option.
and verify the call-put parity.

$$C_0 = 58.96\, e^{-0.05 \times 0.25} N(d_1) - 60\, e^{-0.06 \times 0.25} N(d_2).$$
$$= 1.9260.$$

$P_0 = 2.8051$.

Call-put parity $\boxed{C_0 - P_0} = \boxed{S_0 e^{-\delta T} - K e^{-rT}}$

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(4) One euro is currently trading for $0.92.
The \$- denominated C.C.I.R is 6% and the ~~denom~~
€- denominated CCIR is 3.2%, $\sigma = 10\%$.

Question: Find the 1- year dollar-denominated euro call with strike price of \$0.9/€.

$$C_0 = S_0 e^{-r_f T} N(d_1) - K e^{-rT} N(d_2).$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - r_f + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

for the put option

$$P_0 = K e^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1).$$

or from the call put parity:

$$P_0 = C_0 + K e^{-rT} - S_0 e^{-r_f T}.$$

In our case: we found $C_0 = 0.0606$.

and $P_0 = 0.01719$.

* Test next Sunday April 16 from "7-9".

The end.

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Options on stocks with discrete dividents:

When a stock makes discrete divident payements, the prepaid forward price is given by:

$$F^P_{0,T}(S) = S_0 - PV_{0,T}(Div), \text{ where } PV_{0,T}(Div).$$

is the present value of dividends payable over the life of the option.

<u>Example</u>: $S_0 = 41$, $K = 40$, $\sigma = 30\%$, $r = 8\%$

$T = 3$ months $= 0.25$ years.

The stock will a \$3 dividend in one month, but makes no other payouts over the life of the option.

Find the price of this call option.

$C_0 = 1.7628$ for the put $P_0 = 2.9509$.

~~number~~ Remember that when there were no dividend we found $C_0 = 3.399$.

In the case we have on other payement in 2 months with \$2, then $PV_{0,T}(Div) = 3e^{-0.08 \times \frac{1}{12}} + 2e^{-0.08 \times \frac{2}{12}}$

$$PV_{0,T}(K) = Ke^{-rT}.$$

Options on currencies:

We can price an option on a currency by replacing the divident yield with the foreign interest rate.

If the spot exchange rate is x (expressed as domestic currency per unit of foreign currency) and the foreign currency interest rate is r_f, the prepaid forward price for the currency is:

$F^P_{0,T}(x) = x_0 e^{-r_f T}$, so using the Black-Scholes formula we have:

$$C(x, K, \sigma, r, T, r_f) = x e^{-r_f T} N(d_1) - K e^{-rT} N(d_2):$$

$$d_1 = \frac{\ln\left(\frac{x}{K}\right) + \left(r - r_f + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

This formula for the price of a European call on currencies is called the Garman-Kohlhagen model.

The price of a European currency put is obtained using call-put parity:

$$P(x, K, \sigma, r, T, r_f) = C(x, K, \sigma, r, T, r_f) + K e^{-rT} - x e^{-r_f T}$$

Example: $x = \$1.25/€$, $K = \$1.20$, $\sigma = 10\%$, $r = 1\%$; $T = 1$.
the euro-denominated interest rate is 3%.

The price of a dollar-denominated euro call is: 0.0614
and the " " " " " put is: 0.0364.

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