Consider a financial instruments (a derivative) F.
with the following possible values: F^u and F^d

risky

$$\begin{array}{ccc} & \nearrow & S_0 u = S_1^u \\ S_0 & & \\ & \searrow & S_0 d = S_1^d \\ t=0 & & t=1 \end{array}$$

riskless.

$$\begin{array}{ccc} 1 & \longrightarrow & e^r = S_1^0 \\ t=0 & & t=1 \end{array}$$

$$\begin{array}{ccc} & \nearrow & f_1 = F^u \\ F_0 = ? & & \\ & \searrow & f_2 = F^d \end{array}$$

If: $F = (S_1 - K)^+$

then $f_1 = S_0 u - K$.

$f_2 = 0$.

<u>Question:</u> Find the price at time 0 of F and

Find the replicating portfolio.

The seller of the financial instrument F will get a premium F_0 which will be invested in the market to generate at time 1 the payoff F.

in such away:

$F_0 = \alpha_0 + \Delta_0 S_0$, $\phi = (\alpha_0, \Delta_0)$

$V_1^\Phi = \boxed{F = \alpha_0 S_1^0 + \Delta_0 S_1}$.

$$\begin{cases} \alpha_0 e^r + \Delta_0 S_0 u = F^u \\ \alpha_0 e^r + \Delta_0 S_0 d = F^d . \end{cases} \quad \Bigg| \quad \Delta_0 = \frac{F^u - F^d}{S_0 u - S_0 d}.$$

and $\alpha_0 = \dfrac{F^u - \Delta_0 S_0 u}{e^r} = \dfrac{F^d - \Delta_0 S_0 d}{e^r}$.

W3-L1-P9

Then the ~~problem~~ premium F_0 is equal?

$$F_0 = \frac{F^u - \Delta_0 S_0 u}{e^r} + \Delta_0 S_0$$

$$= \frac{F^u}{e^r} + \frac{\Delta_0 S_0 (e^r - u)}{e^r}$$

$$= \frac{F^u}{e^r} + \frac{F^u - F^d}{u-d} \cdot \left(1 - \frac{u}{e^r}\right) \overset{?}{=} q\frac{F^u}{e^r} + (1-q)\frac{F^d}{e^r}$$

$$= \frac{F^u}{e^r} + \frac{e^r - u}{e^r} \cdot \frac{F^u}{u-d} - \frac{e^r - u}{u-d} \cdot \frac{F^d}{e^r}$$

$$= \frac{F^u}{e^r}\left(1 + \frac{e^r - u}{u-d}\right) + \frac{F^d}{e^r} \cdot \frac{u - e^r}{u-d}$$

$$= \frac{e^r - d}{u-d} \cdot \frac{F^u}{e^r} + \frac{u - e^r}{u-d} \cdot \frac{F^d}{e^r}$$

In order to avoid arbitrage opportunities we should: $\boxed{d < e^r < u}$. This implies that

$$0 < q = \frac{e^r - d}{u-d} < 1 \text{ and } 0 < \frac{u - e^r}{u-d} = 1 - q < 1$$

If we set $Q = (q, 1-q)$:

F_0 can be written as the expectation under Q of the discounted payoff $\frac{F}{e^r}$

That: $F_0 = E_Q\left[\frac{F}{e^r}\right]$ with:

$$q = Q(F = F^u)$$

W3-L1-P10

$$F = \begin{cases} F^u & \rightarrow \text{ with proba } q := \dfrac{e^r - d}{u - d} \\ F^d & \rightarrow \text{ `` `` } 1 - q = \dfrac{u - e^r}{u - d} \end{cases}$$

$$E_Q\left[\frac{F}{e^r}\right] = \frac{F^u}{e^r}\left(\frac{e^r - d}{u - d}\right) + \frac{F^d}{e^r}\left(\frac{u - e^r}{u - d}\right) = F_0.$$

<u>Comments:</u> The portfolio (α_0, Δ_0) is called the replicating (or hedging) portfolio.

<u>Example:</u> Find a corresponding example.

S_0, u, d, r, F^u, F^d.

Then find the premium and the replicating portfolio.

$$\begin{tikzcd}[row sep=small] & 110 \; (u) \\ 100 \arrow[ur] \arrow[dr] & \\ & 90 \; (d) \end{tikzcd} \qquad 1 \longrightarrow 102 \; (r)$$

$$F = \begin{cases} 3 \\ 1 \end{cases} \Bigg| \; F \begin{cases} 2 \\ 5 \end{cases} \Bigg| \; F \begin{cases} 1/2 \\ 4 \end{cases}$$

To be discussed in exercise sessions.

End

W3-L1-P11

In the one period binomial model on a non-divident-paying stock: we found $q = \frac{e^r - d}{u - d}$ $\quad 0 < q < 1.$

with the non-arbitrage condition: $\boxed{d < e^r < u}$.

$$S_0 \begin{array}{l} \nearrow S_0 u \\ \searrow S_0 d \end{array}$$

$u = 1 +$ return of the stock in the rise case.
$d = 1 +$ return of the stock in fall case.

denote by R_1 the return of the stock: that is

$$R_1 = \frac{S_1 - S_0}{S_0} = \frac{S_1}{S_0} - 1 = \begin{cases} u - 1 \\ d - 1 \end{cases}$$

$$E_Q[R_1] = (u-1)q + (d-1)(1-q) = qu + (1-q)d - q - (1-q)$$

$$= q(u-d) + d - 1$$
$$= e^r - d + d - 1 = e^r - 1$$

$$E_Q\left[\frac{S_1}{S_0}\right] = E_Q(1 + R_1) = 1 + E_Q(R_1) = e^r.$$

$$\Leftrightarrow E_Q\left[\frac{S_1}{e^r}\right] = S_0 \Leftrightarrow E_Q(S_1) = e^r E_Q S_0 = e^r S_0.$$

c/c: $\boxed{E_Q\left[\frac{S_1}{e^r}\right] = S_0}$.

W3-L1-P12

Question Find Q s.t. $E_Q\left[\frac{S_1}{e^r}\right] = S_0$ ⊛

Our model
$$S_0 \begin{array}{l} \nearrow S_0 u \\ \searrow S_0 d \end{array} \qquad 1 \longrightarrow e^r ; \qquad \boxed{d < u}$$

$\exists ?\ Q = (q, 1-q) \quad q = Q(S_1 = S_0 u)$.

$$⊛ \Leftrightarrow S_0 = \frac{S_0 u}{e^r} q + \frac{S_0 d}{e^r}(1-q) \Leftrightarrow 1 = \left(\frac{u-d}{e^r}\right) q + \frac{d}{e^r}$$

$$q = \frac{e^r - d}{u - d} \quad ; \quad 0 < q < 1 \Leftrightarrow d < e^r < u.$$

$(q, 1-q)$ is called Risk-neutral-probability measure.

Def: Arbitrage strategy: is a portofolio Φ such that its value at time zero V_0^{Φ} is $= 0$.

and its value at time T, $V_T^{\Phi} \geq 0$.

and $\mathbb{P}(V_T^{\Phi} > 0) > 0$.

Assume $u < e^r = 1 + i \Leftrightarrow u - 1 < i$.

This means that the return of the ~~ri~~ riskless asset is always greather than the return of the stock.

W3-P13

Consider the following strategy:

$\underline{t=0}$:
- short sell the stock: that is borrow one share and sell it in the market for S_0.
- Invest S_0 in a saving ~~acont~~ acount with interest rate r.

<u>at time</u>: we will get $S_0 e^r$ from the bank and then buy 1 share for S_1 which can be $S_0 u$ or $S_0 d$. and give it back to the lender.

The profit is $S_0 e^r - S_1 \geq S_0 e^r - S_0 d$
$> S_0 e^r - S_0 u > 0$

then we should have $u > e^r$ to avoid A.O.

<u>Now If</u>: $e^r < d$. $\underline{t=0}$ we borrow S_0 from the bank. and buy one share of the stock.

<u>at $t=1$</u>: we sell the share at S_1 and cover the loan.

we end-up with a profit: $S_1 - S_0 e^r > 0$

$$= \begin{cases} S_0 u - S_0 e^r \\ S_0 d - S_0 e^r \end{cases} \geq S_0 (d - e^r) > 0.$$

we should have the $d < e^r$.

c/c $\boxed{d < e^r < u.}$

WIS-P14

Homework Solution

P1:

$$100 \to \begin{cases} 200 & F^u \\ 75 & F^d \end{cases} \qquad 1 \longrightarrow e^r$$

European ~~Call~~ option with $K = 150$.

Q1. $F = (S_1 - 150)^+ = \max(S_1 - 150, 0) = \begin{cases} F^u = 50 \\ F^d = 0 \end{cases}$

Q2. ① $d < e^r < u$ ② $75 < K < 200$ This true since $K = 150$:

$\Updownarrow$

$\ln(d) < r < \ln(u) \Leftrightarrow r \in]\ln(d), \ln(u)[=]-0.28\,;\,0.69[$.

In this $u = 2$, $d = 0.75$.

~~What if~~

Q3: Take $r = 0$. and then take $r = 5\%$; the premium is given by:

$$r = 0 \quad C_0 = E_Q\left[\frac{F}{e^r}\right] = E_Q(F) = F^u \cdot \frac{(1 - 0.75)}{2 - 0.75} + F^d \frac{2-1}{2-0.75}$$

$$C_0 = 10.$$

$$r = 5\% \quad C_0 = \frac{e^{0.05} - 0.75}{2 - 0.75} \cdot \frac{50}{e^{0.05}} = 11.46$$

Q4: $\Delta_0 = \dfrac{F^u - F^d}{S_0 u - S_0 d} = \dfrac{50 - 0}{200 - 75} = \dfrac{4}{10}$ $(r = 0)$ for simplicity

$$\alpha_0 = (F^d - \Delta_0 S_0 d)/e^r = -\frac{4}{10} \cdot \frac{75}{e^r} = -\frac{30}{e^r} = -30$$

<u>Interpretation:</u> The seller will borrow 30 units of money from the bank at the rate $r = 0$, buy $\frac{2}{5}$ shares of the stock at the price 100 which costs 40 units. this is possible since she gets 10 as the premium and 30 from the bank.

At time $T = 1$, maturity: Two possibilities.

If $S_1 = 200$, then the option will be exercised. in cash settlement; Shehana pays 50 to Lamia.

She sells $\frac{2}{5}$ shares of the stock at 200. $\Rightarrow \frac{2}{5} \times 200 = 80$.

After paying Lamia, she has: $80 - 50 = 30$ and will return it to the bank.

Lamia has a net profit $50 - 10 = 40$.

If $S_1 = 75$. the option will not be exercised.

- Lamia: has lost 10.
- Shehana has $\frac{2}{5}$ shares and will sell it for 75. which leads to a cash: $\frac{2}{5} \times 75 = 30$ and then turn it to the bank.

Suppose that: (1) $S_1^u = 160$. $T=1$ exercise of the option:

Shehana will pay 10 to Lamia.

$$\frac{2}{5} \times 160 = \frac{320}{5} = 64 \begin{cases} \to 10 \\ \to 30 \end{cases}$$

Shehana has then got a net profit 24.

(2) $S_1^u = 220 \longrightarrow$ Shehana $\to$ 70 Lamia.

$$\frac{2}{5} \times 220 = \frac{440}{5} = 88 \begin{cases} \to 70 \\ \to 18 + 12 \end{cases}$$

WB-P16

if $S_1^d = 130$. Option not exercised:

$$Sh \rightarrow \frac{2}{5} \times 130 = \frac{260}{5} = 52$$

bank: 30

22 net profit.

If $S_1^d = 150$. option worthless:

$$Sh \longrightarrow \frac{2}{5} \times 150 = \frac{300}{5} = 60$$

30 bank

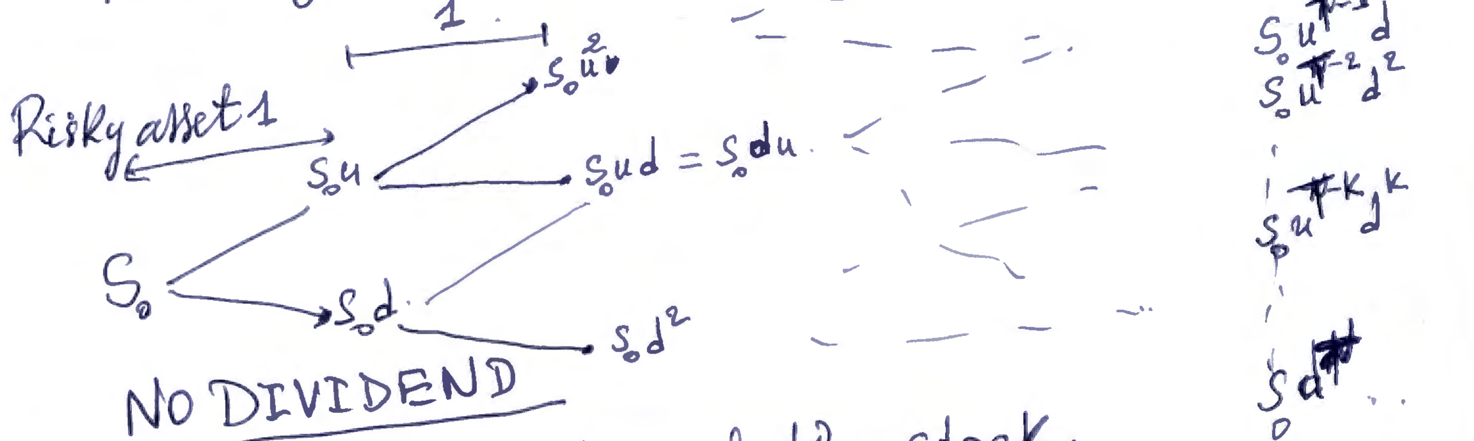
30 net profit.

Lecture :

Multiperiod binomial model.

The model: Riskless asset $\quad t=0 \quad 1 \longrightarrow e^{r} \longrightarrow e^{2r} \longrightarrow e^{nr} \quad t = \frac{N}{n}$

maturity T. $\quad t=0 \quad S_0^0 = 1 \longrightarrow e^{rt} \longrightarrow e^{Tr}$



NO DIVIDEND

This is the binomial tree of the stock.

let us consider a derivative with payoff F of the form:

$F = f(S_T)$ is a function of the price of the underlying at maturity.

Our goal is to build the tree of the financial derivative.

Examples of f: $\quad f(x) = (x-K)^+$ or $(K-x)^+$ / options call / put

& $f(x) = 1\!\!1_{[K,+\infty[}(x)$ or $1\!\!1_{]-\infty,K]}(x)$ digital options

$$1\!\!1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

In fact the tree will gives us the premium of the option and also the dynamic hedging portfolio.

$T=2$:

$F_0 = \frac{F^u}{e^r}q + \frac{F^d}{e^r}(1-q)$ $\to F^u$, F^d $\qquad e^r \xleftrightarrow[e^r]{} e^{2r}\, f(S_T) = V_2$

$q = \frac{e^r - d}{u - d}$

V_1

$V_1^u = \frac{f(S_0u^2)}{e^r}q + \frac{f(S_0ud)}{e^r}(1-q) \to f(S_0u^2)$

V_0

$\to f(S_0ud) = f(S_0du)$

$\bar{q} = 1 - q$

$V_1^d = \frac{f(S_0ud)}{e^r}q + \frac{f(S_0d^2)}{e^r}\bar{q} \to f(S_0d^2)$

$$V_0 = \frac{V_1^u}{e^r}q + \frac{V_1^d}{e^r}(1-q).$$

This is the replicating method to price a F.D.

In general we can use the following algorithm to price european type options.

Let V_t be the value of the F.D. at time t:

We have from the replication principle:

$$\boxed{V_t = f(t, S_t)} \text{ for any } t \in \{0, 1, 2, \dots, T\}.$$

where
$$\begin{cases} f(t,x) = q\,\frac{f(t+1, xu)}{e^r} + (1-q)\,\frac{f(t+1, xd)}{e^r}. \\ f(T,x) = f(x). \end{cases}$$
where $q = \frac{e^r - d}{u - d}$.

$T=2 \qquad T=3$

$T=3$.

$f(S_0u^3) = f(3, S_0u^3)$

$f(S_0u^2d) = f(3, S_0u^2d)$

$f(S_0ud^2) = f(3, S_0ud^2)$

$f(S_0d^3) = f(3, S_0d^3)$

V_2^{uu}

$f(2, S_0 u^2) = q \frac{f(3, S_0 u^3)}{e^r} + (1-q) \frac{f(3, S_0 u^2 d)}{e^r}$

$f(0, S_0)$

$f(1, S_0 u)$

$f(1, S_0 d)$

$\cancel{f(2, S_0 ud)}$ — $f(2, S_0 ud)$

$f(2, S_0 d^2)$. $f(2, S_0 d^2)$

$f(3, S_0 u^3)$

$f(3, S_0 u^2 d) = f(3, S_0 udu)$

$f(3, S_0 u d^2) = f(3, S_0 d^2 u)$

$f(3, S_0 d^3)$:

KV3-20