Int. J. Contemp. Math. Sciences, Vol. 9, 2014, no. 5, 195 - 213
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2014.4217

On Approach Limit Groups and Their Uniformization

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Abstract

We discuss the compatibilities between approach limit structures and group structures; present some basic facts, provide several natural examples, and include some characterization theorems. Introducing the notions of approach pre-Cauchy group and approach Cauchy group, we look at the relationship of these notions with approach limit group. In particular, we consider uniformization of approach limit groups.

1. Introduction

The category of convergence approach spaces and contractions **CAP** [14] is Cartesian closed and contains the category of convergence spaces **CONV** as a simultaneously bireflective and bicoreflective subcategory. In [25] approach uniform convergence spaces were introduced and in [13] the subcategory of ultra approach uniform convergence spaces **uAUCS** was shown to be Cartesian closed. In [19, 17] the categories of approach Cauchy spaces and of ultra approach Cauchy spaces **uACS** were introduced, the latter being a Cartesian closed topological category. As an application of the notion of approach

convergence space, R. Lowen and B. Windels introduced the notion of convergence approach group [20], and showed that every convergence approach group gives rise to an approach uniform convergence structure. In [21] approach convergence vector spaces were studied. In 1997, P. Brock and D. C. Kent [3] introduced the category **LTS** of limit tower spaces and proved that **CAP** and **LTS** are isomorphic.
Motivated by the preceding developments, we propose a more general definition of approach limit group. Besides presenting natural examples, we introduce the notions of approach pre-Cauchy group, approach Cauchy group and their relationship with approach limit group. Further, we present two constructions of approach uniform convergence structures for approach limit groups. We also provide various functorial relations between the categories discussed in this paper. We arrange our work as follows: In Section 2, we fix the notation and provide some preliminary theory and definitions. Some basic facts on approach limit groups including characterization theorems are given in Sections 3 while natural examples are accomodated in Section 4. In Section 5, we study the uniformization of approach limit groups.

2. Preliminaries

We denote the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on a set X by $\mathbb{F}(X)$. The *point filter* of a point $x \in X$ is defined by $[x] = \{A \subseteq X \ : \ x \in A\}$. The set $\mathbb{F}(X)$ is ordered by set inclusion, i.e. we write $\mathbb{F} \leq \mathbb{G}$ if $\mathbb{F} \subseteq \mathbb{G}$.

A *limit space* [6], (X, q) is a non-void set X together with a mapping $q : \mathbb{F}(X) \longrightarrow 2^X$ that satisfies the following axioms.
(L1) $x \in q([x])$ for all $x \in X$;
(L2) $\mathbb{F} \leq \mathbb{G}$ implies $q(\mathbb{F}) \subseteq q(\mathbb{G})$;
(L3) $q(\mathbb{F}) \cap q(\mathbb{G}) \subseteq q(\mathbb{F} \wedge \mathbb{G})$.
A mapping between two limit spaces (X, q) and (X', q'), $f : X \longrightarrow X'$, is called *continuous* if $f(q(\mathbb{F})) \subseteq q'(f(\mathbb{F}))$ for all $\mathbb{F} \in \mathbb{F}(X)$. The category of limit spaces is denoted by **LIM**. This category is topological over **SET** and Cartesian closed. For two limit structures p, q on a set X we call q *finer than* p if $id_X : (X, q) \longrightarrow (X, p)$ is continuous and we write $p \leq q$ in this case.

Definition 2.1. [6] Let $(X, \cdot)$ be a group and $(X, q) \in |\mathbf{LIM}|$. We call $(X, \cdot, q)$ a *limit group* if the group operations $m : X \times X \longrightarrow X, (x, y) \mapsto xy$ and $\jmath : X \longrightarrow X, x \mapsto x^{-1}$ are continuous. The category of limit groups and continuous group homomorphisms is denoted by **LIMGRP**.

We consider in the sequel a group $(X, \cdot)$. For $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ we define $\mathbb{F} \odot \mathbb{G} = m(\mathbb{F} \times \mathbb{G})$ and $\mathbb{F}^{-1} = \jmath(\mathbb{F})$. Noting that $m(F \times G) = \{xy | x \in F, y \in G\} = F \odot G$, we have $\mathbb{F} \odot \mathbb{G} = [\{F \odot G | F \in \mathbb{F}, G \in \mathbb{G}\}]$. Similarly, we find $\mathbb{F}^{-1} = [\{F^{-1} | F \in \mathbb{F}\}]$, where $F^{-1} = \{x^{-1} | x \in F\}$. Throughout the text for a group $(X, \cdot)$, we consider e as the identity element.

Lemma 2.2. *Let X and Y be groups, $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathbb{F}(X)$ and $f : X \longrightarrow Y$ a group homomorphism, then we have*

(i) $\mathbb{F} \odot \mathbb{F}^{-1} \leq [e]$ *and* $\mathbb{F}^{-1} \odot \mathbb{F} \leq [e]$*;*
(ii) $[x] \odot [x]^{-1} = [x]^{-1} \odot [x] = [e]$*;*
(iii) $[x^{-1}] = [x]^{-1}$*;*
(iv) $[x \cdot y] = [x] \cdot [y]$*;*
(v) $(\mathbb{F} \odot \mathbb{G}) \odot \mathbb{H} = \mathbb{F} \odot (\mathbb{G} \odot \mathbb{H})$*;*
(vi) $(\mathbb{F}^{-1})^{-1} = \mathbb{F}$*;*
(vii) $(\mathbb{F} \odot \mathbb{G})^{-1} = \mathbb{G}^{-1} \odot \mathbb{F}^{-1}$*;*
(viii) $[e] \odot \mathbb{F} = \mathbb{F} \odot [e] = \mathbb{F}$*;*
(ix) $(\mathbb{F} \wedge \mathbb{G})^{-1} = \mathbb{F}^{-1} \wedge \mathbb{G}^{-1}$*;*
(x) $(\mathbb{F} \wedge \mathbb{G}) \odot \mathbb{H} = (\mathbb{F} \odot \mathbb{H}) \wedge (\mathbb{G} \odot \mathbb{H})$*;*
(xi) $f(\mathbb{F} \odot \mathbb{G}) = f(\mathbb{F}) \odot f(\mathbb{G})$*;*
(xii) $f(\mathbb{F}^{-1}) = (f(\mathbb{F}))^{-1}$.

Hence we can characterize a limit group as follows.

Lemma 2.3. $(X, \cdot, q)$ *is a limit group if and only if* $(X, q) \in |\boldsymbol{LIM}|$ *and the following axioms are true.*
(LM) $x \in q(\mathbb{F})$, $y \in q(\mathbb{G})$ *implies* $xy \in q(\mathbb{F} \odot \mathbb{G})$*;*
(LI) $x \in q(\mathbb{F})$ *implies* $x^{-1} \in q(\mathbb{F}^{-1})$.

Following an idea introduced in [5], we consider a binary operation $\boxplus$: $[0, \infty] \times [0, \infty] \longrightarrow [0, \infty]$ which is commutative and associative such that

$$0 \boxplus \alpha = \alpha, \tag{1}$$

$$\alpha \boxplus \bigwedge_{\beta_k \in K} \beta_k = \bigwedge_{\beta_k \in K} (\alpha \boxplus \beta_k), K \subseteq [0, \infty]. \tag{2}$$

Note that, as a consequence of (2), the operation $\boxplus$ is order-preserving in both arguments. Moreover $\boxplus = \vee$ (the maximum operation) is the pointwise smallest such operation.
Thus $([0, \infty], \boxplus)$ is a dual quantale in the preceding definition. We consider in this paper often two such *dual quantale operations*, $\boxplus, \boxtimes$, on $[0, \infty]$, the most prominent examples being the sum, $\alpha \boxplus \beta = \alpha + \beta$, and the maximum, $\alpha \boxtimes \beta = \alpha \vee \beta$.

Example 2.4. Define for a fixed $n \in \mathbb{N}$ the operation $\boxplus_n : [0, \infty] \times [0, \infty] \longrightarrow [0, \infty]$ by

$$\alpha \boxplus_n \beta = (\alpha^n + \beta^n)^{1/n}.$$

This example includes the two standard examples $\boxplus = +$ for $n = 1$ and $\boxplus = \vee$ as $\lim_{n \to \infty} (\alpha \boxplus_n \beta) = \alpha \vee \beta$.

The distributivity property (2) immediately implies the following result.

Lemma 2.5. *Let* $L = [0, \infty]$ *and* $\alpha_i, \beta_j \in L$ *for* $i \in I, j \in J$*. Then*

$$(\bigwedge_{i \in I} \alpha_i) \boxplus (\bigwedge_{j \in J} \beta_j) = \bigwedge_{(i,j) \in I \times J} (\alpha_i \boxplus \beta_j).$$

Definition 2.6. [3] A $\boxplus$-*limit tower space* is a pair $\left(X, \overline{p} = (p_\epsilon)_{\epsilon \in [0,\infty]}\right)$ of a set X and a family of mappings $\left(p_\epsilon : \mathbb{F}(X) \longrightarrow 2^X\right)_{\epsilon \in [0,\infty]}$ which satisfies the following conditions:

(LT1) $x \in p_\epsilon([x])$, $\forall \epsilon \in [0,\infty], x \in X$.
(LT2) $\mathbb{F} \subseteq \mathbb{G}$ implies $p_\epsilon(\mathbb{F}) \subseteq p_\epsilon(\mathbb{G})$.
(LT3$\boxplus$) $\forall \epsilon, \delta \in [0,\infty]$, $x \in p_\epsilon(\mathbb{F}), x \in p_\delta(\mathbb{G})$ implies $x \in p_{\epsilon \boxplus \delta}(\mathbb{F} \wedge \mathbb{G})$.
(LT4) $\epsilon \le \delta$ implies $p_\delta \le p_\epsilon$, i.e., $\forall \mathbb{F} \in \mathbb{F}(X)$, $p_\epsilon(\mathbb{F}) \subseteq p_\delta(\mathbb{F})$.
(LT5) p_∞ is indiscrete, i.e. $\forall \mathbb{F} \in \mathbb{F}(X)$, $p_\infty(\mathbb{F}) = X$.
(LT6) $\bigcap_{\gamma > \epsilon} p_\gamma(\mathbb{F}) = p_\epsilon(\mathbb{F})$, $\forall \epsilon, \delta \in [0,\infty]$,$\forall \mathbb{F} \in \mathbb{F}(X)$.

The property (LT6) is called *left-continuity*. A mapping between two $\boxplus$-limit tower spaces, $f : (X, \overline{p}) \longrightarrow (Y, \overline{q})$ is called *continuous* if for all $\epsilon \in [0,\infty]$, and $\forall \mathbb{F} \in \mathbb{F}(X)$, we have $f(p_\epsilon(\mathbb{F})) \subseteq q_\epsilon(f(\mathbb{F}))$.

Note that if $\boxplus = \vee$, then the axiom (LT3$\vee$) is equivalent to
(LT3$\vee'$) $x \in p_\epsilon(\mathbb{F}), x \in p_\epsilon(\mathbb{G})$ implies $x \in p_\epsilon(\mathbb{F} \wedge \mathbb{G})$.
Let $\boxplus$-**LTS** denote the category of all $\boxplus$-limit tower spaces and morphisms the continuous mappings between the objects.

Definition 2.7. [14] We call a pair (X, λ) with a non-void set X and a *limit* $\lambda : \mathbb{F}(X) \longrightarrow [0,\infty]^X$ a $\boxplus$-*approach limit space* if the following axioms are satisfied.
(AL1) $\lambda([x])(x) = 0$ for all $x \in X$;
(AL2) $\lambda(\mathbb{G}) \le \lambda(\mathbb{F})$ whenever $\mathbb{F} \le \mathbb{G}$;
(AL3$\boxplus$) For all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ we have $\lambda(\mathbb{F} \wedge \mathbb{G}) \le \lambda(\mathbb{F}) \boxplus \lambda(\mathbb{G})$.
A mapping between two $\boxplus$-approach limit spaces, $f : (X, \lambda) \longrightarrow (Y, \lambda')$ is called a *contraction* if for all $\mathbb{F} \in \mathbb{F}(X)$ and for all $x \in X$, $\lambda'(f(\mathbb{F}))(f(x)) \le \lambda(\mathbb{F})(x)$.

The category with objects all the $\boxplus$-approach limit spaces and as morphisms the contractions is denoted by $\boxplus$-**ALS**. In [14] $\vee$-approach limit spaces are called *convergence approach spaces* and in [18] $+$-approach limit spaces are calle *weak convergence approach spaces*. Note that (AL3$\vee$) is the strongest possible condition and that $\vee$-**ALS** is a Cartesian closed category [14].

The following two functors provide an isomorphism between the categories $\boxplus$-**LTS** and $\boxplus$-**ALS** [3].

$$\mathfrak{F} : \begin{cases} \boxplus - \mathbf{ALS} & \longrightarrow & \boxplus - \mathbf{LTS} \\ (X, \lambda) & \longmapsto & (X, \mathfrak{F}\lambda) \\ f & \longmapsto & f \end{cases},$$

where $x \in \mathfrak{F}\lambda_\epsilon(\mathbb{F}) \iff \lambda(\mathbb{F})(x) \le \epsilon$, and

$$\mathfrak{G} : \begin{cases} \boxplus - \mathbf{LTS} & \longrightarrow & \boxplus - \mathbf{ALS} \\ (X, \overline{p}) & \longmapsto & (X, \mathfrak{G}\overline{p}) \\ f & \longmapsto & f \end{cases},$$

where $\mathfrak{G}\overline{p}(\mathbb{F})(x) = \bigwedge\{\epsilon \in [0,\infty] : x \in p_\epsilon(\mathbb{F})$. Thus in view of Theorem 3.3 [11], we have the following:

Corollary 2.8. *The categories* $\boxplus$-***ALS*** *and* $\boxplus$-***LTS*** *are isomorphic.*

For filters $\Phi, \Psi \in \mathbb{F}(X \times X)$ we use the notation $\Phi^{-1} = \{F^{-1} | F \in \Phi\}$, where $F^{-1} = \{(x, y) \in X \times X | (y, x) \in F\}$ and $\Phi \circ \Psi = [\{F \circ G | F \in \Phi, G \in \Psi\}]$, where $F \circ G = \{(x, y) \in X \times X | \, \exists z \in X \ni (x, z) \in F, (z, y) \in G\}$. Note that only if $F \circ G \neq \emptyset$ for all $F \in \Phi, G \in \Psi$, then $\Phi \circ \Psi \in \mathbb{F}(X)$. In this case, we also say that $\Phi \circ \Psi$ exists.

Definition 2.9. A $\boxtimes\boxplus$-*approach uniform convergence structure* on a set X is a mapping $\Upsilon : \mathbb{F}(X \times X) \to [0, \infty]$ satisfying the following conditions:
(AUC1) $\forall x \in X$: $\Upsilon([x] \times [x]) = 0$;
(AUC2) $\Phi \leq \Psi$ implies $\Upsilon(\Phi) \geq \Upsilon(\Psi)$;
(AUC3$\boxplus$) $\Upsilon(\Phi \wedge \Psi) \leq \Upsilon(\Phi) \boxplus \Upsilon(\Psi)$;
(AUC4) $\Upsilon(\Phi) = \Upsilon(\Phi^{-1})$;
(AUC5$\boxtimes$) $\forall \Phi, \Psi \in \mathbb{F}(X \times X)$, $\Upsilon(\Phi \circ \Psi) \leq \Upsilon(\Phi) \boxtimes \Upsilon(\Psi)$, whenever $\Phi \circ \Psi$ exists.
The pair (X, Υ) is called an $\boxtimes\boxplus$-*approach uniform convergence space*. If (X, Υ) and (Y, Υ') are two $\boxtimes\boxplus$-approach uniform convergence spaces, then a mapping $f\colon (X, \Upsilon) \longrightarrow (Y, \Upsilon')$ is called a *uniform contraction* if and only if for any $\Phi \in \mathbb{F}(X \times X)$, $\Upsilon'((f \times f)(\Phi)) \leq \Upsilon(\Phi)$. The category of $\boxtimes\boxplus$-approach uniform convergence spaces and uniform contractions is denoted by $\boxtimes\boxplus$-**AUCS**.

For $\boxplus = \boxtimes = +$, a $\boxplus\boxplus$-approach uniform convergence space is known as *approach uniform convergence space* while for $\boxplus = \boxtimes = \vee$, it is known as *ultra approach uniform convergence space* [13].

Definition 2.10. [13] A pair $\left(X, \Upsilon = (\Upsilon_\epsilon)_{\epsilon \in [0,\infty]}\right)$, is called a $\boxtimes\boxplus$-*uniform convergence tower space*, where Υ_ϵ satisfies the following conditions:
(UCTS1) $\forall x \in X$, $\forall \epsilon \in [0, \infty]$, $\dot{x} \times \dot{x} \in \Upsilon_\epsilon$;
(UCTS2) $\Phi \in \Upsilon_\epsilon$ with $\Psi \leq \Phi$, implies $\Psi \in \Upsilon_\epsilon$;
(UCTS3) if $\epsilon \leq \delta$, then $\Upsilon_\epsilon \leq \Upsilon_\delta$;
(UCTS4) $\Upsilon_\infty = \mathbb{F}(X \times X)$;
(UCTS5$\boxplus$) $\forall \epsilon, \delta \in [0, \infty]$, if $\Phi \in \Upsilon_\epsilon$ and $\Psi \in \Upsilon_\delta$, then $\Phi \wedge \Psi \in \Upsilon_{\epsilon \boxplus \delta}$;
(UCTS6) if $\Phi \in \Upsilon_\epsilon$, then $\Phi^{-1} \in \Upsilon_\epsilon$;
(UCTS7$\boxtimes$) if $\Phi \in \Upsilon_\epsilon$, $\Psi \in \Upsilon_\delta$, and $\Phi \circ \Psi$ exists, then $\Phi \circ \Psi \in \Upsilon_{\epsilon \boxtimes \delta}$;
(UCTC) $\Upsilon_\epsilon = \cap_{\epsilon < \delta} \Upsilon_\delta$.
A mapping $f\colon (X, \Upsilon) \longrightarrow (Y, \Upsilon')$ between $\boxtimes\boxplus$-uniform convergence tower spaces is called uniformly continuous if and only if for all $\epsilon \in [0, \infty]$, $(f \times f)(\Upsilon_\epsilon) \subseteq \Upsilon'_\epsilon$. We denote $\boxtimes\boxplus$-**UCTS** the category of $\boxtimes\boxplus$-uniform convergence tower spaces as objects and uniformly continuous functions between $\boxtimes\boxplus$-uniform convergence tower spaces as morphisms (see also [13], pp. 708).

Note again that (UCTS7$\vee$) is equivalent to the axiom
(UCTS7$\vee'$) if $\Phi \in \Upsilon_\epsilon$, $\Psi \in \Upsilon_\epsilon$, and $\Phi \circ \Psi$ exists, then $\Phi \circ \Psi \in \Upsilon_\epsilon$.

3. $\boxtimes\boxplus$-limit tower groups and $\boxtimes\boxplus$-approach limit groups

Definition 3.1. Let $(X, \cdot)$ be a group and $(X, \overline{p}) \in |\boxplus-\mathbf{LTS}|$. We call the triple $(X, \cdot, \overline{p})$ a $\boxtimes\boxplus$-*limit tower group* if and only if the following axioms are fulfilled:
(LTM$\boxtimes$) $\forall \epsilon, \delta \in [0, \infty]$, $\mathbb{F}_\epsilon, \mathbb{G}_\delta \in \mathbb{F}(X)$, $\forall x, y \in X :\, x \in p_\epsilon(\mathbb{F})$ and $y \in p_\delta(\mathbb{G})$ implies $xy \in p_{\epsilon\boxtimes\delta}(\mathbb{F} \odot \mathbb{G})$.
(LTI) $\forall \mathbb{F} \in \mathbb{F}(X), \forall \epsilon \in [0, \infty], \forall x \in X :\, x \in p_\epsilon(\mathbb{F})$ implies $x^{-1} \in p_\epsilon(\mathbb{F}^{-1})$.

We denote the category with objects the $\boxtimes\boxplus$-limit tower groups and morphisms the continuous group homomorphisms by $\boxtimes\boxplus$-**LTG**.

Lemma 3.2. *Let $(X, \overline{p}) \in |\boxtimes\boxplus - \boldsymbol{LTG}|$. Let $\mathbb{F} \in \mathbb{F}(X)$, $\epsilon \in [0, \infty]$, and $x \in X$. Then $x \in p_\epsilon(\mathbb{F}) \Leftrightarrow e \in p_\epsilon\left([x^{-1}] \odot \mathbb{F}\right) \Leftrightarrow e \in p_\epsilon\left(\mathbb{F} \odot [x^{-1}]\right)$.*

Proof. Let $\mathbb{F} \in \mathbb{F}(X)$, $\epsilon \in [0, \infty]$, and $x \in X$. Let $x \in p_\epsilon(\mathbb{F})$. By (LTI), $x^{-1} \in p_0([x^{-1}])$ and hence, by (LTM$\boxtimes$), $e = xx^{-1} \in p_{\epsilon\boxtimes 0}\left([x^{-1}] \odot \mathbb{F}\right) = p_\epsilon\left([x^{-1}] \odot \mathbb{F}\right)$. Conversely, assume that $e \in p_\epsilon\left([x^{-1}] \odot \mathbb{F}\right)$. Then since $x \in p_0([x])$, again applying (LMT$\boxtimes$) in conjunction with Lemma 2.2(ii) and (v), we have $x = xe \in p_{\epsilon\boxtimes 0}\left([x] \odot [x^{-1}] \odot \mathbb{F}\right) = p_\epsilon(\mathbb{F})$. □

Definition 3.3. [20] Let $(X, \cdot)$ be a group. A triple $(X, \cdot, \lambda)$ is called a $\boxtimes\boxplus$-*approach limit group* if and only if the following conditions are fulfilled:
(ALS$\boxplus$) (X, λ) is a $\boxplus$-approach limit space.
(ALM$\boxtimes$) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X), x, y \in X :\, \lambda(\mathbb{F} \odot \mathbb{G})(xy) \leq \lambda(\mathbb{F})(x) \boxtimes \lambda(\mathbb{G})(y)$.
(ALI) $\forall \mathbb{F} \in \mathbb{F}(X), x \in X :\, \lambda(\mathbb{F}^{-1})(x^{-1}) \leq \lambda(\mathbb{F})(x)$.
We denote the category with objects the $\boxtimes\boxplus$-approach limit groups and contractive homomorphisms as morphisms by $\boxtimes\boxplus$-**ALG**.

It is not difficult to show that functors $\mathfrak{F}, \mathfrak{G}$, are isomorphism functors when restricted to $\boxtimes\boxplus$-**LTG** and $\boxtimes\boxplus$-**ALG**, i.e. that $\boxtimes\boxplus$-**LTG** and $\boxtimes\boxplus$-**ALG** are isomorphic and this is we put in the following

Theorem 3.4. *The functors $\mathfrak{F} : \boxplus-\boldsymbol{LTS} \longrightarrow \boxplus-\boldsymbol{ALS}$ and $\mathfrak{G} : \boxplus-\boldsymbol{ALS} \longrightarrow \boxplus-\boldsymbol{LTS}$, restricted in domain and codomain to $\boxtimes\boxplus$-$\boldsymbol{LTG}$ and $\boxtimes\boxplus$-$\boldsymbol{ALG}$, are isomorphism functors between $\boxtimes\boxplus$-$\boldsymbol{LTG}$ and $\boxtimes\boxplus$-$\boldsymbol{ALG}$.*

Lemma 3.5. *Let $(X, \lambda) \in |\boxtimes\boxplus - \boldsymbol{ALG}|, \mathbb{F} \in \mathbb{F}(X)$ and $x \in X$.
Then $\lambda(\mathbb{F})(x) = \lambda\left([x^{-1}] \odot \mathbb{F}\right)(e) = \lambda\left(\mathbb{F} \odot [x^{-1}]\right)(e)$.*

Proof. This is quite similar to the proof of Lemma 3.2 and is left to the reader. □

We present below characterizations of $\boxtimes\boxplus$-approach limit groups.

Theorem 3.6. *Let $(X,\cdot)$ be a group and λ be a $\boxplus$-approach limit structure on X. Then $(X,\cdot,\lambda)$ is an $\boxtimes\boxplus$-approach limit group if and only if the following axioms are fulfilled:*
(1) $\lambda\left([e]\right)(e)=0$.
(2) If $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ *with* $\mathbb{F}\leq\mathbb{G}$, *then* $\lambda\left(\mathbb{G}\right)(e)\leq\lambda\left(\mathbb{F}\right)(e)$.
(3) $\forall\,\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, $\lambda\left(\mathbb{F}\wedge\mathbb{G}\right)(e)\leq\lambda\left(\mathbb{F}\right)(e)\boxplus\lambda\left(\mathbb{G}\right)(e)$.
(4) $\lambda\left(\mathbb{F}\odot\mathbb{G}\right)(e)\leq\lambda\left(\mathbb{F}\right)(e)\boxtimes\lambda\left(\mathbb{G}\right)(e)$, $\forall\,\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$.
(5) $\lambda\left(\mathbb{F}^{-1}\right)(e)\leq\lambda\left(\mathbb{F}\right)(e)$, $\forall\,\mathbb{F}\in\mathbb{F}(X)$.
(6) $\forall\,\mathbb{F}\in\mathbb{F}(X)$ *and* $x\in X$*:* $\lambda\left(\mathbb{F}\right)(x)=\lambda\left([x^{-1}]\odot\mathbb{F}\right)(e)=\lambda\left(\mathbb{F}\odot[x^{-1}]\right)(e)$.

Proof. If $(X,\cdot,\lambda)$ is a $\boxtimes\boxplus$-approach limit group, then (1), (2) and (3) are special cases of (AL1), (AL2) and (AL3$\boxplus$). Items (4) and (5) follow from the definition of multiplication and inversion for the identity element, while (6) is just Lemma 3.5. Conversely, assume that (1)-(6) are true. First, we prove (ALM$\boxtimes$). Let $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ and $x,y\in X$. Then by using Lemma 2.2 several times, we have the following:
$\lambda\left(\mathbb{F}\odot\mathbb{G}\right)(xy)=\lambda\left([e\odot(\mathbb{F}\odot\mathbb{G})\right)(xy)=\lambda\left([(xy)^{-1}]\odot\underbrace{(\mathbb{F}\odot\mathbb{G})}\right)(e)$ (by (6))
$=\lambda\left([y^{-1}]\odot[x^{-1}]\odot(\mathbb{F}\odot\mathbb{G})\right)(e)=\lambda\left([y^{-1}]\odot\overbrace{\left(\left([x^{-1}]\odot\mathbb{F}\right)\odot\mathbb{G}\right)}\right)(e)$
$=\lambda\left(\overbrace{\left(\left([x^{-1}]\odot\mathbb{F}\right)\odot\mathbb{G}\right)}\odot[y^{-1}]\right)(e)$ (by (6))
$=\lambda\left(\left([x^{-1}]\odot\mathbb{F}\right)\odot\left(\mathbb{G}\odot[y^{-1}]\right)\right)(e)$ (by Lemma 2.2(v))
$\leq\lambda\left([x^{-1}]\odot\mathbb{F}\right)(e)\boxtimes\lambda\left(\mathbb{G}\odot[y^{-1}]\right)(e)$ (by (4))
$=\lambda\left(\mathbb{F}\right)(x)\boxtimes\lambda\left(\mathbb{G}\right)(y)$ (by (6))
(ALI) Let $\mathbb{F}\in\mathbb{F}(X)$ and $x\in X$. Then
$\lambda\left(\mathbb{F}^{-1}\right)(x^{-1})=\lambda\left([x^{-1}]^{-1}\odot\mathbb{F}^{-1}\right)(e)$ (by (6))
$=\lambda\left(\left(\mathbb{F}\odot[x^{-1}]\right)^{-1}\right)(e)$ (by Lemma 2.2(vii))
$\leq\lambda\left(\mathbb{F}\odot[x^{-1}]\right)(e)$ (by (5))
$=\lambda\left(\mathbb{F}\right)(x)$ (by (6)). □

Theorem 3.7. *Let $(X,\cdot)$ be a group and $\lambda':\mathbb{F}(X)\to[0,\infty]^X$ be a map such that the following axioms are fulfilled:*
(1) $\lambda'\left([e]\right)(e)=0$.
(2) If $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ *with* $\mathbb{F}\leq\mathbb{G}$, *then* $\lambda'\left(\mathbb{G}\right)(e)\leq\lambda'\left(\mathbb{F}\right)(e)$.
(3) $\forall\,\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, $\lambda'\left(\mathbb{F}\wedge\mathbb{G}\right)(e)\leq\lambda'\left(\mathbb{F}\right)(e)\boxplus\lambda'\left(\mathbb{G}\right)(e)$.
(4) $\lambda'\left(\mathbb{F}\odot\mathbb{G}\right)(e)\leq\lambda'\left(\mathbb{F}\right)(e)\boxtimes\lambda'\left(\mathbb{G}\right)(e)$, $\forall\,\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$.
(5) $\lambda'\left(\mathbb{F}^{-1}\right)(e)\leq\lambda'\left(\mathbb{F}\right)(e)$, $\forall\,\mathbb{F}(X)$.
Then there exists a unique $\boxplus$-approach limit structure $\lambda:\mathbb{F}(X)\to[0,\infty]^X$ satisfying $\lambda\left(\mathbb{F}\right)(e)=\lambda'\left(\mathbb{F}\right)(e)$, such that $(X,\cdot,\lambda)$ is a (homogeneous) $\boxtimes\boxplus$-approach limit group.

Proof. Define the map $\lambda:\mathbb{F}(X)\to[0,\infty]^X$ by $\lambda\left(\mathbb{F}\right)(x)=\lambda'\left([x]^{-1}\odot\mathbb{F}\right)(e)$. Then the result follows from straightforward calculation. □

4. Examples of ⊠⊞-approach limit groups

Example 4.1 ([14])**.** For a limit group $(X, \cdot, q)$ its embedding in ⊠⊞-**ALG**, $(X, \cdot, \lambda_q)$, is an ⊠⊞-approach limit group. Here,

$$\lambda_q(\mathbb{F})(x) = \begin{cases} 0, & \text{if } x \in q(\mathbb{F}); \\ \infty, & \text{otherwise.} \end{cases}$$

Example 4.2. This example generalizes Example 3.2 in [20]. Let X be a vector space and let $\| \ \|$ be a norm with the triangle inequality

$$\|x + y\| \le \|x\| \boxtimes \|y\|$$

for all $x, y \in X$. Note that for all norms, no matter what triangular inequality we have, we obtain a $\vee$-approach limit space. A typical example for the case $\boxtimes = \vee$ is the space of p-adic numbers that can be seen in [12]. Also, further examples are given by commutative Hausdorf topological groups, where the neighborhood filter of the zero element has a base consisting of subgroups, see e.g. [22].

Every norm induces a metric $d(x, y) = \|x - y\|$ for all $x, y \in X$. Hence we can consider such a normed space as an approach space [16] with limit function

$$\lambda_{\| \|}(\mathbb{F})(x) = \bigvee_{\mathbb{U} \ge \mathbb{F} \text{ ultra}} \ \bigvee_{U \in \mathbb{U}} \bigwedge_{z \in U} \|x - z\|.$$

If we define for $\epsilon \ge 0$ the ϵ-neighborhood filter at x by $\mathbb{U}^x_\epsilon = [\{B(x, \alpha) \,:\, \alpha > \epsilon\}]$ with the balls $B(x, \alpha) = \{y \in X \,:\, \|x - y\| \le \alpha\}$, then it is not difficult to show that

$$\mathbb{U}^x_\epsilon = \bigwedge_{\lambda_{\| \|}(\mathbb{F})(x) \le \epsilon} \mathbb{F}.$$

Hence this concept of neighborhood filter coincides with the ϵ-neighborhood filter introduced in [8, 9]. We therefore have $\lambda(\mathbb{F})(x) \le \epsilon \iff \mathbb{F} \ge \mathbb{U}^x_\epsilon$. In case of a normed space we can then show that $\lambda(\mathbb{F})(x) \le \epsilon$ and $\lambda(\mathbb{G})(y) \le \delta$ implies $\lambda(\mathbb{F} \oplus \mathbb{G})(x + y) \le \epsilon \boxtimes \delta$. In fact, it suffices to prove that $\mathbb{U}^x_\epsilon \oplus \mathbb{U}^y_\delta \ge \mathbb{U}^{x+y}_{\epsilon \boxtimes \delta}$. To show this, let $U \in \mathbb{U}^{x+y}_{\epsilon \boxtimes \delta}$. Then there is $\alpha > \epsilon \boxtimes \delta$ such that $B(x + y, \alpha) \subseteq U$. If $z \in B(x, \epsilon) \oplus B(y, \delta)$, then $z = u + v$ with $\|x - u\| \le \epsilon$ and $\|y - v\| \le \delta$. It follows from this that $\|(x + y) - (u + v)\| \le \|x - u\| \boxtimes \|y - v\| \le \epsilon \boxtimes \delta < \alpha$, and hence $z \in B(x + y, \alpha)$. We have thus shown that $B(x, \epsilon) \oplus B(y, \delta) \subseteq B(x + y, \alpha) \subseteq U$ and therefore $U \in \mathbb{U}^x_\epsilon \oplus \mathbb{U}^y_\delta$. Hence $\big(X, +, \lambda_{\| \|}\big)$ a $\boxtimes\vee$-approach limit group.

Example 4.3. For $(X, \lambda) \in |\vee\text{-}ALS|$ Lowen and Windels in [20] define

$$\mathcal{A}_\lambda(x) = \{\varphi \ :\ \forall \mathbb{F} \in \mathbb{F}(X) : \bigwedge_{F \in \mathbb{F}} \bigvee_{y \in \mathbb{F}} \varphi(y) \le \lambda(\mathbb{F})(x)\}.$$

This implies, in particular, for $\mathbb{F} \in \mathbb{F}(X)$ and for $x \in X$ that

$$\bigvee_{\varphi \in \mathcal{A}_\lambda(x)} \bigwedge_{F \in \mathbb{F}} \bigvee_{y \in F} \varphi(y) \le \lambda(\mathbb{F})(x).$$

They further define the approach limit $\lambda_\otimes$ on $X \times X$ by

$$\lambda_\otimes(\Phi)(a,b) = \bigvee_{\varphi_a \in \mathcal{A}_\lambda(a)} \bigvee_{\varphi_b \in \mathcal{A}_\lambda(b)} \bigwedge_{\phi \in \Phi} \bigvee_{(x,y)\in\phi} (\varphi_a(x) + \varphi_b(y)).$$

For technical reason we use the symbol $\oplus$ for the group operation. Then the additive group $(X, \oplus)$, is a Lowen-Windels approach convergence group if

$$\oplus : \begin{cases} (X \times X, \lambda_\otimes) & \longrightarrow & (X, \lambda) \\ (x,y) & \longmapsto & x \oplus y \end{cases}$$

and $- : (X,\lambda) \longrightarrow (X,\lambda), x \longmapsto -x$ are contractions. In particular then

$$\lambda(\mathbb{F} \oplus \mathbb{G})(x \oplus y) \leq \lambda_\otimes(\mathbb{F} \times \mathbb{G})(x,y) \leq \lambda(\mathbb{F})(x) + \lambda(\mathbb{G})(y).$$

Hence $(X, \oplus, \lambda)$ is a $+\vee$-approach limit group. Because every approach group in the definition of Lowen-Windels is an approach convergence group, also Lowen-Windels approach groups are $+\vee$-approach limit groups.

Example 4.4. Let μ be the Lebesgue measure on $[0,1]$ and let further τ be the usual (metric) topology on $[0,1]$. Denote further $X = \{f : [0,1] \longrightarrow \Re \ : \ f \text{ measurable}\}$ and define the pointwise (group) operations on X by $(f \oplus g)(x) = f(x) \oplus g(x)$ and $(-f)(x) = -f(x)$. For $\mathbb{F} \in \mathbb{F}(X)$ and $f \in X$ we define

$$\lambda(\mathbb{F})(f) = \bigwedge \{\alpha \in [0,1] \ : \ \exists A \subseteq [0,1], \mu(A) \leq \alpha, \mathbb{F}(x) \xrightarrow{\tau} f(x), \forall x \notin A\}.$$

Then $(X, \oplus, \lambda)$ is a $+$-approach limit space. We only prove condition (AL3+). If $\lambda(\mathbb{F})(f) < \delta$ and $\lambda(\mathbb{G})(f) < \gamma$, then there are $\alpha < \delta, \beta < \gamma$, $A, B \subseteq [0,1]$ with $\mu(A) < \alpha, \mu(B) < \beta$ such that $\mathbb{F}(x) \xrightarrow{\tau} f(x)$ for all $x \notin A$ and $\mathbb{G}(x) \xrightarrow{\tau} f(x)$ for all $x \notin B$. Then $\mu(A \cup B) < \alpha + \beta < \delta + \gamma$ and for all $x \notin A \cup B$ we have $(\mathbb{F} \wedge \mathbb{G})(x) = \mathbb{F}(x) \wedge \mathbb{G}(x) \xrightarrow{\tau} f(x)$. Hence $\lambda(\mathbb{F} \wedge \mathbb{G})(f) < \delta + \gamma$.
We further consider addition. We have if $\lambda(\mathbb{F})(f) < \delta$ and $\lambda(\mathbb{G})(g) < \gamma$ that there are $\alpha < \delta, \beta < \gamma$ and $A, B \subseteq [0,1]$ with $\mu(A) < \alpha$, $\mu(B) < \beta$ such that $\mathbb{F}(x) \xrightarrow{\tau} f(x)$ for all $x \notin A$ and $\mathbb{G}(x) \xrightarrow{\tau} g(x)$ for all $x \notin B$. Then for all $x \notin A \cup B$ we conclude that $(\mathbb{F} \oplus \mathbb{G})(x) = \mathbb{F}(x) \oplus \mathbb{G}(x) \xrightarrow{\tau} f(x) \oplus g(x) = (f \oplus g)(x)$. Because $\mu(A \cup B) \leq \mu(A) + \mu(B) < \alpha + \beta < \delta + \gamma$ we conclude $\lambda(\mathbb{F} \oplus \mathbb{G}) < \delta + \gamma$. Contractivity of the inverse is easy and not presented. Hence $(X, \oplus, \lambda)$ is a $++$-approach limit group.
Note that $\lambda(\mathbb{F})(f) = 0$ if and only if $\mathbb{F} \to f$ μ-almost everywhere. Note further that $0 \leq \lambda(\mathbb{F})(f) \leq \mu([0,1]) = 1$ in this example.

Example 4.5. For two $\vee$-approach limit spaces $(X, \lambda_X), (Y, \lambda_Y)$ we denote $C(X,Y) = \{h : X \longrightarrow Y \ : \ h \text{ contraction}\}$ and for $f, g \in C(X,Y)$ we define the mappings fg and f^{-1} pointwise by $(fg)(x) = f(x)g(x)$ and $f^{-1}(x) = (f(x))^{-1}$ with the group operations in Y. It is not difficult to show that fg and f^{-1} are contractions and that $(C(X,Y), \cdot)$ is a group. We will show that this group, endowed with the natural function space structure of $\vee$-**ALS** [14] is an

$\vee\vee$-approach limit group. With the evaluation mapping $ev : C(X,Y)\times X \longrightarrow Y, (f,x)\longmapsto f(x)$ we define for $\mathbb{F}\in\mathbb{F}(C(X,Y))$ and $f\in C(X,Y)$ the interval

$$\mathcal{L}_{\vee}(\mathbb{F},f)=\{\alpha\in[0,\infty] : \forall \mathbb{H}\in\mathbb{F}(X), x\in X, \lambda_Y(ev(\mathbb{F}\times\mathbb{H}))(f(x))\le\lambda_X(\mathbb{F})(x)\vee\alpha\}$$

and with this then $\lambda_c(\mathbb{F})(f)=\bigwedge\mathcal{L}_{\vee}(\mathbb{F},f)$. It is shown in Theorem 4.2 [14] that $(C(X,Y),\lambda_c)\in|\vee-\mathbf{ALS}|$. We show that it is also a $\vee\vee$-approach limit group with the group operations defined above. We first show that the multiplication in $C(X,Y)$ is a contraction. To this end, let $\mathbb{H}\in\mathbb{F}(X)$ and let $x\in X$. If for $\mathbb{F},\mathbb{G}\in\mathbb{F}(C(X,Y))$ and $f,g\in C(X,Y)$ we have that $\alpha\in\mathcal{L}_{\vee}(\mathbb{F},f)$ and $\beta\in\mathcal{L}_{\vee}(\mathbb{G},g)$, then $\lambda_Y(ev(\mathbb{F}\times\mathbb{H}))(f(x))\le\lambda_X(\mathbb{H})(x)\vee\alpha$ and $\lambda_Y(ev(\mathbb{G}\times\mathbb{H}))(g(x))\le\lambda_X(\mathbb{H})(x)\vee\beta$. It is not difficult to show that $ev(\mathbb{F}\times\mathbb{H})\odot ev(\mathbb{G}\times\mathbb{H})\le ev((\mathbb{F}\odot\mathbb{G})\times\mathbb{H})$. Hence we conclude

$$\begin{aligned}\lambda_Y(ev((\mathbb{F}\odot\mathbb{G})\times\mathbb{H}))((fg)(x)) &\le \lambda_Y(ev(\mathbb{F}\times\mathbb{H})\odot ev(\mathbb{G}\times\mathbb{H}))((fg)(x))\\ &\le \lambda_Y(ev(\mathbb{F}\times\mathbb{H}))(f(x))\vee\lambda_Y(ev(\mathbb{G}\times\mathbb{H}))(g(x))\\ &\le \lambda_X(\mathbb{H})(x)\vee\alpha\vee\lambda_X(\mathbb{H})(x)\vee\beta\\ &= \lambda_X(\mathbb{H})(x)\vee(\alpha\vee\beta).\end{aligned}$$

Hence $\alpha\vee\beta\in\mathcal{L}_{\vee}(\mathbb{F}\odot\mathbb{G},fg)$. Using Lemma 2.5, we conclude that

$$\lambda_c(\mathbb{F}\odot\mathbb{G})(fg)\le(\bigwedge_{\alpha\in\mathcal{L}_{\vee}(\mathbb{F},f)}\alpha)\vee(\bigwedge_{\beta\in\mathcal{L}_{\vee}(\mathbb{G},g)}\beta)=\lambda_c(\mathbb{F})(f)\vee\lambda_c(\mathbb{G})(g).$$

To show that the inverse f^{-1} is a contraction, let $\alpha\in\mathcal{L}_{\vee}(\mathbb{F},f)$. Then for all $\mathbb{H}\in\mathbb{F}(X), x\in X$ we have $\lambda_Y(ev(\mathbb{F}\times\mathbb{H}))(f(x))\le\lambda_X(\mathbb{H})(x)\vee\alpha$. It is not difficult to show that $ev(\mathbb{F}^{-1}\times\mathbb{H})=(ev(\mathbb{F}\times\mathbb{H})^{-1})$ and hence we conclude

$$\lambda_Y(ev(\mathbb{F}^{-1}\times\mathbb{H}))(f^{-1}(x))\le\lambda_Y(ev((\mathbb{F}\times\mathbb{H})^{-1})(f(x))\le\lambda_X(\mathbb{H})(x)\vee\alpha,$$

and $\alpha\in\mathcal{L}_{\vee}(\mathbb{F}^{-1},f^{-1})$. From this it follows that $\lambda_c(\mathbb{F}^{-1})(f^{-1})\le\lambda_c(\mathbb{F})(f)$ and therefore $(C(X,Y),\cdot,\lambda_c)$ is a $\vee\vee$-approach limit group.

Note that if (X,λ_X) and (Y,λ_Y) are $\vee$-approach limit groups and if Y is commutative, then the set $C_h(X,Y)$ of contractive group homomorphisms is a subgroup of $C(X,Y)$. Hence, $(C_h(X,Y),\cdot,\lambda_c)$ is a $\vee\vee$-approach limit group using the structure λ_c. However, this function space does not make $\vee$-**ALG** Cartesian closed. This is due to the fact that the evaluation mapping, $ev : C_h(X,Y)\times X\longrightarrow Y, (f,x)\longmapsto f(x)$ is not a group homomorphism.

5. Relationship between approach pre-Cauchy groups, ⊠⊞-approach Cauchy groups and ⊠⊞-approach limit groups, and uniformization of ⊠⊞-approach limit groups

Definition 5.1. A map $\Gamma:\mathbb{F}(X)\longrightarrow[0,\infty]$ is called an *approach pre-Cauchy structure* on X if and only if the following axioms are fulfilled:
(PACS1) $\forall x\in X$: $\Gamma([x])=0$;
(PACS2) $\forall\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ with $\mathbb{F}\le\mathbb{G}$, $\Gamma(\mathbb{G})\le\Gamma(\mathbb{F})$,

and the pair (X,Γ) is called an *approach pre-Cauchy space*. It is called a $\boxplus$-*approach Cauchy space* if it additionally fulfills the following Cauchy condition:
(AChy$\boxplus$) $\forall \mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, if $\mathbb{F}\vee\mathbb{G}$ exists, then $\Gamma(\mathbb{F}\wedge\mathbb{G})\leq\Gamma(\mathbb{F})\boxplus\Gamma(\mathbb{G})$.

A mapping $f:X\longrightarrow Y$ between approach pre-Cauchy spaces (resp. between $\boxplus$-approach Cauchy spaces) (X,Γ) and (Y,Γ') is called a *Cauchy contraction* if and only if $\forall\mathbb{F}\in\mathbb{F}(X)$, $\Gamma'(f(\mathbb{F}))\leq\Gamma(\mathbb{F})$. The categories of approach pre-Cauchy spaces (resp. $\boxplus$-approach Cauchy spaces) and Cauchy contractions as morphisms are denoted by **APRCHYS** (resp. $\boxplus$-**ACHYS**).

Let $(X,\cdot,\lambda)\in|\boxtimes\boxplus-\mathbf{ALG}|$. Define for $\mathbb{F}\in\mathbb{F}(X)$, $\Gamma_l(\mathbb{F})=\lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e)$. We call $\Gamma_l(\mathbb{F})$ the *left-Cauchy degree of* $\mathbb{F}$.

Proposition 5.2. *If* $(X,\cdot,\lambda)\in|\vee\boxplus-\boldsymbol{ALG}|$, *then* $(X,\Gamma_l)\in|\boldsymbol{APRCHYS}|$. *If* $\boxtimes=\boxplus=\vee$, *then* $(X,\Gamma_l)\in|\vee-\boldsymbol{ACHYS}|$.

Proof. (PACS1) $\Gamma_l([x])=\lambda([x]^{-1}\odot[x])(e)=\lambda([e])(e)=0$;
(PACS2) $\mathbb{F}\leq\mathbb{G}$ implies $\Gamma_l(\mathbb{F})=\lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e)\geq\lambda(\mathbb{G}^{-1}\odot\mathbb{G})(e)=\Gamma_l(\mathbb{G})$;
(PACS3$\vee$) If $\mathbb{F}\vee\mathbb{G}$ exists, then it can be shown that $\mathbb{F}\odot\mathbb{G}^{-1}\leq[e]$. Hence

$$\begin{aligned}\Gamma_l(\mathbb{F})\vee\Gamma_l(\mathbb{G}) &= \lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e)\vee\lambda(\mathbb{G}^{-1}\odot\mathbb{G})(e)\\ &\overset{(ALM\vee)}{\geq} \lambda(\mathbb{F}^{-1}\odot\mathbb{F}\odot\mathbb{G}^{-1}\odot\mathbb{G})(e)\geq\lambda(\mathbb{F}^{-1}\odot\mathbb{G})(e)\end{aligned}$$

Similarly, one can show that $\Gamma_l(\mathbb{G})\vee\Gamma_l(\mathbb{F})\geq\lambda(\mathbb{G}^{-1}\odot\mathbb{F})(e)$. Hence

$$\begin{aligned}\Gamma_l(\mathbb{F}\wedge\mathbb{G}) &= \lambda((\mathbb{F}\wedge\mathbb{G})^{-1}\odot(\mathbb{F}\wedge\mathbb{G}))(e)\\ &= \lambda((\mathbb{F}^{-1}\odot\mathbb{F})\wedge(\mathbb{F}^{-1}\odot\mathbb{G})\wedge(\mathbb{G}^{-1}\odot\mathbb{F})\wedge(\mathbb{G}^{-1}\odot\mathbb{G}))(e)\\ &\leq \lambda(\mathbb{F}^{-1}\odot\mathbb{F})(e)\vee\lambda(\mathbb{F}^{-1}\odot\mathbb{G})(e)\vee\lambda(\mathbb{G}^{-1}\odot\mathbb{F})(e)\vee\lambda(\mathbb{G}^{-1}\odot\mathbb{G})(e)\\ &\leq \Gamma_l(\mathbb{F})\vee(\Gamma_l(\mathbb{F})\vee\Gamma_l(\mathbb{G}))\vee(\Gamma_l(\mathbb{F})\vee\Gamma_l(\mathbb{G}))\vee\Gamma_l(\mathbb{G}).\\ &= \Gamma_l(\mathbb{F})\vee\Gamma_l(\mathbb{G}).\end{aligned}$$

□

Definition 5.3. Let $\Gamma:\mathbb{F}(X)\to[0,\infty]$ be a approach pre-Cauchy structure (resp. a $\boxplus$-approach Cauchy structure) on a group $(X,\cdot)$, then the triple $(X,\cdot,\Gamma)$ is called a $\boxtimes$-*approach pre-Cauchy group* (resp. a $\boxtimes\boxplus$-*approach Cauchy group*) if and only if the following axioms are satisfied:
(ACGM$\boxtimes$) $\forall\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, $\Gamma(\mathbb{F}\odot\mathbb{G})\leq\Gamma(\mathbb{F})\boxtimes\Gamma(\mathbb{G})$;
(ACGI) $\forall\mathbb{F}\in\mathbb{F}(X)$, $\Gamma(\mathbb{F}^{-1})\leq\Gamma(\mathbb{F})$).

Note that (ACGM$\boxtimes$) and (ACGI) are equivalent to the single condition:
(ACGM$\boxtimes'$) $\forall\mathbb{F}\in\mathbb{F}(X)$, $\Gamma(\mathbb{F}\odot\mathbb{G}^{-1})\leq\Gamma(\mathbb{F})\boxtimes\Gamma(\mathbb{G})$.

In fact, if $\mathbb{F}\in\mathbb{F}(X)$, then $\Gamma(\mathbb{F}^{-1})=\Gamma([e]\odot\mathbb{F}^{-1})\leq\Gamma([e])\boxtimes\Gamma(\mathbb{F})=0\boxtimes\Gamma(\mathbb{F})=\Gamma(\mathbb{F})$, and if $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, then $\Gamma(\mathbb{F}\odot\mathbb{G})=\Gamma\left(\mathbb{F}\odot{\mathbb{G}^{-1}}^{-1}\right)\leq\Gamma(\mathbb{F})\boxtimes\Gamma(\mathbb{G}^{-1})\leq\Gamma(\mathbb{F})\boxtimes\Gamma(\mathbb{G})$. Conversely, if $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$, then $\Gamma(\mathbb{F}\odot\mathbb{G}^{-1})\leq\Gamma(\mathbb{F})\boxtimes\Gamma(\mathbb{G}^{-1})\leq\Gamma(\mathbb{F})\boxtimes\Gamma(\mathbb{G})$.
The category of $\boxtimes$-approach pre-Cauchy groups (resp. $\boxtimes\boxplus$-approach Cauchy

groups) and contractive group homomorphisms is denoted by $\boxtimes$-**APRCHYG** (resp. $\boxtimes\boxplus$-**ACHYG**).

Proposition 5.4. *If* $(X,\cdot,\Gamma)\in|\vee-\boldsymbol{APRCHYG}|$, *then it fulfills the* $\vee$-*approach Cauchy condition (AChy*$\vee$*) and hence is* $\in|\vee\vee-\boldsymbol{ACHYG}|$.

Proof. If $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ such that $\mathbb{F}\vee\mathbb{G}$ exists, then $\mathbb{F}\wedge\mathbb{G}\geq\mathbb{F}\odot\mathbb{G}^{-1}\odot\mathbb{G}$. Then using (PACS2) and (ACGM$\vee'$), we get $\Gamma(\mathbb{F}\wedge\mathbb{G})\leq\Gamma(\mathbb{F}\odot\mathbb{G}^{-1}\odot\mathbb{G})\leq\Gamma(\mathbb{F}\odot\mathbb{G}^{-1})\vee\Gamma(\mathbb{G})\leq\Gamma(\mathbb{F})\vee\Gamma(\mathbb{G})\vee\Gamma(\mathbb{G})=\Gamma(\mathbb{F})\vee\Gamma(\mathbb{G})$. □

Proposition 5.5. *If* $(X,\cdot,\Gamma)\in|\vee-\boldsymbol{APRCHYG}|$, *then* $(X,\cdot,\lambda_\Gamma)\,|\vee\vee-\boldsymbol{ALG}|$.

Proof. Every approach pre-Cauchy space (X,Γ) gives rise to a $\vee$-approach limit structure $\lambda_\Gamma:\mathbb{F}(X)\to[0,\infty]^X$ defined by $\lambda_\Gamma(\mathbb{F})(x)=\Gamma(\mathbb{F}\wedge[x])$ ([18]). It is then sufficient to verify the condition (ALM$\vee$), while the condition (ALI) is easy. If $\mathbb{F},\mathbb{G}\in\mathbb{F}(X)$ and $x,y\in X$, then by the Lemma 2.2(iv) and (v), we have
$(\mathbb{F}\odot\mathbb{G}^{-1})\wedge[xy^{-1}]=(\mathbb{F}\odot\mathbb{G}^{-1})\wedge([x]\odot[y]^{-1})\geq(\mathbb{F}\wedge[x])\odot(\mathbb{G}^{-1}\wedge[y]^{-1})$.
Upon using (PACS2), we get
$\Gamma((\mathbb{F}\odot\mathbb{G}^{-1})\wedge([x]\odot[y]^{-1}))\leq\Gamma((\mathbb{F}\wedge[x])\odot(\mathbb{G}^{-1}\wedge[y]^{-1}))$
$=\Gamma\left((\mathbb{F}\wedge[x])\odot(\mathbb{G}\wedge[y])^{-1}\right)$ (by Lemma 2.2(vii))
$\leq\Gamma(\mathbb{F}\wedge[x])\vee\Gamma(\mathbb{G}\wedge[y])$ (by using condition (ACGM$\vee'$))
$=\lambda_\Gamma(\mathbb{F})(x)\vee\lambda_\Gamma(\mathbb{G})(y)$.
Consequently, we have
$\lambda_\Gamma(\mathbb{F}\odot\mathbb{G}^{-1})(xy^{-1})=\Gamma((\mathbb{F}\odot\mathbb{G}^{-1})\wedge[xy^{-1}])\leq\lambda_\Gamma(\mathbb{F})(x)\vee\lambda_\Gamma(\mathbb{G})(y)$. □

Proposition 5.6. *If* $f:\ (X,\cdot,\Gamma)\longrightarrow(Y,\cdot,\Gamma')$ *is a contractive group homomorphism between* $\vee$-*approach pre-Cauchy groups, then* $f:\ (X,\cdot,\lambda_\Gamma)\longrightarrow(Y,\cdot,\lambda_{\Gamma'})$ *is a contraction.*

We deduce from Propositions 5.5 and 5.6, the following

Corollary 5.7.

$$\mathfrak{T}:\begin{cases}\vee-\boldsymbol{PRACHYG} & \longrightarrow & \vee\vee-\boldsymbol{ALG}\\ (X,\cdot,\Gamma) & \longmapsto & (X,\cdot,\lambda_\Gamma)\\ f & \longmapsto & f\end{cases},$$

is a functor.

Now we are going to introduce a compatible finest $\boxtimes\boxplus$-approach uniform convergence structure which has for all $\mathbb{F}\in\mathbb{F}(X)$ the same left-Cauchy degree as $(X,\cdot,\lambda)$.

We define the following mapping $\Lambda_l:\mathbb{F}(X\times X)\longrightarrow[0,\infty]$ by

$$\Lambda_l(\Phi)=\bigwedge_{\Phi\geq(\mathbb{F}_1\times\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n\times\mathbb{F}_n)}\Gamma_l(\mathbb{F}_1)\boxplus\cdots\boxplus\Gamma_l(\mathbb{F}_n).$$

Theorem 5.8. *If* $(X, \cdot, \lambda) \in |\boxplus\boxplus - \boldsymbol{ALG}|$, *then* (X, Λ_l) *is a* $\boxplus\boxplus$*-approach uniform convergence space.*

We use the following results from e.g. [6, 7].

Lemma 5.9. *(i) Let* $\Phi, \Psi \in \mathbb{F}(X \times X)$, $\Phi \geq \Phi_1$, $\Psi \geq \Psi_1$ *and let* $\Phi \circ \Psi$ *exist. Then also* $\Phi_1 \circ \Psi_1$ *exists.*
(ii) If $(\bigwedge_{i\in I} \Phi_i) \circ (\bigwedge_{j\in J} \Psi_j)$ *exists, then there is* $(i, j) \in I \times J$ *such that* $\Phi_i \circ \Psi_j$ *exists.*

Proof of Theorem 5.8. (AUC1): We have $\Lambda_l([x] \times [x]) \leq \Gamma_l([x]) = 0$.
(AUC2) follows as $\Phi \geq (\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots \wedge (\mathbb{F}_n \times \mathbb{F}_n)$ implies $\Psi \geq (\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots \wedge (\mathbb{F}_n \times \mathbb{F}_n)$ for $\Phi \leq \Psi$.
(AUC3$\boxplus$) follows directly with Lemma 2.5.
(AUC4) follows from the fact that $\Phi \geq (\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots (\mathbb{F}_n \times \mathbb{F}_n)$ implies $\Phi^{-1} \geq ((\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots \wedge (\mathbb{F}_n \times \mathbb{F}_n))^{-1} = (\mathbb{F}_1 \times \mathbb{F}_1)^{-1} \wedge \cdots (\mathbb{F}_n \times \mathbb{F}_n)^{-1}$.
(AUC5$\boxplus$) In view of the preceding lemma, we conclude from $\Phi \geq (\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots (\mathbb{F}_n \times \mathbb{F}_n)$ and $\Psi \geq (\mathbb{G}_1 \times \mathbb{G}_1) \wedge \cdots (\mathbb{G}_m \times \mathbb{G}_m)$ and $\Phi \circ \Psi$ exists that $((\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots (\mathbb{F}_n \times \mathbb{F}_n)) \circ ((\mathbb{G}_1 \times \mathbb{G}_1) \wedge \cdots (\mathbb{G}_m \times \mathbb{G}_m))$ exists and is $= \bigwedge_{(i,j)\in K} (\mathbb{F}_i \times \mathbb{F}_i) \circ (\mathbb{G}_j \times \mathbb{G}_j)$, where K is the set of all (i, j) such that $(\mathbb{F}_i \times \mathbb{F}_i) \circ (\mathbb{G}_j \times \mathbb{G}_j)$ exists. According to results of W. Gähler [7] we have that, in case of existence, $(\mathbb{F}_i \times \mathbb{F}_i) \circ (\mathbb{G}_j \times \mathbb{G}_j) = \mathbb{F}_i \times \mathbb{G}_j$ and that $\mathbb{F}_i \vee \mathbb{G}_j$ exists. Hence in this case $\Gamma_l(\mathbb{F}_i \wedge \mathbb{G}_j) \leq \Gamma_l(\mathbb{F}_i) \boxplus \Gamma_l(\mathbb{G}_j)$. We conclude from this

$$\begin{aligned}
\Lambda_l(\Phi) \boxplus \Lambda_l(\Psi) &\geq \bigwedge_{\Phi\circ\Psi \geq \bigwedge_{(i,j)\in K} \mathbb{F}_i \times \mathbb{G}_j \geq \bigwedge_{(i,j)\in K} (\mathbb{F}_i \wedge \mathbb{G}_j) \times (\mathbb{F}_i \wedge \mathbb{G}_j)} \; \bigvee_{(i,j)\in K} \Gamma_l(\mathbb{F}_i \wedge \mathbb{G}_j) \\
&\geq \bigwedge_{\Phi\circ\Psi \geq (\mathbb{H}_1 \times \mathbb{H}_1) \wedge \cdots \wedge (\mathbb{H}_p \times \mathbb{H}_p)} \Gamma_l(\mathbb{H}_1) \boxplus \cdots \boxplus \Gamma_l(\mathbb{H}_p) \\
&= \Lambda_l(\Phi \circ \Psi).
\end{aligned}$$

□

For a $\boxtimes\boxplus$-approach uniform convergence space (X, Λ) we define, for $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$ the limit map by $\lambda_\Lambda(\mathbb{F})(x) = \Lambda(\mathbb{F} \times [x])$. It is straightforward to see that then (X, λ_Λ) is an $\boxplus$-approach limit space.

Lemma 5.10. *Let* $(X, \cdot, \lambda)$ *be a* $\vee\vee$*-approach limit group. Then* $\lambda = \lambda_{\Lambda_l}$.

Proof. Let $\mathbb{F} \in \mathbb{F}(X)$ and $x \in X$. If $\lambda(\mathbb{F})(x) \leq \delta$, then $\lambda(\mathbb{F} \wedge [x])(x) \leq \lambda(\mathbb{F})(x) \vee \lambda([x])(x) \leq \delta \vee 0 = \delta$. Hence, by (ALI), $\lambda((\mathbb{F} \wedge [x])^{-1})(x^{-1}) \leq \delta$ and with (ALM$\vee$) we conclude

$$\Gamma_l(\mathbb{F} \wedge [x]) = \lambda((\mathbb{F} \wedge [x]) \odot (\mathbb{F} \wedge [x])^{-1})(\underbrace{e}_{=xx^{-1}}) \leq \lambda(\mathbb{F} \wedge [x])(x) \vee \lambda((\mathbb{F} \wedge [x])^{-1})(x^{-1}) \leq \delta.$$

Hence $\lambda_{\Lambda_l}(\mathbb{F})(x) = \Lambda_l(\mathbb{F} \times [x]) \leq \Gamma_l(\mathbb{F} \wedge [x]) \leq \delta$.
Conversely, let $\lambda_{\Lambda_l}(\mathbb{F})(x) \leq \delta$. Then $\Lambda_l(\mathbb{F} \times [x]) \leq \delta$. It can be shown that

$\mathbb{F}\times[x] \geq (\mathbb{F}_1\times\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n\times\mathbb{F}_n)$ implies $[x^{-1}]\odot\mathbb{F} \geq (\mathbb{F}_1^{-1}\odot\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n^{-1}\odot\mathbb{F}_n)$ and hence

$$\lambda([x^{-1}] \odot \mathbb{F})(e) \leq \lambda(\mathbb{F}_1^{-1} \odot \mathbb{F}_1)(e) \vee \cdots \vee \lambda(\mathbb{F}_n^{-1} \odot \mathbb{F}_n)(e) = \Gamma_l(\mathbb{F}_1) \vee \cdots \vee \Gamma_l(\mathbb{F}_n).$$

This implies

$$\delta \geq \bigwedge_{\mathbb{F}\times[x]\geq(\mathbb{F}_1\times\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n\times\mathbb{F}_n)} \Gamma_l(\mathbb{F}_1) \vee \cdots \vee \Gamma_l(\mathbb{F}_n) = \lambda([x^{-1}] \odot \mathbb{F})(e) = \lambda(\mathbb{F})(x).$$

□

For a ⊠⊞-approach uniform convergence space (X, Λ) we define the Cauchy grade of a filter [17] by $\Gamma_\Lambda(\mathbb{F}) = \Lambda(\mathbb{F} \times \mathbb{F})$. Then (X, Γ_Λ) is a ⊞-approach Cauchy space (see also [13]).

Lemma 5.11. *Let $(X, \cdot, \lambda)$ be an ⊞⊞-approach limit group. Then $\Gamma_l = \Gamma_{\Lambda_l}$.*

Proof. Let $\Gamma_l(\mathbb{F}) \leq \delta$. Then $\Gamma_{\Lambda_l}(\mathbb{F}) = \Lambda_l(\mathbb{F} \times \mathbb{F}) \leq \Gamma_l(\mathbb{F}) \leq \delta$. Conversely, let $\Gamma_{\Lambda_l}(\mathbb{F}) \leq \delta$. Then $\Lambda_l(\mathbb{F} \times \mathbb{F}) \leq \delta$. It can be shown that $\mathbb{F}\times\mathbb{F} \geq (\mathbb{F}_1\times\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n\times\mathbb{F}_n)$ implies $\mathbb{F}^{-1}\odot\mathbb{F} \geq (\mathbb{F}_1^{-1}\odot\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n^{-1}\odot\mathbb{F}_n)$. Hence $\Gamma_l(\mathbb{F}) = \lambda(\mathbb{F}^{-1} \odot \mathbb{F})(e) \leq \lambda\left((\mathbb{F}_1^{-1} \odot \mathbb{F}_1) \wedge \cdots \wedge (\mathbb{F}_n^{-1} \odot \mathbb{F}_n)\right)$ $\leq \lambda(\mathbb{F}_1^{-1}\odot\mathbb{F}_1)(e)\boxplus\cdots\boxplus\lambda(\mathbb{F}_n^{-1}\odot\mathbb{F}_n)(e) = \Gamma_l(\mathbb{F}_1)\boxplus\cdots\boxplus\Gamma_l(\mathbb{F}_n)$ and we conclude $\delta \geq \Lambda_l(\mathbb{F} \times \mathbb{F}) \geq \Gamma_l(\mathbb{F})$. □

Lemma 5.12. *Let $(X, \cdot, \lambda) \in |\boxtimes\boxplus - \boldsymbol{ALG}|$ and let Λ be a ⊞⊞-approach uniform convergence structure on X with $\Gamma_\Lambda = \Gamma_l$. Then $\Lambda \leq \Lambda_l$.*

Proof. This follows from $\Gamma_l(\mathbb{F}) = \Lambda(\mathbb{F} \times \mathbb{F})$ and the axioms (AUC3⊞) and (AUC2). □

Lemma 5.13. *Let $(X, \lambda), (Y, \eta) \in |\boxplus\boxplus - \boldsymbol{ALG}|$ and let $f : X \longrightarrow Y$ be a group homomorphism. Then $f : (X, \lambda) \longrightarrow (Y, \eta)$ is a contraction if and only if $f : (X, \Lambda_l^\lambda) \longrightarrow (Y, \Lambda_l^\eta)$ is a uniform contraction.*

Proof. Let first $f : (X, \lambda) \longrightarrow (Y, \eta)$ be a contraction and let $\Lambda_l^\lambda(\Phi) \leq \delta$. If $\Phi \geq (\mathbb{F}_1 \times \mathbb{F}_1) \wedge \cdots \wedge (\mathbb{F}_n \times \mathbb{F}_n)$, then $(f \times f)(\Phi) \geq (f(\mathbb{F}_1) \times f(\mathbb{F}_1)) \wedge \cdots \wedge (f(\mathbb{F}_n) \times f(\mathbb{F}_n))$. Moreover, since ⊞ preserves order, we have by using Lemma 2.2(xi),

$$\begin{aligned} & \Gamma_l^\lambda(\mathbb{F}_1) \boxplus \cdots \boxplus \Gamma_l^\lambda(\mathbb{F}_n) \\ = \;& \lambda(\mathbb{F}_1^{-1} \odot \mathbb{F}_1)(e_X) \boxplus \cdots \boxplus \lambda(\mathbb{F}_n^{-1} \odot \mathbb{F}_n)(e_X) \\ \geq \;& \eta(f(\mathbb{F}_1^{-1} \odot \mathbb{F}_1))(f(e_X)) \boxplus \cdots \boxplus \eta(f(\mathbb{F}_n^{-1} \odot \mathbb{F}_n))(f(e_X)) \\ = \;& \eta(f(\mathbb{F}_1^{-1}) \odot f(\mathbb{F}_1))(e_Y) \boxplus \cdots \boxplus \eta(f(\mathbb{F}_n^{-1}) \odot f(\mathbb{F}_n))(e_Y) \\ = \;& \Gamma_l^\eta(f(\mathbb{F}_1)) \boxplus \cdots \boxplus \Gamma_l^\eta(f(\mathbb{F}_n)). \end{aligned}$$

Hence we conclude

$$\begin{aligned}
&\Lambda_l^\eta((f\times f)(\Phi))\\
\leq &\bigwedge_{(f\times f)(\Phi)\geq(f(\mathbb{F}_1)\times f(\mathbb{F}_1))\wedge\cdots\wedge(f(\mathbb{F}_n)\times f(\mathbb{F}_n))} \Gamma_l^\eta(f(\mathbb{F}_1))\boxplus\cdots\boxplus\Gamma_l^\eta(f(\mathbb{F}_n))\\
\leq &\bigwedge_{\Phi\geq(\mathbb{F}_1\times\mathbb{F}_1)\wedge\cdots\wedge(\mathbb{F}_n\times\mathbb{F}_n)} \Gamma_l^\lambda(\mathbb{F}_1)\boxplus\cdots\boxplus\Gamma_l^\lambda(\mathbb{F}_n) \;=\; \Lambda_l^\lambda(\Phi)
\end{aligned}$$

and $f:(X,\Lambda_l^\lambda)\longrightarrow(Y,\Lambda_l^\eta)$ is a uniform contraction.
Conversely, let $f:(X,\Lambda_l^\lambda)\longrightarrow(Y,\Lambda_l^\eta)$ be a uniform contraction. Then

$$\lambda(\mathbb{F})(x)=\lambda_{\Lambda_l^\lambda}(\mathbb{F})(x)=\Lambda_l^\lambda(\mathbb{F}\times[x])$$

$$\geq\Lambda_l^\eta(f(\mathbb{F})\times[f(x)])=\lambda_{\Lambda_l^\eta}(f(\mathbb{F}))(f(x))=\eta(f(\mathbb{F}))(f(x))$$

and $f:(X,\lambda)\longrightarrow(Y,\eta)$ is a contraction. □

In view of the Theorem 5.8 and Lemma 5.13, we have the following

Corollary 5.14.

$$\mathfrak{H}:\begin{cases}\boxplus\boxplus-\boldsymbol{ALG} & \longrightarrow & \boxplus\boxplus-\boldsymbol{AUCS}\\ (X,\cdot,\lambda) & \longmapsto & (X,\Lambda_l)\\ f & \longmapsto & f\end{cases},$$

is a functor.

In the rest of the section, we consider the mapping $\omega_l:X\times X\longrightarrow X,(x,y)\mapsto x^{-1}y$. Based on this mapping, we show that every $\boxtimes\boxplus$-approach limit group gives rise to a $\boxtimes\boxplus$-approach uniform convergence structure. Note that if $\boxplus=\boxtimes=+$, then with a slight modification, the Theorem 5.16 below coincides with Proposition 6.9 [20].

The following lemma follows by easy algebraic manipulations.

Lemma 5.15. *Let $(X,\cdot)$ be a group. Then for all $\Phi,\Psi\in\mathbb{F}(X\times X)$:*
(a) $\omega_l(\Phi\wedge\Psi)=\omega_l(\Phi)\wedge\omega_l(\Psi)$;
(b) $\omega_l(\Phi\times\Psi)=\Phi^{-1}\odot\Psi$;
(c) $\omega_l(\Phi^{-1})=(\omega_l(\Phi))^{-1}$;
(d) $\omega_l(\Phi)\odot\omega_l(\Psi)\leq\omega_l(\Phi\circ\Psi)$;
(e) $f(\omega_l(\Phi))=\omega_l((f\times f)(\Phi))$.

Let $(X,\cdot,\lambda)$ be a $\boxtimes\boxplus$-approach limit group. For $\Phi\in\mathbb{F}(X\times X)$ we define $\Lambda_l':\mathbb{F}(X\times X)\longrightarrow[0,\infty]$ by $\Lambda_l'(\Phi)=\lambda(\omega_l(\Phi))(e)$.

Theorem 5.16. *Let $(X,\cdot,\lambda)\in|\boxtimes\boxplus-ALG|$. Then (X,Λ_l') is a $\boxtimes\boxplus$-approach uniform convergence space.*

Proof. (AUC1) follows from $\omega_l([x]\times[x]) = [x]^{-1}\odot[x] = [e]$.
(AUC2) If $\Phi \le \Psi$ then $\omega_l(\Phi) \le \omega_l(\Psi)$ and hence $\Lambda_l'(\Phi) = \lambda(\omega_l(\Phi))(e) \ge \lambda(\omega_l(\Psi))(e) = \Lambda_l'(\Psi)$.
(AUC3⊞) $\Lambda_l'(\Phi\wedge\Psi) = \lambda(\omega_l(\Phi\wedge\Psi))(e) = \lambda(\omega_l(\Phi)\wedge\omega_l(\Psi))(e) \le \lambda(\omega_l(\Phi))(e)\boxplus\lambda(\omega_l(\Psi))(e) = \Lambda_l'(\Phi)\boxplus\Lambda_l'(\Psi)$.
(AUC4) $\Lambda_l'(\Phi^{-1}) = \lambda(\omega_l(\Phi^{-1}))(e) = \lambda(\omega_l((\Phi))^{-1})(e) \le \lambda(\omega_l(\Phi))(e) = \Lambda_l'(\Phi)$.
(AUC5⊠) $\Lambda_l'(\Phi\circ\Psi) = \lambda(\omega_l(\Phi\circ\Psi))(e) \le \lambda(\omega_l(\Phi)\odot\omega_l(\Psi))(e) \le \lambda(\omega_l(\Phi))(e)\boxtimes\lambda(\omega_l(\Psi))(e) = \Lambda_l'(\Phi)\boxtimes\Lambda_l'(\Psi)$. □

The following example shows that the preceding uniformization Λ_l' is quite natural.

Example 5.17. We consider a normed vector space $(X, \|\cdot\|)$ and the $\vee$-ALS-structure $\lambda = \lambda_{\|\,\|}$ as in Example 4.2. For the uniformization Λ_l' we obtain

$$\begin{aligned} & \Lambda_l'(\Phi) = \lambda(\omega_l(\Phi))(0) \le \epsilon \\ \iff\ & \omega_l(\Phi) \ge \mathbb{U}_\epsilon^0 \\ \iff\ & \forall\phi\in\Phi\exists\alpha>\epsilon : \omega_l(\phi)\subseteq B(0,\alpha) \\ \iff\ & \forall\phi\in\Phi\exists\alpha>\epsilon :\ y\in\omega_l(\phi)\Rightarrow \|y\|\le\alpha \\ \iff\ & \forall\phi\in\Phi\exists\alpha>\epsilon :\ (a,b)\in\phi\Rightarrow\|a-b\|\le\alpha \\ \iff\ & \forall\phi\in\Phi\exists\alpha>\epsilon :\ N_\alpha = \{(a,b)\ :\ \|a-b\|\le\alpha\}\subseteq\phi. \end{aligned}$$

Hence, if we define the ϵ-entourage filter $\mathbb{N}_\epsilon = [\{N_\alpha\ :\ \alpha>\epsilon\}]$ with $N_\alpha = \{(x,y)\ :\ \|x-y\|\le\alpha\}$, then we have $\Lambda_l'(\Phi)\le\epsilon \iff \Phi\ge\mathbb{N}_\epsilon$.

Lemma 5.18. *Let* $(X,\cdot,\lambda)\in|\boxtimes\boxplus-\boldsymbol{ALG}|$. *Then* $\lambda = \lambda_{\Lambda_l'}$.

Proof. Let $\mathbb{F}\in\mathbb{F}(X)$ and $x\in X$. Then
$\lambda_{\Lambda_l'}(\mathbb{F})(x) = \Lambda_l'(\mathbb{F}\times[x]) = \Lambda_l'([x]\times\mathbb{F}) = \lambda(\omega_l([x]\times\mathbb{F}))(e) = \lambda([x]\odot\mathbb{F})(e) = \lambda([x^{-1}]\odot\mathbb{F})(e) = \lambda(\mathbb{F})(x)$. □

Lemma 5.19. *Let* $(X,\cdot,\lambda)\in|\boxtimes\boxplus-\boldsymbol{ALG}|$. *Then* $\Gamma_l = \Gamma_{\Lambda_l'}$.

Proof. $\Gamma_{\Lambda_l'}(\mathbb{F}) = \Lambda_l'(\mathbb{F}\times\mathbb{F}) = \lambda_l(\mathbb{F}^{-1}\odot\mathbb{F})(e) = \Gamma_l(\mathbb{F})$. □

Corollary 5.20. *Let* $(X,\cdot,\lambda)\in|\boxtimes\boxplus-\boldsymbol{ALG}|$. *Then* $\Lambda_l'\le\Lambda_l$.

Lemma 5.21. *Let* $(X,\lambda),(Y,\eta)\in|\boxtimes\boxplus-\boldsymbol{ALG}|$ *and let* $f\colon X\longrightarrow Y$ *be a group homomorphism. Then* $f\colon(X,\lambda)\longrightarrow(Y,\eta)$ *is a contraction if and only if* $f:(X,\Lambda_l'^{\lambda})\longrightarrow(Y,\Lambda_l'^{\eta})$ *is a uniform contraction.*

Proof. Let $f:(X,\lambda)\longrightarrow(Y,\eta)$ be a contractive group homomorphism and $\Phi\in\mathbb{F}(X\times X)$. Then $\Lambda_l'^{\lambda}(\Phi) = \lambda(\omega_l(\Phi))(e_X)\ge\eta(f(\omega_l(\Phi)))(e_Y)$. Hence in view of Lemma 5.15(e), we have $\Lambda_l'^{\lambda}(\Phi)\ge\eta(\omega_l((f\times f)(\Phi)))(e_Y) = \Lambda_l'^{\eta}((f\times f)(\Phi))$.
To show the converse, assume that $f:(X,\Lambda_l'^{\lambda})\longrightarrow(Y,\Lambda_l'^{\eta})$ is a uniform contraction. Let $\mathbb{F}\in\mathbb{F}(X)$ and $x\in X$. Then
$\lambda(\mathbb{F})(x) = \lambda_{\Lambda_l'^{\lambda}}(\mathbb{F})(x) = \Lambda_l'^{\lambda}(\mathbb{F}\times[x]) \ge \Lambda_l'^{\eta}((f\times f)(\mathbb{F}\times[x])) = \Lambda_l'^{\eta}(f(\mathbb{F})\times[f(x)])$
$= \eta(\omega_l(f(\mathbb{F})\times[f(x)]))(e_Y) = \eta((f(\mathbb{F}))^{-1}\odot[f(x)])(e_Y) = \eta(f(\mathbb{F}))(f(x))$.
□

In view of the Theorem 5.16 and Lemma 5.21, we deduce the following

Corollary 5.22.

$$\mathfrak{V}:\begin{cases} \boxtimes\boxplus-\boldsymbol{ALG} & \longrightarrow & \boxtimes\boxplus-\boldsymbol{AUCS} \\ (X,\cdot,\lambda) & \longmapsto & (X,\Lambda_l') \\ f & \longmapsto & f \end{cases},$$

is a functor.

Let $\left(X,\cdot,\overline{p}=(p_\epsilon)_{\epsilon\in[0,\infty]}\right)$ be a $\boxtimes\boxplus$-limit tower group. Define for $\Phi\in\mathbb{F}(X\times X)$, $\Omega_l(\Phi)=\bigwedge\{\epsilon\in[0,\infty]:e\in p_\epsilon(\omega_l(\Phi))\}$.

Theorem 5.23. *If* $\left(X,\cdot,\overline{p}=(p_\epsilon)_{\epsilon\in[0,\infty]}\right)$ *is a* $\boxtimes\boxplus$*-limit tower group, then* (X,Ω_l) *is a* $\boxtimes\boxplus$*-approach uniform convergence space.*

Proof. Since $\Omega_l=(\Lambda_l')^{\lambda_{p_\epsilon}}$, the result follows from the Theorems 3.4 and 5.16. □

Proposition 5.24. *Let* $\left(X,\cdot,\overline{p}=(p_\epsilon)_{\epsilon\in[0,\infty]}\right)\in|\boxtimes\boxplus-\boldsymbol{LTG}|$*. Then* $p_\epsilon=p_{\epsilon\Omega_l}$.

Proof. We show for any $\mathbb{F}\in\mathbb{F}(X)$, $p_\epsilon(\mathbb{F})=p_{\epsilon\Omega_l^\epsilon}(\mathbb{F})$. Let $x\in p_{\epsilon\Omega_l^\epsilon}(\mathbb{F})$. Then $[x]\times\mathbb{F}\in\Omega_l^\epsilon$, implying $\Omega_l([x]\times\mathbb{F})\le\epsilon$. This yields that $\bigwedge\{\alpha\in[0,\infty]:e\in p_\alpha(\omega_l([x]\times\mathbb{F}))\}\le\epsilon$. This means that for all $\alpha>\epsilon, e\in p_\alpha(\omega_l([x]\times\mathbb{F}))$, which in turn gives that $e\in\bigcap_{\alpha>\epsilon}p_\alpha(\omega_l([x]\times\mathbb{F}))=p_\epsilon(\omega_l([x]\times\mathbb{F}))=p_\epsilon([x^{-1}]\odot\mathbb{F})$. It then follows from Lemma 3.2 that $x\in p_\epsilon(\mathbb{F})$. Converse follows similarly. □

Corollary 5.25. *Let* $\left(X,\cdot,\overline{p}=(p_\epsilon)_{\epsilon\in[0,\infty]}\right)\in|\boxtimes\boxplus-\boldsymbol{LTG}|$*. Then* $\left(X,\left(\Omega_l^{\overline{p},\epsilon}\right)_{\epsilon\in[0,\infty]}\right)$ *is a* $\boxtimes\boxplus$*-uniform convergence tower space, where* $\Omega_l^{\overline{p},\epsilon}=\{\Phi\in\mathbb{F}(X\times X)\mid\Omega_l^{\overline{p},\epsilon}(\Phi)\le\epsilon\}$.

Proposition 5.26. *Let* $(X,\overline{p}),(Y,\overline{q})\in|\boxtimes\boxplus-\boldsymbol{LTG}|$ *and* $f:X\to Y$ *a group homomorphism. Then* $f:(X,\overline{p})\to(Y,\overline{q})$ *is continuous if and only if* $f:(X,\Omega_l)\to(Y,\Omega_l)$ *is a uniform contraction.*

Proof. Since $\Omega_l=\Lambda_l^{\lambda_{p_\epsilon}}$, the results follows from Lemma 5.21. □

We now state the following functorial relation between the category of $\boxtimes\boxplus$-limit tower groups and the category of $\boxtimes\boxplus$-approach uniform convergence space which follows from the Theorem 5.21 and Proposition 5.23.

Corollary 5.27.

$$\mathfrak{U}:\begin{cases} \boxtimes\boxplus-\boldsymbol{LTG} & \longrightarrow & \boxtimes\boxplus-\boldsymbol{AUCS} \\ \left(X,\cdot,\overline{p}=(p_\epsilon)_{\epsilon\in[0,\infty]}\right) & \longmapsto & (X,\Omega_l) \\ f & \longmapsto & f \end{cases},$$

is a functor.

In view of proposition 5.24, Corollary 5.25, and the definitions of continuity between ⊞-limit tower spaces, we have the following

Corollary 5.28.

$$\mathfrak{W}:\left\{\begin{array}{ccc} \boxtimes\boxplus-\boldsymbol{LTG} & \longrightarrow & \boxtimes\boxplus-\boldsymbol{UCTS} \\ \Big(X,\cdot,\overline{p}=(p_\epsilon)_{\epsilon\in[0,\infty]}\Big) & \longmapsto & \Big(X,\big(\Omega_l^{\overline{p},\epsilon}\big)_{\epsilon\in[0,\infty]}\Big) \\ f & \longmapsto & f \end{array}\right. ,$$

is a functor.

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Received: February 15, 2014