In the frame work of the binomial model:

$$q = \frac{a-d}{u-d} \qquad d < a < u:$$

$$a = \begin{cases} e^{rh} & \text{No dividend } h \text{ step lenght} \\ e^{(r-\delta)h} & \delta - \text{dividend.} \\ e^{(r-r_f)h} & r_f \text{ foreign rate;} \end{cases}$$

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma\sqrt{h}}.$$

σ is the volatility.

$S_0 = 40$, 35-call option, $T=1$, $r=5\%$, $\sigma=30\%$:

The initial price 40 is expected to appreciate at a rate of 20% per annum.

Build a trees for stock and option with two period model:

$t=0$: 40

$t=\frac{1}{2}$: 50.704 ; 33.173

$t=1$: 64.272 $\boxed{29.272}$; 42.051 $\boxed{7.051}$; 27.512 $\boxed{0}$

(2) What would be the price of this option be, in the event that the market expects XYZ to pay a \$2 dividend in early October (we are in late ~~April~~ March).

Let S'_t be the ex-dividend price:

$$S'_t = S_t - PV_t(D).$$

$$S_1^u = S_0 e^{(r-\delta)h}$$

$$S'_0 = 40 - 2e^{-0.05\times 0.5} = 40 - 1.95 \qquad S'_1 = S_1 - \underbrace{PV_1(D)}_{=0}$$

$$S'_{0.5} = S_{0.5} - PV_{0.5}(D) = S_{0.5} - 2$$

W7.P46.

Remark.

There are many choices for the estimatio[n] of the parameters u and d.

① $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$ $\left(u = \frac{1}{d}\right)$.

$10 \to 10u \to 10u^2 \to 10u^3,\ 10u^2 d = 10u$

$10u \to 10ud = 10du = 10$

$10 \to 10d \to 10d^2 \to 10d^2u = 10d$

$$q = \frac{a-d}{u-d} = \frac{a - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

② $u = e^{(r-\delta)h + \sigma\sqrt{h}}$, $d = e^{(r-\delta)h - \sigma\sqrt{h}}$.

$u = a e^{\sigma\sqrt{h}}$ and $d = a e^{-\sigma\sqrt{h}}$.

$$q = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

$$\frac{S_2^1 + S_2^2}{2}$$

4 periods.

h

$5 \to 7 \to 7\cdot\frac{7}{5}$

$7 \to 7\cdot\frac{6}{5} = 6\cdot\frac{7}{5}$

$5 \to 6 \to 6\cdot\frac{6}{5}$

$\max_{0\le t\le 3} S_t - \cancel{S_3}$

$\max_{0\le t\le 3} S_t - S_3$

W7 P.47

$$F_1 = (S_2^1 - S_2^2)^+$$

$$F_2 = (S_2^2 - S_2^1)^+$$

$$F_3 = S_2^1 - S_2^2$$

$$(S_t^0, S_t^1, S_t^2)_{0 \le t \le 2}$$

$$S = (S_t^0, S_t^1, S_t^2, S_t^3)_{t \in \{0,1\}}$$

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

F is attainable in this market:

$\exists \Phi = ($ s.t. $\boxed{V_1^{\Phi}(\omega) = F(\omega)}$

$\omega \in \{\omega_1, \omega_2, \omega_3, \omega_5, \ldots \omega_{100}\}$

$$\alpha S_1^0 + \Delta^1 S_1^1 + \Delta^2 S_1^2 + \Delta^3 S_1^3 = \boxed{\Phi \cdot S(\omega) = F(\omega)}$$

W7.P48

$$V_0 = \Phi_0 \cdot S_0 = (\alpha_0, \Delta^1_0, \Delta^2_0, \dots, \Delta^d_0) \cdot (S^0_0, S^1_0, \dots, S^d_0)$$

at time zero Φ_0.

at time 1; $V_1^{\Phi_0} = \Phi_0 \cdot S_1 = (\alpha_0, \Delta^1_0, \Delta^2_0, \dots \Delta^d_0) \times (S^0_1, S^1_1, S^2_1 \dots, S^d_1)$.

$$V_1^{\Phi} = \Phi_1 \cdot S_1 = \sum_{j=1}^{d} \Delta^j_1 S^j_1 + \alpha_1 S^0_1.$$

Self financing:

$$\boxed{\phi_{n-1} \cdot S_n = \phi_n \cdot S_n}$$

$$\cancel{\forall n > 0}$$

$$\cancel{\phi_{n-1} \cdot S_{n-1} = \Phi_n \cdot S_{n-1}}$$

$$V_1^{\Phi_0} = \Phi_0 \cdot S_1 = \Phi_1 \cdot S_1 \qquad V_2^{\Phi_1} = \Phi_1 \cdot S_2 =$$

W7.P49

$$C_0^a \geq C_0^e \geq \underbrace{(S_0 - Ke^{-rT})^+ \geq (S_0 - K)^+}.$$

If $S_0 > K$.. $\quad S_0 - K \underbrace{e^{-(r-\delta)T}} < S_0 - K$ //

Different versions of the binomial model.

The binomial model for the stock price is

$$S_{t+h} = \begin{cases} S_t u \\ S_t d \end{cases} \quad \text{where } u = e^{(r-\delta)h + \sigma\sqrt{h}}, \quad d = e^{(r-\delta)h - \sigma\sqrt{h}}.$$

We can write $S_{t+h} = S_t e^{(r-\delta)h \pm \sigma\sqrt{h}}$.

$\ln\left(\frac{S_{t+h}}{S_t}\right) = r_{t,t+h}$. log return of the stock:

Then $\quad r_{t,t+h} = (r-\delta)h \pm \sigma\sqrt{h}$.

The return has two parts, one of which is certain $(r-\delta)h$, and the other of which is uncertain and generates the up and down stock price moves $\pm\sigma\sqrt{h}$.

If we denote by $a = e^{(r-\delta)h}$ or $e^{(r-r_f)h}$ the RNPM

$$q = \frac{a - ae^{-\sigma\sqrt{h}}}{ae^{\sigma\sqrt{h}} - ae^{-\sigma\sqrt{h}}} = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{1}{1 + e^{\sigma\sqrt{h}}}$$

In this q depends only on σ and h.

WP50

We have: $\frac{u}{d} = e^{2\sigma\sqrt{h}} > 1$ since $\sigma > 0$ and $h > 0$.

Alternative binomial models.

The Cox – Ross – Rubinstein binomial tree.

(*) It is Know in the litterature that the best way to construct a binomial tree is that of C.R.R: in which the tree is constructed ~~~~ using the following parametrization of u and d.

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}}. \quad (du = 1)$$

$$\frac{u}{d} = e^{2\sigma\sqrt{h}} > 1.$$

The C.R.R approach is often used in practice.

But remark that if h is large ~~~~ or σ is small: it is possible that $e^{rh} > e^{\sigma\sqrt{h}} = u$. in which case the binomial tree violates the non arbitrage condition.

In real aplications h would be small.

(*) The lognormal tree:

Another alternative tree is to construct the model using:

$$u = e^{(r-\delta-\frac{\sigma^2}{2})h+\sigma\sqrt{h}}$$

and

$$d = e^{(r-\delta-\frac{\sigma^2}{2})h-\sigma\sqrt{h}}$$

$$\frac{u}{d} = e^{2\sigma\sqrt{h}} > 1.$$

$$q = \frac{a-d}{u-d}.$$

W7.P51

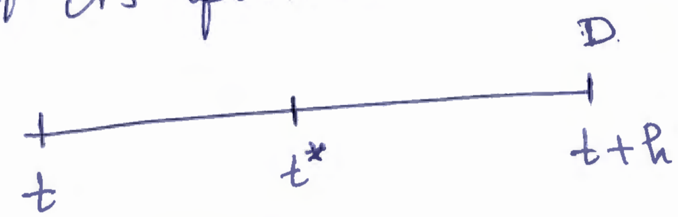
Remark: All the three methods of constructing a binomial tree yield different option prices for finite steps. While the different trees all have different up and down movements. but all have the ration of u to d:

$$\frac{u}{d} = e^{2\sigma\sqrt{h}} \quad \text{or} \quad \ln\left(\frac{u}{d}\right) = 2\sigma\sqrt{h}.$$

Stocks paying discrete dividends:

Modeling discrete dividends:

Suppose a dividend will be paid between t and $t+h$ and that its future value at time $t+h$ is D.



The time forward price for delivery at $t+h$ is then:

$$F_{t,t+h} = S_t e^{r(t+h-t)} - D = S_t e^{rh} - D.$$

Since the stock price at time $t+h$ will be ex-dividend we create the up and down moves based on the ex-dividend stock price:

$$S_t^u = F_{t,t+h} e^{\sigma\sqrt{h}} ; \quad S_t^d = F_{t,t+h} e^{-\sigma\sqrt{h}}.$$

That is $S_t^u = (S_t e^{rh} - D)e^{\sigma\sqrt{h}}$, $S_t^d = (S_t e^{rh} - D)e^{-\sigma\sqrt{h}}$.

Question: How does option replication when a dividend is imminent?

Trees: stock S_{t+h}

$$\begin{tikzcd}[row sep=small] & S_t^u \\ S_t \arrow[ur] \arrow[dr] & \\ & S_t^d \end{tikzcd}$$

option:

$$\begin{tikzcd}[row sep=small] & C^u \\ C_0 ? \arrow[ur] \arrow[dr] & \\ & C^d \end{tikzcd}$$

The payoff of the option is $C = \begin{cases} C^u \\ C^d \end{cases}$ is replicated if ~~B~~ we can find a ~~port~~ trading strategy $(\alpha, \Delta) = \Phi$ such that

$$V^{\Phi}_{t+h} = C = \alpha e^{rh} + \Delta S_{t+h}.$$

Find out α and Δ for the next lecture.

When a dividend is paid we have to account for the fact that the stock earns the dividend.