American options:

If we denote: C_0^a the premium of ~~the~~ American call option with strike price K and maturity T.
Early exercise means that the option is exercised at time zero. Here its profit is given:

$$\max(S_0 - K, 0) - C_0^a$$

Therefore ~~the~~ the early exercise may happen when $S_0 > K$.

Consider first non dividend paying stocks.

① $C_0^a \geq (S_0 - K)^+ = \max(S_0 - K, 0)$.

If not, that is $C_0^a < (S_0 - K)^+$.

Assume that $S_0 > K$.
We shall make the following strategy:

Ⓐ buy a call at C_0^a

Ⓑ early exercise (exercise immediately the option)

Then we get $S_0 - K - C_0^a > 0$. This an arbitrage opportunity. So $C_0^a \geq (S_0 - K)^+$.

② $S_0 > C_0^a$ (if not) $C_0^a > S_0$

a). sell the call option

b). buy one share at S_0

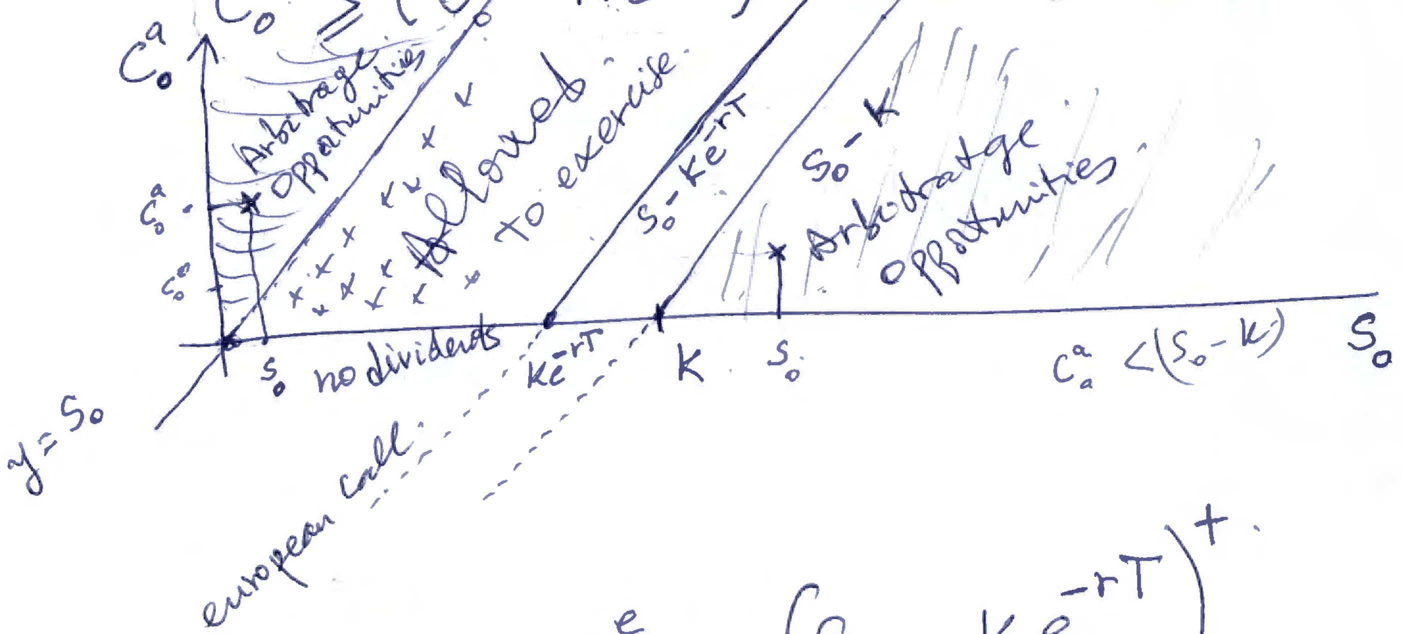
c) we may invest in a saving account $C_0^a - S_0$

This position will always cover the option. WSP31

③ $C_0^a \geq C_0^e$. (options with the same maturity and strike)

for the european option we have:

$C_0^e \geq (S_0 - Ke^{-rT})^+$ and $C_a^e \leq S_0$.



Then $C_0^a \geq C_0^e \geq (S_0 - Ke^{-rT})^+$

and $C_0^a \geq (S_0 - K)^+$ | where $\boxed{x^+ = \max(x, 0)}$.

④ If the stock pays no-dividend and the risk-free rate is positive "$r > 0$". $e^{-rt} < 1 \ \forall t > 0$

We have $C_0^a \geq C_0^e \geq (S_0 - Ke^{-rT})^+ > (S_0 - K)^+$

early exercise induces a net loss for the buyer.

Assume that we ~~can~~ exercise $t^0 < T$.

$C_{t^0}^a = (S_{t^0} - K)^+$

but we have $C_{t^0}^a \geq C_{t^0}^e \geq (S_{t^0} - Ke^{-r(T-t^0)})^+ > (S_{t^0} - K)^+$

$C_{t^0}^a \geq C_{t^0}^e > C_{t^0}^a \Rightarrow \boxed{C_{t^0}^a = C_{t^0}^e}$

W.S. P32.

S_0 stock S_h no-dividends bond.

$$\begin{tikzcd}[row sep=small] & S_0 u \\ S_0 \arrow[ur] \arrow[dr] & \\ & S_0 d \end{tikzcd} \qquad \underbrace{1 \longrightarrow e^{rh}}_{h\text{-period}}$$

American call option.

$$\begin{tikzcd}[row sep=small] & S_0 u - K \\ C_0^a \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$

$$q = \frac{e^{rh} - d}{u - d}$$

$$\Updownarrow$$

$$\boxed{E_Q S_h = e^{rh} S_0}$$

$$q = Q(S_h = S_0 u)$$

a) If the american option is exercised early.

$$C_0^a = (S_0 - K)^+.$$

b) If the option is ~~not~~ exercised at $t = h$ = maturity

$$C_0^a = q(S_0 u - K)e^{-rh}.$$

<u>Conclusion:</u> $$C_0^a = \max\left((S_0 - K)^+, q(S_0 u - K)^+ e^{-rh}\right)$$

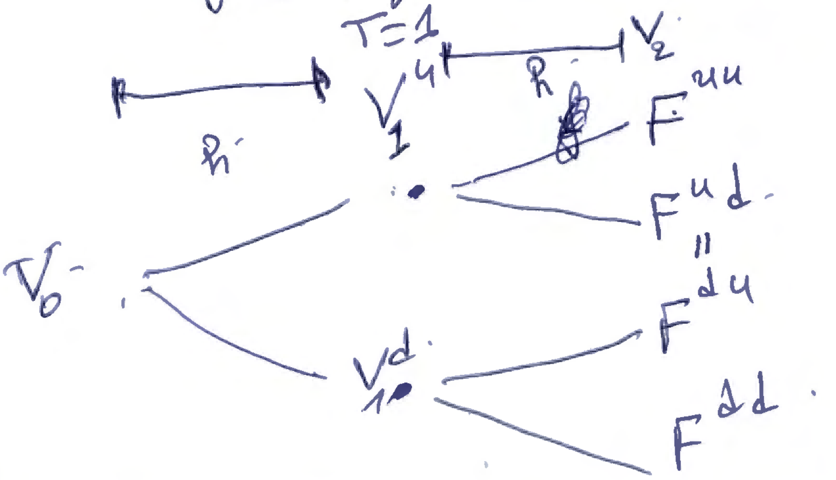
Now consider multiperiod binomial model. ([illegible]).

with ~~period~~ step period h.

$$\begin{tikzcd}[row sep=small] & & S_0 uu \\ & S_0 u \arrow[ur] \arrow[r] & S_0 ud = S_0 du \\ S_0 \arrow[ur] \arrow[dr] & & \\ & S_0 d \arrow[uur] \arrow[r] & S_0 d^2 \end{tikzcd}$$

$S_0 u^j d^{n-j}$

Derivative $F = g(S_T)$. where g is a given function



$T = 2h$

where: $V_1^u = \max\left((S_0 u - K)^+ ; (qF^{uu} + (1-q)F^{ud})e^{-rh}\right)$.

$V_1^d = \max\left((S_0 d - K)^+ ; (qF^{du} + (1-q)F^{dd})e^{-rh}\right)$.

and $V_0 = \max\left((S_0 - K)^+ ; (qV_1^u + (1-q)V_1^d)e^{-rh}\right)$.

HW 2. Solutions

P1 θ_1 See the spread sheet in excel form in the Bb.

$\Phi_0 = (\alpha_0, \Delta_0) = (28.719; -0.294)$ portofolio at time zero.

$$\Phi_1 = (\alpha_1, \Delta_1) = \begin{cases} (20.768; -0.133) \\ (54.516 \;; -1). \end{cases}$$

θ_2

$P_0 = 14.$ Assume that you sold 1000 put option.

will be invested in the bank at 2.5%.

- 14×1000 will be invested in the bank at 2.5%.
- Short sell 294 shares at price 50 ⟶ 14700. invst ~~at bank~~ also the this amount in a saving account at 25%.

$\boxed{28700}$

$$\Phi_0^S = (\underset{\alpha_0}{28719}; \underset{\Delta_0}{-294}), \quad V_0 = 28719 \times 1 + (-294) \times 50 = \text{premium} \simeq 14000.$$

$$\boxed{V_1 = \alpha_0 e^r + \Delta_0 S_1} \Rightarrow \begin{cases} 7435. & \text{if } S_1^u = 100 \\ 29516. & \text{if } S_1^d = 25. \end{cases}$$

if $\underline{\underline{S_1 = 25}}$, shortsell 1000 shares at 25 ⇒ 25000: invest it in the bank at 25%, hence the whole amount she has at the bank is $29516 + 25000 = 54516$.

$$\Phi_1^S = (54516, -1000) \longrightarrow V_2$$

W5.P35

If $S_1 = 100$, $\Phi_1^S = (20768 \;\; ; -133)$.

Short sell 133 shares at 100 ⟿ 13300, invest it in a bank:

$V_1^{S,u} = 7.435 \longrightarrow 7435 + 13300 = 20735$

$\boxed{83.1}$ if 1000 call. $\downarrow \Delta_0$

$(\alpha_0, \Delta_0) = (-766, 709)$, premium = 80 $\quad \frac{136}{97}$

a. we borrow 766 at 5%

b. and buy 709 at 1.2 $\longrightarrow$ 850.8