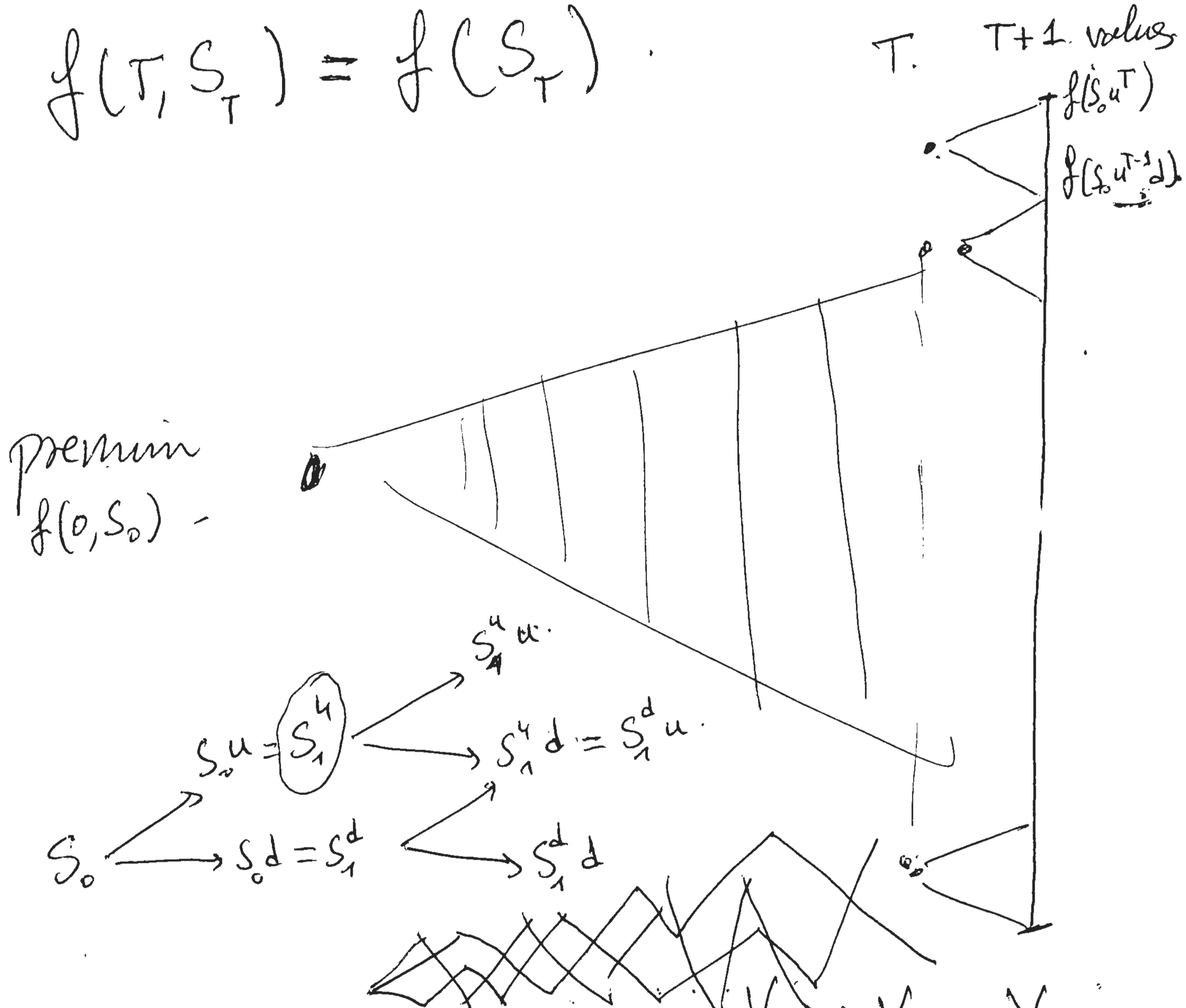
The payoff is of the form: $f(S_T)$:

The possible values of $f(S_T)$ are $f(S_0 u^k d^{T-k})$ $0 \le k \le T$.

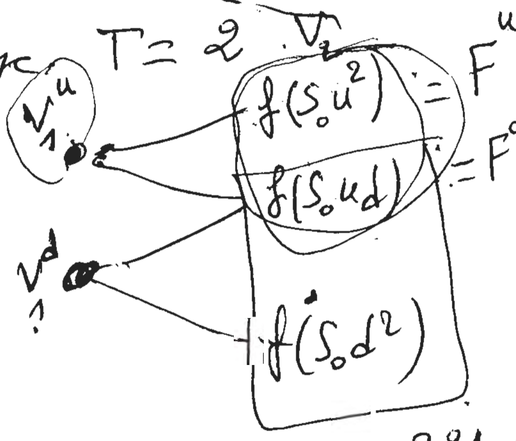
We have $T+1$ values of the payoff.

$f(T, S_T) = f(S_T)$.



$V_0 = f(0, S_0)$ ... V_{T-2} V_{T-1} V_T

consider the sample case where $T = 2$



$V_1^u = \alpha_1^u S_1^0 + \Delta_1^u S_1^u \rightarrow$

$\Phi_1^u = (\alpha_1^u, \Delta_1^u) \rightarrow V_2^{\Phi_1^u} = \alpha_1^u S_2^0 + \Delta_1^u S_2$

This leads to the following system:

$$\begin{cases} \alpha_1^u + \Delta_1^u S_0 u^2 = f(S_0 u^2) \\ \alpha_1^u + \Delta_1^u S_0 u d = f(S_0 u d) \end{cases}$$

$$\Downarrow \quad \begin{cases} \alpha_1^u + \Delta_1^u S_1^u u = f(S_1^u u) \\ \alpha_1^u + \Delta_1^u S_1^u d = f(S_1^u d) \end{cases}$$

$$\Delta_1^u = \frac{f(S_1^u u) - f(S_1^u d)}{S_1^u u - S_1^u d} \; ; \; \alpha_1^u = \left(f(S_1^u u) - \Delta_1^u S_1^u u\right)$$

Similarly

$$\Delta_1^d = \frac{f(S_1^d u) - f(S_1^d d)}{S_1^d (u-d)} \; , \; \alpha_1^d = f(S_1^d d) - \Delta_1^d S_1^d d$$

at any time t we can write the value of the portfolio at time t as $\boxed{V_t = \alpha_t + \Delta_t S_t}$

but we know that $V_t = f(t, S_t)$.

then $\boxed{\alpha_t + \Delta_t S_t = f(t, S_t)}$ $(*)$

Remark that $S_t = \begin{cases} S_{t-1} u \\ S_{t-1} d \end{cases} \quad \forall t \geq 1.$

Then we get: $\begin{cases} \alpha_t + \Delta_t S_{t-1} u = f(t, S_{t-1} u) \\ \alpha_t + \Delta_t S_{t-1} d = f(t, S_{t-1} d) \end{cases}$

$$\underline{\Delta_t} = \frac{f(t, S_{t-1} u) - f(t, S_{t-1} d)}{S_{t-1} u - S_{t-1} d}$$

N.A.C.

$d < e^r < u$

and $\underline{\alpha_t} = f(t, S_{t-1} u) - \Delta_t S_{t-1} u$.

$\forall\, t \in [0, T]$.

The hedging portfolio $\Phi_t = (\alpha_t, \Delta_t)_{t \geq 0}$ is completely determined.

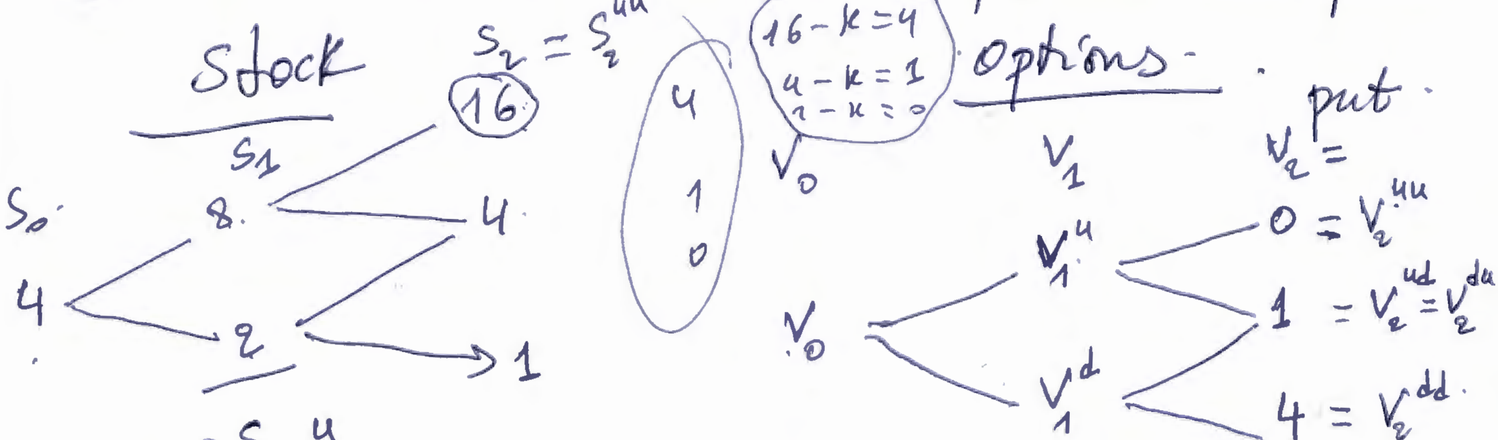
Where $f(t, x)$ is given by the following algorithm: $\begin{cases} f(T, x) = f(x) \\ f(t, x) = q \dfrac{f(t+1, xu)}{e^r} + (1-q) \dfrac{f(t+1, xd)}{e^r} \end{cases}$

and $q = \dfrac{e^r - d}{u - d}$ in the case of non-dividend paying stock.

Example: $S_0 = 4,\ K = 5,\ u = 2,\ d = \frac{1}{2},\ T = 2$

$\boxed{\delta = 0}$ $\quad e^r = 1 + \frac{1}{4} = \frac{5}{4}$. In general $\boxed{e^{rt} = (1+i)^t}$.

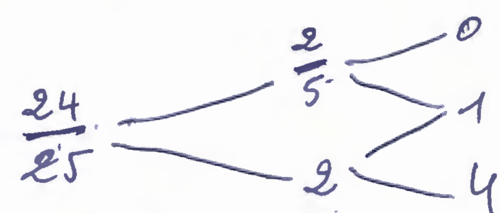
Built the trees for the stock and put and call options:



$$S_t = \begin{cases} S_{t-1}u \\ S_{t-1}d \end{cases}$$

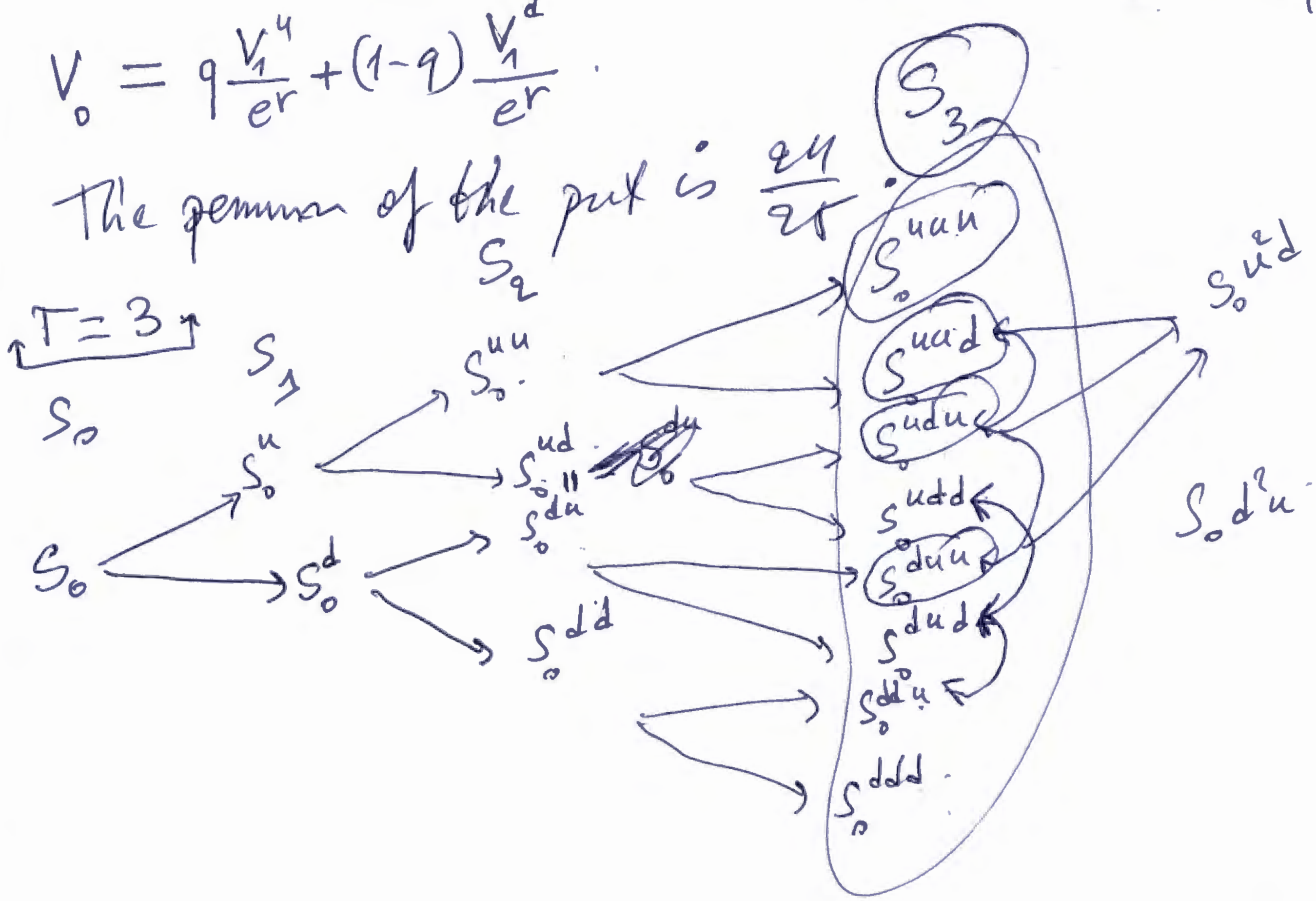
Where. $V_1^u = q\dfrac{V_2^{u\boxed{u}}}{e^r} + (1-q)\dfrac{V_2^{u\boxed{d}}}{e^r}$ and $V_1^d = q\dfrac{V_2^{d\boxed{u}}}{e^r} + (1-q)\dfrac{V_2^{d\boxed{d}}}{e^r}$

and $q = \dfrac{e^r - d}{u - d}$: then



$$V_0 = q\frac{V_1^u}{e^r} + (1-q)\frac{V_1^d}{e^r}.$$

The premium of the put is $\frac{24}{25}$.



Three of the derivative V_3

$f(S_3)$.

$T=0$ $\quad T=1$ $\quad T=2$ $\quad T=3$.

(α_0, Δ_0) $\quad V_1(\alpha_1, \Delta_1)$ $\quad V_2$

$f(S_0^{uuu})$.
$f(S_0^{uud})$.
$f(S_0^{udu})$.
$f(S_0^{udd})$.
$f(S_0^{ddd})$
$f(S_0^{ddu})$.
$f(S_0^{duu})$
$f(S_0^{ddd})$.

$$V_2 = \begin{cases} V_2^{uu} \\ V_2^{ud} \\ V_2^{du} \\ V_2^{dd} \end{cases}, \quad V_1 = \begin{cases} V_1^{u} \\ V_1^{d} \end{cases}$$

For $T=2$ and $T=3$.
Find the replicating portfolio.

$T=2$ $\quad \Delta_1^u, \Delta_1^d, \alpha_1^u, \alpha_1^d, \alpha_0 \Delta_0$

$T=3$ $(\alpha_0, \Delta_0), (\alpha_1, \Delta_1), (\alpha_2, \Delta_2)$.

$T=3$ as homework.

$$V_2^{uu} = q\frac{V_3^{uuu}}{e^r} + (1-q)\frac{V_3^{uud}}{e^r}.$$

$$V_2^{du} = q\frac{V_3^{duu}}{e^r} + (1-q)\frac{V_3^{dud}}{e^r}.$$

for $n \geq 0$

$$V_n^{\boxed{ududud}} = q\frac{V_{n+1}^{\boxed{ududud}u}}{e^r} + (1-q)\frac{V_{n+1}^{\boxed{ududud}d}}{e^r}.$$

W4. P2

Exercises.

P1 Q5. $r = 0$; $C_0 = 10$.

If ~~$C_0 = 8$~~. a call is ~~availa~~ available for 8.

A time 0: LAMA $\longleftrightarrow$ A.

a. buy a call option for 8. '' $\longleftrightarrow$ B.

b. sell a call option for 10.

c. Invest $10-8=2$ in the risk-free asset. with $r=0$.

It can be consumed.

A time ~~✱~~ $T=1$ (maturity). What is your payoff at T:

$\max(S_1-150,0) - \max(S_1-150,0) = 0$.

LAMA'S net profit 2.

(a, b, c) is an arbitrage strategy.

If a call option is available for 13. In this situation:

ASMAE: at time: a). Buy a call for 10.

b). Sell a call for 13.

c). Invest $13-10$ in the risk-free investment.

Asmae will end up with a net profit of 3.

This is also an arbitrage stategy.

Q6: First recall the call-put parity:

$$C_0 - P_0 = S_0 - Ke^{-Tr}$$ for non-dividend paying stock.

$$\Updownarrow$$

$$C_0 + Ke^{-rT} = P_0 + S_0 \quad (T=1), (r=0)$$

Therefore $\boxed{P_0 = C_0 + K - S_0} = \boxed{10+150-100 = 60}$

Exercises continued.

P1 - Q7. ~~r=0~~ . $r = 5\%$. $u = 2$, $d = 0.75$. $[0.75 < e^r < 2)$.

Find r, u, d s.t $\boxed{d < e^r < u}$ not satisfied that is $e^r < d < u$ (1)

or

$d < u < e^r$ (2)

$r = 5\%$. $e^{0.05} \simeq 1.05$

$$S_0 = 50 \begin{cases} 50 \times 4 \\ 100 = 50 \times 2 \end{cases}$$

(1). $1.05 = \underset{e^r}{e^{0.05}} < \underset{d}{2} < \underset{u}{4} \iff 0.05 < \ln(2) < \ln(u)$. ✓

~~RAN~~ Shehna: at time (0) | a) Short sell the asset: sell one share for 50.
b) Invest 50 in saving account with risk-free rate 5%.

At time (1) . she gets from the bank $50e^{0.05}$.
. cover the short sell by buying one share at 100 or 200.

This is not a good strategy.

The good strategy would be: at time 0:

$t=0$ a) Borrow 50 at 5%.
b) buy one share at 50. | $t=1$ | c). sell one share at S_1
d). reimburse the loan.

$$\text{The net profit} = S_1 - 50e^{0.05} = \begin{cases} 100 - 50e^{0.05} \\ 200 - 50e^{0.05} \end{cases}$$

$$= \begin{cases} 47.436 \\ 147.436 \end{cases}$$

This an arbitrage strategy.

W4. P27

(9) $d < u < e^r$, $r = 0.1 = 10\%$ $u = 1$..

$d = \frac{1}{2}$

$$50 \begin{cases} \nearrow 50 \\ \searrow 25 \end{cases}$$

$t=0$

a. Short sell one share at 50

b. lend to the bank 50 at 10%

$t=1$

a. get $50e^{0.1}$ from the bank.

b. Cover the short sell : that is to buy one share at most for 50.

The net profit $\left(50e^{0.1} - \frac{50}{1}\right) = \begin{cases} 50(e^{0.1} - 1) > 0 \\ 50(e^{0.1} - \frac{1}{2}) > 0 \end{cases}$

This is also an arbitrage strategy.

$\longleftrightarrow$ Let us choose the appropriate parameters r, d and u such that the non arbitratage condition is satisfied:

$r = 1\%$, $d = \frac{1}{u}$; $u = 1.5$, $T = 3$; $S_0 = 10$

$\boxed{\text{Consider a put option with strike price } K =}$

1- Build the tree of the stock:

V_2^{uu}

$V_2^{ud} = q \times 0 + (1-q)\frac{V_3^{udd}}{e^r}$

$V_2^{du} =$

$V_2^{dd} =$

$\boxed{V_0 = \alpha_0 + \Delta_0 S_0}$

$\boxed{\alpha_0 = V_0 - \Delta_0 S_0}$

W4. P28

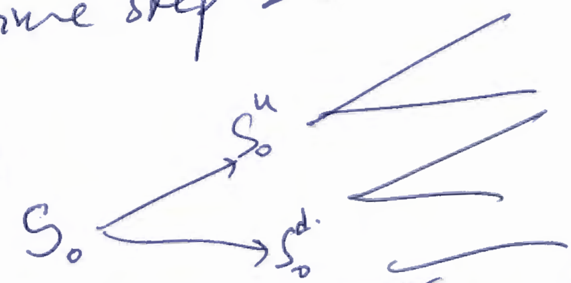
Consider now an asset paying dividends ~~with~~ continuously compounding with rate δ.

$$d < e^r < u. \quad \left(\text{Assume that } d = \frac{1}{u}\right).$$

We consider T-periodic binomial model with time step 1.

$$S_t = \begin{cases} S_{t-1}u \\ S_{t-1}d \end{cases} \quad \forall\, t \geq 1$$

$\boxed{f(S_T)}$



And let $\boxed{f(S_T)}$ be the payoff of contingent claim: that is a payoff depending on the ~~p~~ stock price at time T.

The only difference is the pricing algorithm is the expression of the RNPM Q which has the form.

$$q = Q(S_t = S_{t-1}u) = \frac{e^{r-\delta} - d}{u - d}.$$

In this situation the algorithm is the same as before: That is the value of hedging portfolio denoted V_t, $0 \leq t \leq T$ has the form:

$$\begin{cases} V_T = f(S_T) = f(T, S_T). \\ V_t = f(t, S_t). \quad \forall \quad t \in \{0, 1, 2, \dots, T\} \end{cases}$$

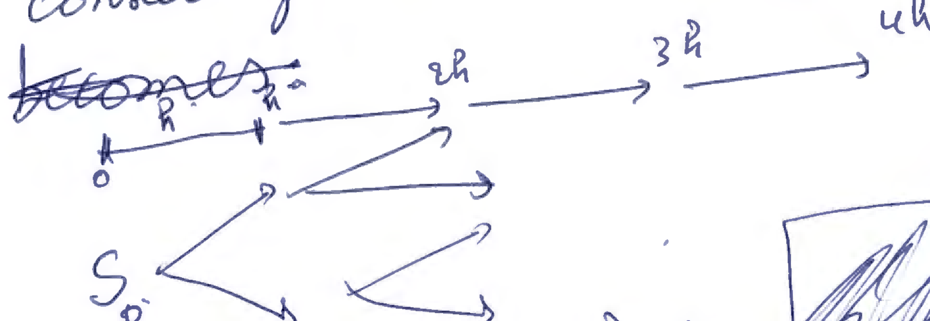
where $f(t,x) = q\,\dfrac{f(t+1, xu)}{e^r} + (1-q)\,\dfrac{f(t+1, xd)}{e^r}, \quad 0 \leq t \leq T-1$

If the stock is a currency with the domestic risk-free rate r_d and the foreign risk-free rate r_f then The RNPM corresponding the binomial model is given by:

$$q = \frac{e^{r_d - r_f} - d}{u - d}.$$

~~Examples~~:

Remark: If the lenght step between consecutif dates is h. ~~then the algorithm becomes~~



In this situation we have

$$q = \frac{a - d}{u - d}$$

where

$$a = \begin{cases} e^{rh} & \text{if the stock pays no dividends} \\ e^{(r-\delta)h} & \text{if the stock pays dividend } (\delta) \\ e^{(r_d - r_f)h} & \text{if the stock is a currency.} \end{cases}$$

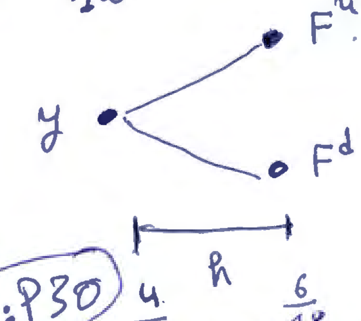
Example: $S_0 = 930$, $K = 900$, $r_d = 5\%$, $r_f = 3\%$.

$T = 6$ months $= \frac{6}{12}$ years, $h = \frac{2}{12}$.

$u = 1.5$, $d = \frac{1}{u}$.

$$q = \frac{e^{(0.05-0.03)\cdot\frac{1}{6}} - \frac{1}{1.5}}{1.5 - \frac{1}{1.5}} = 0.4040$$

$$y = q\frac{F^u}{e^{0.05h}} + (1-q)\frac{F^d}{e^{r_d h}} = (qF^u + (1-q)F^d)e^{-r_d h}$$



W4:P30