Chapter 3 - general discrete time model:

Introduction:

$$
\begin{array}{lcl}
t=0 & & t=1 \\
 & \nearrow & S_0 u \\
S_0 & \rightarrow & S_0 m \\
 & \searrow & S_0 d
\end{array}
\qquad
\begin{array}{l}
1 \longrightarrow e^r \\
1 \longrightarrow 1+r
\end{array}
$$

$$\boxed{d < m < u}$$

Consider a payoff of the form: $F = \begin{cases} F^u \\ F^m \\ F^d \end{cases}$

Question is it possible ~~to~~ to replicate F with a trading strategy?

$$
\begin{array}{lcl}
 & \nearrow & F^u \\
\overset{?}{F_0} & \rightarrow & F^m \\
 & \searrow & F^d
\end{array}
$$

In the binomial we found Q s.t.

$$E_Q[S_1] = e^r S_0$$

(α_0, Δ_0)? st: $F_0 = V_0 = \alpha_0 + \Delta_0 S_0$

and $\boxed{F = \alpha_0 + \Delta_0 S_1}$ this leads to

$$\begin{cases} F^u = \alpha_0 e^r + \Delta_0 S_0 u \\ F^m = \alpha_0 e^r + \Delta_0 S_0 m \\ F^d = \alpha_0 e^r + \Delta_0 S_0 d \end{cases} \iff \begin{cases} \Delta_0 = \dfrac{F^u - F^d}{S_0(u-d)} \\ \alpha_0 = F^u - \Delta_0 S_0 u \\ F^m = F^u - S_0 \Delta_0 (u-m). \end{cases}$$

$\boxed{\text{let's take } r = 0}$

$$\begin{cases} \Delta_0 = \dfrac{F^u - F^d}{S_0(u-d)} \\ \alpha_0 = F^u - u\left(\dfrac{F^u - F^d}{u-d}\right) \\ F^m = F^u - \dfrac{F^u - F^d}{u-d} \cdot (u-m) \end{cases}$$

$$= F^u\left(\frac{\cancel{u} - d \cancel{-u} + m}{u-d}\right) + F^d\left(\frac{u-m}{u-d}\right).$$

$$F^m = \left(\frac{m-d}{u-d}\right)F^u + \left(\frac{u-m}{u-d}\right)F^d.$$

This means that the set of replicated payoff F are of the form: $\left\{ F = \begin{pmatrix} F^u \\ \frac{m-d}{u-d}F^u + \frac{u-m}{u-d}F^d \\ F^d \end{pmatrix} \right\}$.

Therefore: $V_0 = F_0 = \alpha_0 + \Delta_0 S_0$

$$= F^u - \Delta_0 S_0 u + \Delta_0 S_0 = F^u + \Delta S_0 (1-u)$$

$$= F^u + \frac{F^u - F^d}{u-d}(1-u)$$

$$= F^u\left(\frac{1-d}{u-d}\right) + F^d\left(\frac{u-1}{u-d}\right).$$

W6-P38

$$\mathbb{E}_{Q}[S_1] = e^r S_0 \iff q_1 u + q_2 m + q_3 d = e^r$$

$$\iff q_1 u + q_2 m + (1 - q_1 - q_2) d = e^r$$

$$\iff q_1 (u-d) + q_2 (m-d) = e^r - d$$

$$\iff q_1 = \frac{e^r - d}{u-d} - q_2 \frac{m-d}{u-d}$$

$$Q = \left(\frac{e^r - d}{u-d} - q \frac{m-d}{u-d} ; q ; 1 - q - \frac{e^r-d}{u-d} + q \frac{m-d}{u-d} \right)$$

$$1 - q\left(1 - \frac{m-d}{u-d}\right)$$

$$= \left(\underbrace{\frac{e^r - d}{u-d} - q \frac{m-d}{u-d}}_{0 < \ \cdot \ < 1} ; q ; \underbrace{1 - \frac{e^r-d}{u-d} - q \frac{u-m}{u-d}}_{0 < \ \cdot \ < 1} \right);$$

$$0 < q < 1$$

W6-P39

RNPM: Q: $E_Q(S_1^i) = S_1^0 S_0^i \quad \forall i = 1, 2, \dots, d$

$d = 1$ $\quad E_Q[S_1^1] = S_1^0 S_0^1$ $\quad Q(w_i) = q_i \quad i = 1, 2, 3.$

$$\| \quad = \frac{10}{9} \times 5$$

$\tilde{S}_n^i$: the discounted stock i price at time n :

$$\tilde{S}_n^i = \frac{S_n^i}{S_n^0}, \quad i \geq 1$$

$\boxed{\exists Q \text{ RNPM}:}$ $\quad E[\tilde{S}_1^i] = \tilde{S}_0^i = \frac{S_0^i}{S_0^0}.$

$E[\tilde{S}_1^1] = 6(q_1 + q_2) + 3q_3 = 5 \iff 6(1 - q_3) + 3q_3 = 5$

$E[\tilde{S}_1^2] = 12q_1 + 8(q_2 + q_3) = 10 \iff 12q_1 + 8(1 - q_1) = 10$

$$\iff \begin{cases} q_3 = \frac{1}{3} \\ q_1 = \frac{1}{2} \end{cases} \Rightarrow q_2 = \frac{1}{6}.$$

$$Q = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right).$$

consider $S_1 = \begin{cases} S_0(1+\sigma) & \text{on } \{\omega_1, \omega_3\} \\ S_0(1-\sigma) & \text{on } \{\omega_2, \omega_4\} \end{cases}$

and $\sigma = \begin{cases} h & \text{on } \{\omega_1, \omega_2\} \\ \ell & \text{on } \{\omega_3, \omega_4\} \end{cases}$. $\boxed{0 < \ell < h < 1}$

		S_n			
n	S_n^0	ω_1	ω_2	ω_3	ω_4
0	1	S_0	S_0	S_0	S_0
1	$1+r$	$S_0(1+h)$	$S_0(1-h)$	$S_0(1+\ell)$	$S_0(1-\ell)$
F		1	0	1	1

$$F = \mathbb{1}_{]K,+\infty}(S_1) = \begin{cases} 1 & \text{on } \{S_1 > K\} \\ 0 & \text{on } \{S_1 \le K\} \end{cases}$$

$S_0(1-h) < K < S_0(1+h)$

no AC for the ~~strike~~ strike:

$$F(\omega_1) = \mathbb{1}_{]K,+\infty}(S_1(\omega_1)) = \mathbb{1}_{]K,+\infty[}(S_0(1+h)) = 1$$

$$\boxed{F(\omega_3) = \mathbb{1}_{]K,+\infty[}(S_0(1+\ell)) = \begin{cases} 1 & S_0(1+\ell) > K \\ 0 & S_0(1+\ell) \le K \end{cases}}$$

$$F(\omega_4) = \mathbb{1}_{]K,+\infty[}(S_0(1-\ell)) = 1$$

$$F(\omega_3) = 1$$

$F(\omega_1) = 1$,	$F(\omega_3) = 1$
$F(\omega_2) = 0$,	$F(\omega_4) = 0$
$F(\omega_1) = 1$,	$F_3(\omega_3) = 0$
$F(\omega_2) = 0$	$F(\omega_4) = 0$

$S_0(1+h)$

K

$S_0(1+\ell)$

مربع $K \leftarrow$

$S_0(1-\ell)$

سعر $K \leftarrow$

$S_0(1-h)$.

DW6-P41

Assume $S_0(1-\ell) < K < S_0(1+\ell)$.

$$F = \begin{cases} 1 & \omega_1 \\ 1 & \omega_3 \\ 0 & \omega_4 \\ 0 & \omega_2 \end{cases} \quad \text{at time } t = 1.$$

Question; Find the price at zero and find ~~hedging~~ a hedging portfolio if there exists?

(α, Δ) a portfolio at time such that

$$V_1^{(\alpha,\Delta)} = \boxed{F = \alpha S_1^0 + \Delta S_1 = \alpha(1+r) + \Delta S_1} \quad (*)$$

Is it possible to find such portfolio?

$$(*) \begin{cases} \alpha(1+r) + \Delta S_0(1+h) = 1 & 1 \\ \alpha(1+r) + \Delta S_0(1-h) = 0 & 2 \\ \alpha(1+r) + \Delta S_0(1+\ell) = 1 & 3 \\ \alpha(1+r) + \Delta S_0(1-\ell) = 0 & 4 \end{cases}$$

ω_1

$(2-4) \Leftrightarrow \Delta_0 S_0(\ell - h) = 0$

$\Leftrightarrow \Delta_0 = 0$

$\Rightarrow \alpha = 0$

$\Longrightarrow 0 = 1$ impossible (this is a false statement)

This means that there is no hedging portfolio for F. We say that the contingent claim is not ~~attainable~~ attainable.

Hence we say that the market is incomplete.

W6-P4

Find RNPM for financial market:

$(S_t^0)_{t\in\{0,1\}}$ and d risky asset $(S_t^1, S_t^2, \dots, S_t^d)_{t\in\{0,1\}}$

A RNPM is a solution Q to the equations

$$E_Q[S_1^i] = S_1^0 S_0^i \quad \forall\, i = 1, 2, \dots, d.$$

We have: $10 * 1.05 = 12q_1 + 8q_2 + 6(1-q_1-q_2)$. (1)

$2\times 5\times 1.05 = 20q_1 + 8q_2 + 10(1-q_1-q_2)$ (2)

(2) - (1) $\Leftrightarrow 0 = 8q_1 + 4(1-q_1-q_2) = 4q_1 + 4 - 4q_2$

$\Leftrightarrow q_2 = 1 + q_1 > 1$. There is no RNPM.

The model has (or presents) arbitrage opportunities.

W6-P43

$$S_0^1 = 100 \begin{cases} \nearrow 100(1+0.2) & \omega_1 \\ \rightarrow 100(1+0.05) & \omega_2 \\ \searrow 100(1-0.2) & \omega_3 \end{cases} \qquad S_0^2 = 50 \begin{cases} \nearrow & \omega_1 \\ \rightarrow & \omega_2 \\ \searrow & \omega_3 \end{cases}$$

$t=0 \qquad t=1 \qquad\qquad t=0 \qquad t=1.$

$$S_0^0 = 1 \longrightarrow 1+5\%$$

$$R^i = \frac{S_1^i - S_0}{S_0} = \text{returns of the stock:}$$

$$R^1 = \begin{cases} 20\% & \omega_1 \\ 5\% & \omega_2 \\ -20\% & \omega_3 \end{cases} \quad \text{and} \quad R^2 = \begin{cases} -10\% & \omega_1 \\ 8\% & \omega_2 \\ 2\% & \omega_3 \end{cases}$$

Question: is this arbitrage free and complete

RNPM.1 $\boxed{E_Q(S_1^i) = (1+r)S_0^i \quad i = 1,2}$ (1)

$$\Updownarrow \; E_Q(S_1^i - S_0^i) = rS_0^i$$

$$\Leftrightarrow E_Q\left(\frac{S_1^i - S_0^i}{S_0^i}\right) = r \quad i = 1,2$$

$$\Leftrightarrow \boxed{E_Q(R^i) = r \quad i = 1,2} \quad (2)$$

That is $E_Q[R^1] = r$ and $E_Q[R^2] = r$.

$$\begin{cases} 20q_1 + 5q_2 - 20q_3 = 5 \\ -10q_1 + 8q_2 + 2q_3 = 5 \end{cases} \Leftrightarrow \begin{cases} 40q_1 + 25q_2 = 25 \\ 6q_2 - 12q_1 = 3. \end{cases}$$

$$q_1 = \frac{5}{36}; \; q_2 = \frac{7}{9}; \; q_3 = \frac{1}{12}, \; Q = \left(\frac{5}{36}, \frac{7}{9}, \frac{1}{12}\right)$$

Then the market is arbitrage free and complete.

NW6-P44

Consider $F = \max(S_1^1 + S_1^2 - 150, 0)$.

$$F(\omega) = \begin{cases} 15 & \omega_1 \\ 9 & \omega_2 \\ 0 & \omega_3 \end{cases}$$

The premium of this call option is given by:

$$V_0 = E_Q\left[\frac{F}{1+r}\right] = \frac{1}{1.05}\left(15 \cdot \frac{5}{36} + 9 \cdot \frac{7}{9}\right) = \frac{545}{63} \simeq \cancel{10.62}\ 8.65.$$

Find the replicating portfolio of F:

Since the market is complete we know that there exists a trading strategy: $(\alpha, \Delta^1, \Delta^2)$ such that

$V_0 = \alpha + \Delta^1 S_0^1 + \Delta^2 S_0^2$ and

$$\boxed{\alpha(1+r) + \Delta^1 S_1^1 + \Delta^2 S_1^2 = F}$$

$$\alpha \frac{105}{100} + \Delta^1 120 + 45\Delta^2 = 15$$

$$\alpha \frac{105}{100} + \Delta^1 105 + 54\Delta^2 = 9$$

$$\alpha \frac{105}{100} + 90\Delta^1 \quad 51\Delta^2 = 0$$

$$(\alpha, \Delta^1, \Delta^2) = \left(\overset{\text{bank}}{-\frac{530}{21}}, \overset{\text{asset 1}}{\frac{11}{30}}, \overset{\text{Asset 2}}{-\frac{1}{18}}\right).$$

$$\frac{545}{63} + \frac{50}{18} + \frac{530}{21} = \frac{110}{3} = 100 \times \frac{11}{30} \quad \square$$

W6-P45