Options on futures:

The prepaid forward price for a futures contract is just the present value of the futures price.

Thus, we price a European option on a futures contract by using the futures price as the stock price and setting the dividend yield equal to the risk-free rate. The resulting formula is also known as the Black formula:

for a call option we have.

$$C(F, K, \sigma, r, T, r) = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2).$$

where $d_1 = \dfrac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

For the put option on futures contract we have.

$$P(F, K, \sigma, r, T, r) = Ke^{-rT}N(-d_2) - Fe^{-rT}N(-d_1).$$

This formula can be also obtained using the parity relationship: $C_0 - P_0 = F_0 - Ke^{-rT}$

$$= Fe^{-rT} - Ke^{-rT}$$

Then $P = C + Ke^{-rT} - Fe^{-rT}$

$$= C + (K - F)e^{-rT}$$

W11 P66

<u>Example:</u> Suppose the 1-year futures price for natural gas is \$ 6.50. and the volatily is 0.25. $r = 2\%$. Find the price of call and put european option set at the money on a futures contract on gas.

$$C_0 = 6.5\left(N(d_1) - N(d_2)\right)e^{-0.02\times 1} = 0.64660.$$

$$P_0 = C_0 + (K-F)e^{-rT} \quad (\text{but } K = F).$$

Hence $P_0 = C_0 = 0.64660$.

If the option is set in the money (that $F > K$)

then $P_0 < C_0$ since $K - F < 0$.

If the option is set out of the money (that $K > F$)

Then $P_0 > C_0$.

This is trivial for futures option contracts.

W11 P67.

Distribution of the stock price:

In framework of the Black-Schols model the stock price is ~~describe~~ described by the following model of exponential type under the real world probability (or historical probability):

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

where σ is the volatility of the stock per year and μ is the expected rate of return on the stock per year. and for $\forall t$, B_t is normally distributed with mean 0 and variance t: $B_t \hookrightarrow N(0, t)$.

This is equivalent to say that the log-price $\ln(S_t)$ is normally distributed with mean $\ln(S_0) + (\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

Indeed: we have $\ln(S_t) = \ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma B_t$

$$E[\ln(S_t)] = \ln(S_0) + (\mu - \frac{\sigma^2}{2})t \quad (\text{because } E[B_t] = 0)$$

$$Var(\ln(S_t)) = Var(\sigma B_t) = \sigma^2 Var(B_t) = \sigma^2 t.$$

The Black-Scholes model assumes that the log-price follows a normal distribution.

W11P67

Example (1) Consider a stock with initial price of 25, an expected rate of return of 5% per annum and a volatility of 20% per annum. (1) Find the distribution of the stock price S_T in 3 months under the B-S model.
(2) Find the confidence intervale of the stock at 95%.

Solution: We know that $\ln(S_T) \hookrightarrow N\left(\ln(S_0)+(\mu-\frac{\sigma^2}{2})T, \sigma^2 T\right)$

$T = 3\text{ months} = \frac{3}{12} = \frac{1}{4} = 0.25$.

$\ln(S_{0.25}) = N(3.2264; 0.01)$.

the confidence interval at 95% of $\ln(S_{0,25})$ is of the form:

$]3.2264 - 0.10 \times 1.96 ; 3.2264 + 0.10 \times 1.96[$.

$=]3.0304 ; 3.4224[$.

$P(\ln(S_{0,25}) \in]3.0304 ; 3.4224[) = 0.95$.

$= P(S_{0.25} \in]e^{3.0304}, e^{3.4224}[) = 0.95$.

That means there is a 95% probability that the stock price in 3 months will lie between 20.706 and 30.643.

Whenever the log of the stock follows a normal distribution we say the the stock is log-normal distributed: If $\ln(S_T) \hookrightarrow N(m, \sigma^2)$ we say S_T is log-normally distributed.

W11P69.

let $S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T}$.

Find $E[S_T]$ and $Var(S_T)$. If $X \hookrightarrow N(0, \sigma^2)$

Remember that $E[e^X] = e^{\frac{\sigma^2}{2}}$.

$$E[S_T] = S_0 e^{(\mu - \frac{\sigma^2}{2})T} E[e^{\sigma B_T}] = S_0 e^{(\mu - \frac{\sigma^2}{2})T} e^{\sigma^2 \frac{T}{2}}$$

$$= S_0 e^{\mu T}.$$

This formula leads to an estimation of μ.

that $\mu = \dfrac{\ln(E(S_T)) - \ln(S_0)}{T} = \dfrac{\ln(E(\frac{S_T}{S_0}))}{T}$.

and $Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$.

Example: consider a stock where the current price is $\boxed{60}$. the expected rate of return is 20% per annum and the ~~volatil~~ volatility $\sigma = 30\%$ per annum. Find the expected stock price and its variance. in four months. Deduce the standard deviation of the stock price in four months.

Solution: $E[S_T] = 60 e^{0.2 \times \frac{4}{12}} = 60 e^{\frac{0.2}{3}} = 64.136$.

$$Var(S_T) = (60)^2 e^{2 \times 0.2 \times \frac{4}{12}} \left(e^{(0.3)^2 \times \frac{4}{12}} - 1\right) = 125.27.$$

$$sd(S_T) = \sigma_{S_T} = \sigma_{S_{\frac{4}{12}}} = \sqrt{125.27} = 11.192.$$

for 1 year $E[S_1] = 60 e^{0.2} = 73.284$, $Var(S_1) = 505.77$

$$\sigma_{S_1} = 22.489.$$

W11P70

$$P_0 = Ke^{-r_€ T}N(-d_2) - S_0 e^{-r_y T}N(-d_1)$$
$$= 90e^{-0.015\times0.5}\times 0.278 - 95e^{-0.035\times0.5}\times 0,255 = 1.028\text{ Yen.}$$

$$\boxed{C_0 = S_0' e^{-r_y T}N(d_1) - K' e^{-r_€ T}N(d_2)}$$

$$C_€ = \frac{1}{S_0}e^{-r_y}N(d_1') - \frac{1}{K}e^{-r_€ T}N(d_2').$$

b): $S_0' = € \frac{1}{95}/\text{Yen}$, $K' = € \frac{1}{90}/\text{yen}$. | $r_f = r_{yen}$, $r_d = r_€$.

$$d_1' = \frac{\ln\left(\frac{90}{95}\right) + \left(r_€ - r_y + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2' = d_1' - \sigma\sqrt{T}.$$

$$\text{Yen } 95/€ \iff 1€ = 95\text{ yen} \Rightarrow 1\text{yen} = \frac{1}{95}€.$$

$$S_0' = €\frac{1}{95}/\text{yen}.$$

$$C_€ = \frac{1}{95}e^{-0.015\times0.5}N(d_1) - \frac{1}{90}e^{-0.035\times0.5}N(d_2).$$
$$= \frac{1}{95}e^{-0.015\times0.5}\times 0.278 - \frac{1}{90}e^{-0.035\times0.5}\times 0.255 = \frac{1.2027}{10^4}€$$

$$\begin{cases} P_0 = Ke^{-r_€ T}N(-d_2) - S_0 e^{-r_y T}N(-d_1) \\ C_0 = \frac{1}{S_0}e^{-r_€ T}N(+d_1') - \frac{1}{K}e^{-r_y T}N(+d_2') \end{cases} \quad \begin{aligned} d_1' &= -d_2 \\ d_2' &= -d_1 \end{aligned}$$

$$= \frac{1}{S_0}e^{-r_€ T}N(-d_2) - \frac{1}{K}e^{-r_y T}N(-d_1)$$

$$\boxed{C_0 S_0 K = Ke^{-r_€ T}N(-d_2) - S_0 e^{-r_y T}N(-d_1) = P_0}$$

Yen $S_0 K/€$.

Then: $P_{yen} = C_{Euro}\, S_0 K$
$$= \frac{1.2027}{10000}\cdot 95\times 90 = 1.028\text{ Yen}$$

W11P71

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(1) $C_0 = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) = C(S_0, K, \sigma, r, T, \delta)$.

$$= 100\, e^{-0.03\times 0.75}\, N\left(\frac{\ln\left(\frac{100}{95}\right)+\left(0.08-0.03+\frac{(0.3)^2}{2}\right)0.75}{0.3\sqrt{0.75}}\right)$$

$$- 95\, e^{-0.08\times 0.75}\, N\left(d_1 - \sigma\sqrt{T}\right).$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

$$C_0' = S_0 e^{-\delta T} N(d_1') - K e^{-rT} N(d_2')$$

$$d_1' = \frac{\ln\left(S_0 e^{-\delta T}/K e^{-rT}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2' = d_1' - \sigma\sqrt{T}.$$

$d_1' = d_1$ and $d_2' = d_2$.

Then $C_0 = C_0'$ ☐

$$S_0' = 100\, e^{-0.03\times 0.75} = S_0 e^{-\delta T}, \quad K' = 95\, e^{-0.08\times 0.75} = K e^{-rT}$$

(2) ~~$F^P_{0,T}(S) =$~~

$$C_0' = C(S_0', K', \sigma, 0, T, 0) = S_0' N(d_1') - K' N(d_2').$$

$$d_1' = \frac{\ln\left(S_0'/K'\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2' = d_1' - \sigma\sqrt{T}.$$

$$= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1$$

C/C: $\boxed{C(S_0 e^{-\delta T}, K e^{-rT}, \sigma, 0, T, 0) = C(S_0, K, \sigma, r, T, \delta)}$.

Q2) $F^P_{0,T}(S_0) = S_0 e^{-\delta T} = 100\, e^{-0.03\times 0.75}$

$F^P_{0,T}(K) = K e^{-rT} = 95\, e^{-0.08\times 0.75}$

W11P7

$$C_0^F = F e^{-rT} N(d_1) - K e^{-rT} N(d_2).$$

where $d_1 = \dfrac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Distubution of the rate of return:

Let R denotes the continuously compounded rate of return ~~per annum~~ realized by the stock between 0 and T.

That is $S_T = S_0 e^{RT}$.

From this formula we can find R: that.

$$R = \frac{1}{T} \ln\left(\frac{S_T}{S_0}\right).$$

Question: What is the distribution of R.

It is clear that the distribution of R depends on the distribution of S_T.

If the stock price $(S_t)_{0 \le t \le T}$ follows the Black-Scholes model, then $S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T}$.

There for $\ln\left(\frac{S_T}{S_0}\right) = (\mu - \frac{\sigma^2}{2})T + \sigma B_T$

Consequentlly: $R = \mu - \frac{\sigma^2}{2} + \frac{\sigma}{T} B_T$. $(B_T \hookrightarrow N(0,T))$

hence $R \hookrightarrow N\left(\mu - \frac{\sigma^2}{2}; \frac{\sigma^2}{T}\right)$. $\boxed{X \text{ normal distr. } \; a+bX}$

~~This means~~

If $X_1, X_2 \hookrightarrow N(\cdot\,\cdot) \Rightarrow \alpha X_1 + \beta X_2 \; N(\cdot\,,\,)$.

$E[R] = \mu - \frac{\sigma^2}{2} + \frac{\sigma}{T} E[B_T] = \mu - \frac{\sigma^2}{2}$.

$Var(R) = Var\left(\mu - \frac{\sigma^2}{2} + \frac{\sigma}{T} B_T\right)$ $(Var(a+X) = Var(X))$

$= Var\left(\frac{\sigma}{T} B_T\right) = \frac{\sigma^2}{T^2} Var(B_T) = \frac{\sigma^2}{T}$.

WM1P73

$R \hookrightarrow N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$: This means that the continuously compounded rates of return per annum is normally distribution with mean $\mu - \frac{\sigma^2}{2}$ and standard deviation $\frac{\sigma}{\sqrt{T}}$ whenever the stock price is modeled by the Black-Scholes formula.

<u>Example</u>: consider a stock with an expected return of 13.15% per annum and a volatility of 25% /year

① Find the distribution of the average rate of return continuously compounded realized over 3 years.

② Give the confidence interval at 95% of R.

<u>Solution</u> ① $R \hookrightarrow N(10\%, (0.1443)^2)$.

② $C.I(R) =]-0.1826, 0.383[$.

This means that there is 95% probability that the rate of return belongs to $]-0{,}1826; 0.383[$.

<u>Remark</u>: $E[R] = \mu - \frac{\sigma^2}{2} < \mu$.

<u>Volatility</u>: Suppose that: $\sigma = 0.3$ per annum and the current stock price is 70. The standard deviation of the percentage chance in the stock price in one week is approximately: $\sigma\sqrt{\frac{1}{52}} = 0.3\sqrt{\frac{1}{52}} = \underline{0.0416}$

this means that a 1-standard deviation move in the stock price in one week is therefore $70 \times 0.0416 = 2.9120$.

W11P74