Greeks and sensitivity analysis.

$C_0 = C(S_0, K; \sigma, r, T, \delta)$ at time $= 0$

$C_t = C(S_t, K, \sigma, r, T-t, \delta)$ at time t

If we denote by $V_t = V(S_t, K, \sigma, T-t, \delta)$:

Sensitivity analysis deals with the variation of V_t when one the parameters of the model is changing.

$$\Delta = \frac{\partial V_t}{\partial S_t}, \quad \Gamma = \frac{\partial \Delta}{\partial S_t}.$$

This means: $\delta \cancel{V}_t = V_{t+h} - V_t$. $h \lll 1$

$$\delta V_t \simeq \Delta \, \delta S_t$$

$$\delta V_t = \Delta (S_{t+h} - S_t).$$ If $\boxed{\delta S_t = 1}$

then $\delta V_t = \Delta$.

Example: if $\Gamma = 0{,}02$ and $\Delta = 0.5$. Find new Δ.

when $\delta S_t = 3$.

$$\delta \Delta_t \simeq \Gamma_t \, \delta S_t \iff \Gamma = \frac{\partial \Delta}{\partial S} \simeq \frac{\delta \Delta}{\delta S}$$

W12P75

$$\Gamma = \frac{\partial \Delta}{\partial S} = \lim_{h \to 0} \frac{\Delta(S+h) - \Delta(S)}{h} \simeq \frac{\Delta(S+\delta S) - \Delta(S)}{\delta S}.$$

$$\Delta(S+\delta S) - \Delta(S) \simeq \Gamma \, \delta S,$$

$$\Rightarrow \Delta(S+\delta S) = \Delta(S) + \Gamma \, \delta S.$$

$$= 0.5 + 0.02 \times 3$$

Then the new delta $= 0.56$

Example for Vega: call option on a currency with $C_0 = \$2$. Assume that $\nu = 0.20$ with the prepaid forward volatility of 30%, the price of a call option.

a) If $\sigma_2 = 31\%$ what will be the price of a call option. $(\delta\sigma = \sigma_2 - \sigma_1)$.

$$\nu = \frac{\partial C}{\partial \sigma} \Rightarrow \delta C \simeq \nu \, \delta\sigma$$

$$\cancel{C_{\sigma_2} = C_{\sigma_1} + \nu(\sigma_2 - \sigma_1) = 2 + 0.02(31\% - 30\%)}$$

$$\cancel{= 2 + 0.02 \times 1\% = 2.2002}$$

$$C_{\sigma_2} = C_{\sigma_1} + \nu(\sigma_2 - \sigma_1) \qquad \sigma_2 - \sigma_1 = 1.$$

$$= 2 + 0.20 \times 1 = 2.20$$

b) If $\sigma_2 = 29\%$:

$$C_{29\%} = C_{30\%} - 0.20 = \$1.80.$$

W12 P76.

(a) $C(S_0, K, \sigma, r, T, \delta) = C(S_0 e^{-\delta T}, Ke^{-rT}, \sigma, 0, T, 0)$ (b)

at any time $0 \leq t \leq T$ we have:

$$\underset{P}{C}(S_t, K, \sigma, r, T-t, \delta) = \underset{P}{C}(S_t e^{-\delta(T-t)}, Ke^{-r(T-t)}, \sigma, 0, T-t, 0).$$

P4-Q21 a): $F_{0,9M} = S_0 e^{9/12(r-\delta)} = 100\, e^{+0.05 \times 0.75} = 103.82 = F_0$

b). $C_0 = C(F_0 e^{-rT}, Ke^{-rT}, \sigma, 0, T, 0)$.

$$= F_0 e^{-rT} N(d_1) - Ke^{-rT} N(d_2).$$

where $d_1 = \dfrac{\ln\left(\frac{F_0}{K}\right) + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

$d_1 = 0.4716$ and $d_2 = 0.2118$

$C_0 = 103.82\, e^{-0.08 \times .75}\, \cancel{\times}\, 0.6814 - 95\, e^{-0.08 \times 0.75} \times 0.5838$

$= 14.392$.

P4 Q3 a) $C_0 = C(F^P_{0,T}(S), \cancel{K} F^P_{0,T}(K), \sigma, 0, T, 0)$.

$= C(S_0 - PV_0(Div), Ke^{-rT}, \sigma, 0, T, 0)$

= "call on a forward contract"

$= (S_0 - De^{-r(T-t)}) N(d_1) - Ke^{-rT} N(d_2)$.

where $d_1 = \dfrac{\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(K)}\right) + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$.

~~d_1~~ $C_0 = 10.327$.

W12 P77.

$S_0 = 60$; $d_1 = 2.0526$ and $d_2 = 1.8404$.

$C_{60} = 19.743$.

$$C_0 = F^P_{0,T}(S_0) N(d_1) - F^P_{0,T}(K) N(d_2)$$

$$F^P_{0,T}(S_0) = \begin{cases} S_0 e^{-\delta T} \\ S_0 - PV_{0,T}(Div) \end{cases}$$

In both cases: $\nearrow S_0$

To know the variation of C in terms of S we have to compute $\frac{\partial C}{\partial S}$.

↳ The prepaid forward price is always increasing when $S_0 \nearrow$.

$\boxed{\text{P4. Q4}}$

$$C_t = S_t e^{-\delta(T-t)} N(d_1(t)) - K e^{-r(T-t)} N(d_2(t)).$$

$$P_t = K e^{-r(T-t)} N(-d_2(t)) - S_t e^{-\delta(T-t)} N(-d_1(t)).$$

We would like to determine δ.

From the call-put parity we have:

$$C_t - P_t = S_t e^{-\delta(T-t)} - K e^{-r(T-t)}.$$

Then $\delta = \frac{1}{t-T} \ln\left(\frac{C_t - P_t + K e^{-r(T-t)}}{S_t}\right)$:

$\delta = 1.9853\% \simeq 0.2 = 2\%$

W12 P78

Exercise: $S_0 = 40$; $\sigma = 30\%$, $r = 8\%$ $T = 91$ days.
$K = 40$, $\delta = 0$, 100 shares, call option.

① $C_0 = 2.7804$. $\Delta = e^{-\delta T} N(d_1)$.

We have: $N(d_1) = 0.5824$, in our case $\Delta = 0.5824$.

Δ corresponds to the quantity to hold on shares (risky asset) in order to hedge his position.

The seller will then receive form the buyer.
$100 C_0 = \$278.04$.
The risk of the seller (market-maker) is that the price is higher than 40. ($S_{91} > K = 40$).
To hedge his position the seller will buy 100Δ shares of the stock at 40 this will costs: $4000\Delta = 2329.60$
Then the seller has to borrow $2329.60 - 278.04 = 2051.56$ from at the rate 8%.
If the market-maker ~~let~~ leave the position un-hedged:
- $S_1 = 40.50 \rightsquigarrow C_1 = S_1 N(d_1(1)) - Ke^{-rT} N(d_2(1))$
$= 3.0621$. $\left(T = \frac{\text{days}}{365}\right)$
- $2.7804 e^{0.08 \times \frac{1}{365}} = 2.7874$.

$2.7874 - 3.0621 = -0.280$.
This means that the seller loses \$28 for 100 shares.

W12P79.

~~Now if $S_1 =$~~

In the case where the position is hedged:

$(-2051,56,\ 58.24) =$ initial portfolio.

If $S_1 = 40.50$. $V_0^{seller} = 0$.

~~$V_0^{seller} = 0$~~ $V_1^{seller} = -2051,56\, e^{0,08 \times \frac{1}{365}} + 58.24 \times 40.50$.

$= \alpha S_1^0 + \Delta S_1^1$.

The finance charge of a day on the loan is:

$2051.65\left(e^{0.08\frac{1}{365}} - 1\right) = 0.45$.

The profit of the seller is:

$\underbrace{58.24\,(40.30 - 40)}_{>0} + (\underbrace{278.04 - 306.21}_{<0}) - 0.45$

$= 0.50$.

at time 1 we have a new Δ.

Q: compute the Δ of day and find the portfolio of the next day.

This method will lead the seller to hedge his/her position on the call

W12. P80