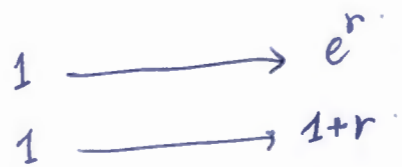
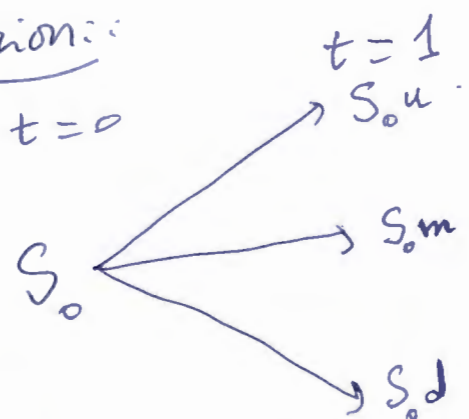


# Chapter 3 - general discrete time model:

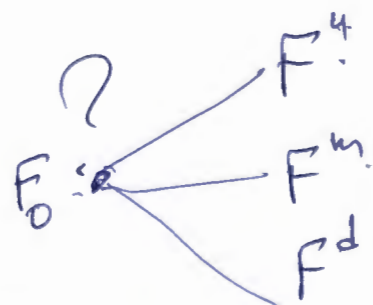
Introduction:



$d < m < u$

Consider a payoff of the form:  $F = \begin{cases} F^u \\ F^m \\ F^d \end{cases}$

Question is it possible to replicate  $F$  with a trading strategy?



In the binomial we found  $\alpha$  st.

$$E_Q[S_1] = e^r S_0$$

$(\alpha_0, \Delta_0)$ ? st:  $F_0 = V_0 = \alpha_0 + \Delta_0 S_0$

and  $F = \alpha_0 + \Delta_0 S_1$  this leads to

$$\begin{cases} F^u = \alpha_0^r + \Delta_0 S_0 u \\ F^m = \alpha_0^r + \Delta_0 S_0 m \\ F^d = \alpha_0^r + \Delta_0 S_0 d \end{cases} \Leftrightarrow \begin{cases} \Delta_0 = \frac{F^u - F^d}{S_0(u-d)} \\ \alpha_0 = F^u - \Delta_0 S_0 u \\ F^m = F^u - \Delta_0 S_0 (u-m) \end{cases}$$

Let us take  $r=0$ .

$$\begin{cases} \Delta_0 = \frac{F^u - F^d}{S_0(u-d)} \\ \alpha_0 = F^u - u \left( \frac{F^u - F^d}{u-d} \right) \\ F^m = F^u - \frac{F^u - F^d}{u-d} (u-m) \\ = F^u \left( \frac{u-d - u+m}{u-d} \right) + F^d \left( \frac{u-m}{u-d} \right) \\ F^m = \left( \frac{m-d}{u-d} \right) F^u + \left( \frac{u-m}{u-d} \right) F^d \end{cases}$$

This means that the set of replicated payoff  $F$  are of the form:  $\left\{ \begin{matrix} F^u \\ F \\ F^d \end{matrix} \right\} = \begin{pmatrix} F^u \\ \left( \frac{m-d}{u-d} F^u + \frac{u-m}{u-d} F^d \right) \\ F^d \end{pmatrix}$ .

Therefore  $V_0 = F_0 = \alpha_0 + \Delta_0 S_0$ .

$$\begin{aligned} &= F^u - \Delta_0 S_0 u + \Delta_0 S_0 = F^u + \Delta_0 S_0 (1-u) \\ &= F^u + \frac{F^u - F^d}{u-d} (1-u) \\ &= F^u \left( \frac{1-d}{u-d} \right) + F^d \left( \frac{u-1}{u-d} \right) \end{aligned}$$

$$\mathbb{E}_Q[S_1] = e^r S_0 \Leftrightarrow q_1 u + q_2 m + q_3 d = e^r$$

$$\Leftrightarrow q_1 u + q_2 m + (1 - q_1 - q_2) d = e^r$$

$$\Leftrightarrow q_1(u - d) + q_2(m - d) = e^r - d$$

$$\Leftrightarrow q_1 = \frac{e^r - d}{u - d} - q_2 \frac{m - d}{u - d}$$

$$Q = \left( \frac{e^r - d}{u - d} - q \frac{m - d}{u - d} ; q ; 1 - q - \frac{e^r - d}{u - d} + q \frac{m - d}{u - d} \right)$$

$$= \left( \frac{e^r - d}{u - d} - q \frac{m - d}{u - d} ; q ; 1 - \frac{e^r - d}{u - d} - q \frac{u - m}{u - d} \right) ;$$

$$0 < q < 1 \quad \downarrow > < 1$$

$$\downarrow > < 1$$

RNPM:  $Q_i: E_Q(S_1^i) = S_1^0 S_0^i \quad \forall i=1,2,\dots,d$

$d=1$   $E_Q(S_1^1) = S_1^0 S_0^1$   $Q(w_i) = q_i \quad i=1,2,3$   
 $\parallel = \frac{10}{9} \times 5$

$\tilde{S}_n^i$ : the discounted stock price at time  $n$

$\tilde{S}_n^i = \frac{S_n^i}{S_n^0} \cdot i \geq 1$

$\exists Q$  RNPM:  $E[\tilde{S}_1^i] = \tilde{S}_0^i = \frac{S_0^i}{S_0^0}$

$E[\tilde{S}_1^1] = 6(q_1 + q_2) + 3q_3 = 5 \Leftrightarrow 6(1 - q_3) + 3q_3 = 5$

$E[\tilde{S}_1^2] = 12q_1 + 8(q_2 + q_3) = 10 \Leftrightarrow 12q_1 + 8(1 - q_1) = 10$

$\Leftrightarrow \begin{cases} q_3 = \frac{1}{3} \\ q_1 = \frac{1}{2} \end{cases} \Rightarrow q_2 = \frac{1}{6}$

$Q = \left( \frac{1}{2}, \frac{1}{6}, \frac{1}{3} \right)$

consider  $S_1 = \begin{cases} S_0(1+r) & \text{on } \{\omega_1, \omega_3\} \\ S_0(1-r) & \text{on } \{\omega_2, \omega_4\} \end{cases}$

and  $\sigma = \begin{cases} h & \text{on } \{\omega_1, \omega_2\} \\ l & \text{on } \{\omega_3, \omega_4\} \end{cases}$

$$0 < l < h < 1$$

$n$	$S_n^0$	$\omega_1$	$\omega_2$	$S_n$	$\omega_3$	$\omega_4$
0	1	$S_0$	$S_0$	$S_0$	$S_0$	$S_0$
1	$1+r$	$S_0(1+r)$	$S_0(1-r)$	$S_0(1+l)$	$S_0(1+l)$	$S_0(1-l)$
F		1	0	1	1	1

$$F = \mathbb{1}_{\{S_1 > K\}} = \begin{cases} 1 & \text{on } \{S_1 > K\} \\ 0 & \text{on } \{S_1 \leq K\} \end{cases}$$

$S_0(1-r) < K < S_0(1+r)$   
no AC for the strike:

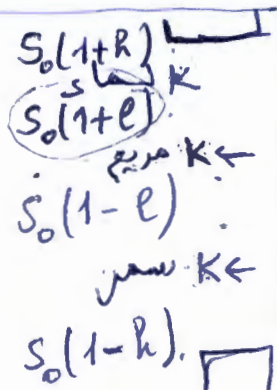
$$F(\omega_1) = \mathbb{1}_{\{S_1(\omega_1) > K\}} = \mathbb{1}_{\{S_0(1+r) > K\}} = 1$$

$$F(\omega_3) = \mathbb{1}_{\{S_0(1+l) > K\}} = \begin{cases} 1 & S_0(1+l) > K \\ 0 & S_0(1+l) \leq K \end{cases}$$

$$F(\omega_4) = \mathbb{1}_{\{S_0(1-l) > K\}} = 0$$

$$F(\omega_2) = 0$$

$F(\omega_1) = 1, F(\omega_3) = 1$
$F(\omega_2) = 0, F(\omega_4) = 0$
$F(\omega_1) = 1, F(\omega_3) = 0$
$F(\omega_2) = 0, F(\omega_4) = 0$



Assume  $S_0(1-l) < K < S_0(1+l)$ .

$$F = \begin{cases} 1 & w_1 \\ 0 & w_2 \end{cases} \quad \text{at time } t=1.$$

Question: Find the price at zero and find ~~the~~ a hedging portfolio if there exists?

$(\alpha, \Delta)$  a portfolio at time 0 such that.

$$V_1^{(\alpha, \Delta)} = F = \alpha(S_1^0) + \Delta S_1 = \alpha(1+r) + \Delta S_1 \quad (*)$$

Is it possible to find such portfolio?

$$(*) \begin{cases} \alpha(1+r) + \Delta S_0(1+R) = 1 & 1 \\ \alpha(1+r) + \Delta S_0(1-h) = 0 & 2 \\ \alpha(1+r) + \Delta S_0(1+l) = 1 & 3 \\ \alpha(1+r) + \Delta S_0(1-l) = 0 & 4 \end{cases} \quad w_1$$

$$(2-4) \Leftrightarrow \Delta S_0(l-h) = 0$$

$$\Leftrightarrow \Delta_0 = 0$$

$$\Rightarrow \alpha = 0$$

$\Rightarrow 0 = 1$  impossible (this is a false statement)

This means that there is no hedging portfolio for  $F$ . We say that the contingent claim is not attainable.

Hence we say that the market is incomplete.

Find RNPM for financial market:

$(S_t^0)_{t \in \{0,1\}}$  and  $d$  risky asset  $(S_t^1, S_t^2, \dots, S_t^d)_{t \in \{0,1\}}$

A RNPM is a solution to the equations

$$E_Q[S_1^i] = S_1^i S_0^i \quad \forall i = 1, 2, \dots, d.$$

We have:  $10 \times 1.05 = 12q_1 + 8q_2 + 6(1 - q_1 - q_2)$  (1)

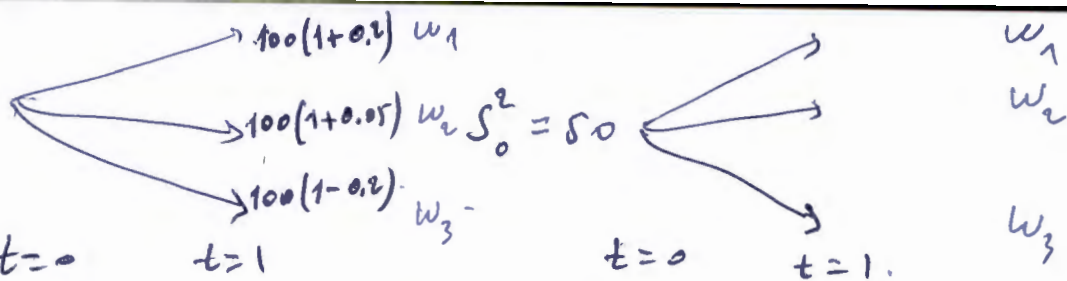
$2 \times 5 \times 1.05 = 20q_1 + 8q_2 + 10(1 - q_1 - q_2)$  (2)

(2) - (1)  $\Leftrightarrow 0 = 8q_1 + 4(1 - q_1 - q_2) = 4q_1 + 4 - 4q_2$

$\Leftrightarrow q_2 = 1 + q_1 > 1$  There is no RNPM.

The model has (or presents) arbitrage opportunities.

$$S_0^1 = 100$$



$$S_0^p = 1 \xrightarrow{t=0} 1+5\% \xrightarrow{t=1}$$

$$R^i = \frac{S_1^i - S_0}{S_0} = \text{return of the stock:}$$

$$R^1 = \begin{cases} 20\% & w_1 \\ 5\% & w_2 \\ -20\% & w_3 \end{cases} \text{ and } R^2 = \begin{cases} -10\% & w_1 \\ 8\% & w_2 \\ 24\% & w_3 \end{cases}$$

Question: is this arbitrage free and complete?

RNPM.1

$$\boxed{E_Q(S_1^i) = (1+r)S_0^i \quad i=1,2} \quad (1)$$

$$\Downarrow E_Q(S_1^i - S_0^i) = rS_0^i$$

$$\Leftrightarrow E_Q\left(\frac{S_1^i - S_0^i}{S_0^i}\right) = r \quad i=1,2$$

$$\Leftrightarrow \boxed{E_Q(R^i) = r \quad i=1,2} \quad (2)$$

that is  $E_Q[R^1] = r$  and  $E_Q[R^2] = r$ .

$$\begin{cases} 20q_1 + 5q_2 - 20q_3 = 5 \\ -10q_1 + 8q_2 + 2q_3 = 5 \end{cases} \Leftrightarrow \begin{cases} 40q_1 + 25q_2 = 25 \\ 6q_2 - 12q_1 = 3 \end{cases}$$

$$q_1 = \frac{5}{36}; \quad q_2 = \frac{7}{9}; \quad q_3 = \frac{1}{12}, \quad Q = \left(\frac{5}{36}, \frac{7}{9}, \frac{1}{12}\right)$$

Then the market is arbitrage free and complete.



Consider  $F = \max(S_1^1 + S_1^2 - 150, 0)$ .

$$F(w) = \begin{cases} 15 & w_1 \\ 9 & w_2 \\ 0 & w_3 \end{cases} \quad \left| \begin{array}{l} \text{The premium of this call option} \\ \text{is given by:} \\ V_0 = E\left[\frac{F}{1+r}\right] = \frac{1}{1.05} \left(15 \cdot \frac{5}{36} + 9 \cdot \frac{7}{9}\right) \\ = \frac{245}{63} \approx 3.89 \end{array} \right.$$

Find the replicating portfolio of  $F$ .

Since the market is complete we know that there exists a trading strategy  $(\alpha, \Delta^1, \Delta^2)$  such that

$$V_0 = \alpha + \Delta^1 S_0^1 + \Delta^2 S_0^2 \text{ - out}$$

$$\alpha(1+r) + \Delta^1 S_1^1 + \Delta^2 S_1^2 = F$$

$$\alpha \frac{105}{100} + \Delta^1 120 + 45 \Delta^2 = 15$$

$$\alpha \frac{105}{100} + \Delta^1 105 + 54 \Delta^2 = 9$$

$$\alpha \frac{105}{100} + 80 \Delta^1 + 51 \Delta^2 = 0$$

$$(\alpha, \Delta^1, \Delta^2) = \left( \overset{\text{bank}}{-\frac{530}{21}}, \overset{\text{asset 1}}{\frac{11}{30}}, \overset{\text{Asset 2}}{-\frac{1}{18}} \right)$$

$$\frac{545}{63} + \frac{50}{18} + \frac{530}{21} = \frac{110}{3} = 100 \times \frac{11}{30}$$

□ ...