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Möbius transformations with *n*-cycles geometric viewpoint

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Abstract. Let $\mathbb{M}$ be the group of Möbius transformations on $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and $\langle f \rangle_{\mathbb{M}} = \{f^n;\ n \in \mathbb{Z}\}$ the cyclic subgroup of $\mathbb{M}$ generated by f, for $f \in \mathbb{M}$. If $\langle f \rangle_{\mathbb{M}}$ is finite of order n, f is called an n-cycle. We prove in the first part that if f is an n-cycle, then for any $\alpha \in \mathbb{C}_\infty$, the set $\{f^n(\alpha);\ n \in \mathbb{Z}\} = \mathcal{O}f(\alpha)$ lies on a circle. Furthermore we characterize with geometric arguments the circles which are invariant under this kind of transformations.

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1. INTRODUCTION

The most well-known characterization of Möbius transformations is that they are the only non-constant meromorphic functions on the complex plane $\mathbb{C}$ which map circles onto circles, including straight lines among circles. Furthermore, Möbius transformations have many important properties. For example: a map is a Möbius transformation if and only if it preserves cross ratios, (cf [3]). Carathédory [8] proved that any injective mapping of a domain Ω of $\mathbb{C}$ to $\mathbb{C}$ is the restriction of a Möbius transformation if the image of any circle contained with its interior in Ω is itself a circle. In [9], Haruki and Rassias gave a new characterization of Möbius transformations by using Apollonius quadrilaterals. They proved that if f is meromorphic and if f sends Apollonius quadrilaterals to Apollonius quadrilaterals, then f is Möbius. On the other hand, Möbius transformations are closely related to hyperbolic geometry since they act as isometries on the hyperbolic space.

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We denote by $\mathbb{M}$ the set of Möbius transformations and for any $f \in \mathbb{M}$, we set $\langle f \rangle_{\mathbb{M}} = \{f^n;\ n \in \mathbb{Z}\}$, the cyclic subgroup of $\mathbb{M}$ generated by f ($f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$ is called the kth iterate of f). If $\langle f \rangle_{\mathbb{M}}$ is finite of order n, f is called an n-cycle.

In this paper, we study this kind of Möbius transformation and we characterize the set of circles which are invariant under f. We prove the following theorems:

Theorem 1.1. *Let f be an n-cycle and $\alpha \in \mathbb{C}_\infty$, then the set $\mathcal{O}f(\alpha) = \{f^n(\alpha);\ n \in \mathbb{Z}\}$ lies on the same circle (the same line if $\infty \in \mathcal{O}f(\alpha)$ and $\mathcal{O}f(\alpha)$ lie on the same circle if not). Otherwise if $n \geqslant 4$, for any $z \in \mathbb{C}_\infty$, the cross-ratio $[z, f(z), f^2(z), f^3(z)]$ is a constant which depends on n and not on f, for all n-cycles f.*

Theorem 1.2. *Let f be a 2-cycle such that ∞ is not a fixed point. Let $\alpha \in \mathbb{C}$ such that $f(\alpha) = \infty$. Let a be a fixed point of f, $\mathcal{C}$ the circle of center α and passing through a, L_1 the straight line (a, α), L_2 the straight line orthogonal to L_1 and passing through α, and assume b is symmetric to a with respect to L_1. Then the only invariant lines under f are L_1, L_2, and the only circles invariant under f are:*

(1) *The circles which pass through a and b.*
(2) *The circles which are symmetric with respect to L_1 and orthogonal to $\mathcal{C}$.*

Theorem 1.3. *Let f be a 3-cycle such that ∞ is not a fixed point. Then there exists only one straight line L invariant under f, and L contains the orbit of ∞. Furthermore the set of circles invariant under f form a pencil of coaxial axis L'. L is the radical axis of these invariant circles and L' contains the fixed points of f.*

Theorem 1.4. *Let f be an n-cycle, $n \geqslant 3$, such that ∞ is not a fixed point. Then there exists only one straight line L invariant with respect to f, and L contains the orbit of ∞. Furthermore the set of circles invariant with respect to f form a pencil of coaxial circles. L is the radical axis of these invariant circles.*

Remark. The results in this paper are proved using geometric arguments. Some of them can be proved using conjugate transformations and the invariance of the results under conjugation.

2. Generalities on Möbius transformations

Möbius transformations are the automorphisms of the extended complex plane $\mathbb{C}_\infty$. Any Möbius transformation f has the form

$$f(z) = \frac{az + b}{cz + d}, \tag{1}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Möbius transformations with $c = 0$ form the subgroup of similarities. Such transformations have the form $f(z) = az + b$, $a, b \in \mathbb{C}$ and $a \neq 0$. The transformation

$I(z) = \frac{1}{z}$ is called an inversion. It is well known that every Möbius transformation f of the form (1) is a composition of finitely many similarities and inversions.

The group $\mathbb{M}$ has the following famous properties: (see for example [1,3–7,10–12]).

Properties 2.1.

(1) *Any Möbius transformation has at most 2 fixed points in* $\mathbb{C}_\infty$.

(2) *Let f be a Möbius transformation which has two fixed points a, b. Then there exists* $q \in \mathbb{C}$ *such that*

$$\frac{f(z) - z_1}{f(z) - z_2} = q\frac{z - z_1}{z - z_2}.$$

$q = f'(z_1) = \frac{1}{f'(z_2)}$. *It follows that for all* $n \in \mathbb{N}$,

$$\frac{f^n(z) - z_1}{f^n(z) - z_2} = q^n \frac{z - z_1}{z - z_2}.$$

Then

$$f^n(z) = \frac{z(z_1 - q^n z_2) - (1 - q^n)z_1 z_2}{(1 - q^n)z - z_2 + q^n z_1}.$$

(3) *The image of a circle under a Möbius transformation is a circle. In what follows, we call any circle or straight line a circle in the complex plane.*

(4) *If* α *and* β *are symmetric with respect to circle* $\mathcal{C}$, *then for any Möbius transformation* f, $f(\alpha)$ *and* $f(\beta)$ *are symmetric with respect to* $f(\mathcal{C})$.

(5) *There is always only one Möbius transformation that maps three given distinct points* z_1, z_2, z_3 *onto three given distinct points* w_1, w_2, w_3, *respectively. This function is denoted by* $w(z) = [z,z_1,z_2,z_3]$. *The number* $[z,z_1,z_2,z_3]$ *is called the cross-ratio of the four complex numbers* z, z_1, z_2, z_3.

(6) *The cross-ratio* $[z_1,z_2,z_3,z_4]$ *of any four complex numbers is invariant under Möbius transformations;(i.e.* $[z_1,z_2,z_3,z_4] = [h(z_1),h(z_2),h(z_3),h(z_4)]$, *for any Möbius transformation* h*).*

(7) *If* $\mathcal{C}$ *is a circle and* $\mathcal{S}$ *is the symmetry with respect to* $\mathcal{C}$, *then the mappings* $z \mapsto \mathcal{S}(\bar{z})$ *and* $z \mapsto \overline{\mathcal{S}(z)}$ *are Möbius transformations.*

(8) *If* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are two circles and* $\mathcal{S}_1$ *(resp* $\mathcal{S}_2$*), is the symmetry with respect to* $\mathcal{C}_1$ *(resp with respect to* $\mathcal{C}_2$*), then* $\mathcal{S}_1 \circ \mathcal{S}_2$ *is a Möbius transformation.*

(9) *Let f be a Möbius transformation,* $f(z) = \frac{az+b}{cz+d}$. *We can suppose that* $ad - bc = 1$, *and we associate with f the matrix* $M_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *This matrix is in* $SL(2,\mathbb{C})$, *the special linear group of* $\mathbb{C}^2$. *Inversely if* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $\mathrm{SL}(2,\mathbb{C})$, *we associate the Möbius transformation* $f(z) = \frac{az+b}{cz+d}$ *and the matrix* $-M_f$ *gives the same Möbius transformation. Thus we can identify the group of Möbius transformations with the projective special linear group* $PSL(2,\mathbb{C})$, *the group of* 2×2 *matrices with complex coefficients, determinant* $= 1$, *modulo the equivalence relation* $A \sim -A$.

3. Properties of the N cycles

Proposition 3.1.

(1) • *A Möbius transformation f is an n-cycle, $n \geqslant 2$, if and only if there exists $a \in \mathbb{C}_\infty$ such that $f^n(a) = a$ and $f^m(a) \neq a$ for all $1 \leqslant m \leqslant n-1$, which is equivalent that the cardinal of $\mathcal{O}f(a)$ is equal to n.*
• *For $n \geqslant 3$, any n-cycle f verify that $f^n = Id$ and $f^m \neq Id$ for all $1 \leqslant m \leqslant n-1$.*
• *If f is an n-cycle and $a \in \mathbb{C}_\infty$ which is not invariant under f, then the cardinal of $\mathcal{O}f(a)$ is equal to n.*

(2) *Let f be an n-cycle, $n \geqslant 3$, such that ∞ is not a fixed point, then: If $\mathcal{F}$ is a circle invariant under f, then $\mathcal{F}$ does not contain any fixed point of f. Then if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different circles invariant under f, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ and there exists only one straight line invariant under f. This line contains the orbit of ∞.*

Proof

(1) Let a, b be the fixed points of f and let $w \neq a$, $w \neq b$.We assume that the cardinal of the orbit of w is equal to p, with $2 \leqslant p \leqslant n-1$. The orbit of w is equal to $\{w, f(w), \ldots, f^{p-1}(w)\}$ and $f^p(w) = w$. It follows that $f^j(w)$ are fixed points of f^p for all $1 \leqslant j \leqslant p-1$. Then $f^p = I$ which is a contradiction.
It follows that $p = n$.We assume that the cardinal of the orbit of a is equal to p, with $2 \leqslant p \leqslant n-1$. We remark that if $f^j(a) = f^{j+1}(a)$ for $1 \leqslant j \leqslant p-1$, then $f(a) = a$ which is in contradiction to the assumption. It follows that the orbit of a is equal to $\{a, f(a), \ldots, f^{p-1}(a)\}$ and $f^p(a) = a$. It follows that $f^j(a)$ are fixed points of f^p and then $f^p = I$ which is a contradiction.

(2) The number of connected components of $\mathcal{F} \setminus \{\alpha_1, \ldots, \alpha_n\}$ is n. f is univalent, and $\mathcal{F}$ is invariant under f, therefore the image of any connected component of $\mathcal{F} \setminus \{\alpha_1, \ldots, \alpha_n\}$ is a connected component, therefore f cannot have an invariant point on $\mathcal{F}$. $\square$

Lemma 3.2. *Let f be a Möbius transformation and A be the matrix of f. If $z \in \mathbb{C}_\infty$ and z, $f(z)$, $f^2(z)$ and $f^3(z)$ are different, then the cross ratio $[z, f(z), f^2(z), f^3(z)]$ is constant and we have the following:*

• *If A has two different eigenvalues, then*

$$[z, f(z), f^2(z), f^3(z)] = 1 + \frac{\det A}{(\operatorname{tr} A)^2 - \det A}.$$

• *If A has only one eigenvalue, then*

$$[z, f(z), f^2(z), f^3(z)] = \frac{4}{3}.$$

Proof

• If A has two different eigenvalues α and β, then there exists a Möbius transformation g, such that $f = g^{-1} \circ h \circ g$, with $h(z) = \frac{\alpha}{\beta} z$. Since the cross ratio is invariant by Möbius transformation, we have

$$\begin{aligned}[z,f(z),f^2(z),f^3(z)] &= [z, g^{-1}\circ h\circ g(z), g^{-1}\circ h^2\circ g(z), g^{-1}\circ h^3\circ g(z)]\\ &= [w, h(w), h^2(w), h^3(w)] = \frac{\beta^2-\alpha^2}{\beta^3-\alpha^3}\cdot\frac{\beta^2-\alpha^2}{\beta-\alpha}\\ &= 1+\frac{\alpha\beta}{(\alpha+\beta)^2-\alpha\beta} = 1+\frac{\det A}{(\mathrm{tr}A)^2-\det A}\end{aligned}$$

with $w = g(z)$.

- If A has only one eigenvalue α, then there exists a Möbius transformation g, such that $f = g^{-1}\circ h\circ g$, with $h(z) = z+\frac{\beta}{\alpha}$. From the assumption $\beta \neq 0$, the result is easily deduced. □

Corollary 3.3. *Let f and g be two Möbius transformations such that their matrices are equivalent, then $[z,f(z),f^2(z),f^3(z)] = [z,g(z),g^2(z),g^3(z)]$, for all $z\in\mathbb{C}_\infty$ such that z, $f(z)$, $f^2(z)$ and $f^3(z)$ are different.*

Theorem 3.4. *Let f be an n-cycle, $n \geqslant 3$, and let $z\in\mathbb{C}_\infty$, then the set $\mathcal{O}f(z) = \{f^n(z);\ n\in\mathbb{Z}\}$ lies on the same straight line if $\infty\in\mathcal{O}f(z)$ and $\mathcal{O}f(z)$ lies on the same circle if not. Otherwise if $n\geqslant 4$, for any $z\in\mathbb{C}_\infty$, the cross-ratio $[z,f(z), f^2(z),f^3(z)]$ is constant which depends on n and not on f, for all n-cycle f.*

Proof. The result is obvious if $n = 3$, so we take $n \geqslant 4$. To prove the result it suffices to prove that $[z,f(z),f^2(z),f^3(z)]$ is real. We associate with f a matrix $A\in \mathrm{SL}(2,\mathbb{C})$ such that $A^n = \mathrm{Id}$. Then A is diagonalizable. There exist $P, D\in \mathrm{GL}(2,\mathbb{C})$ such that $A = P^{-1}DP$, with $D = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$, $\alpha,\beta\in\mathbb{C}$ and $w^n = \frac{\alpha^n}{\beta^n} = 1$. Then from Lemma 3.2

$$\begin{aligned}[z,f(z),f^2(z),f^3(z)] &= \frac{(1+w)^2}{1+w+w^2} = 1+\frac{w}{1+w+w^2} = 1+\frac{1}{1+w+\bar{w}}\\ &= 1+\frac{1}{1+2\cos\left(\frac{2\pi}{n}\right)},\end{aligned}$$

which is real, and independent of f, $[z,f(z),f^2(z),f^3(z)] = 2$ if $n = 4$. □

Proposition 3.5. *Let f be an n-cycle, $n \geqslant 3$. We assume that ∞ is not a fixed point. We denote by:*

- *a, b the fixed points of f,*
- $\{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\infty\}$ the orbit of ∞,
- *$\mathcal{C}_k$ the circle of center α_k and passing through the fixed points a, b*
- *$\mathcal{S}_k$ the symmetry with respect to circle $\mathcal{C}_k$, with $0\leqslant k\leqslant n-1$.*

(1) $f = \mathcal{S}_{k+1}\circ\mathcal{S}_k$.

(2) *Any circle invariant under f contains in its interior a fixed point of f.*

(If F is the circle which contains the orbit of ∞, then the circle $\mathcal{C}_k$ of center α_k and passing through the fixed point a passes through the fixed point b because a and b are symmetric with respect to $\mathcal{F}$. Indeed it suffices to remark that $\mathcal{S} \circ f \circ \mathcal{S}$ and f are equal, where $\mathcal{S}$ is the symmetry with respect to $\mathcal{F}$.)

Proof

(1) $\mathcal{S}_{k+1} \circ \mathcal{S}_k$ is a Möbius transform and a, b are fixed points. Moreover $\mathcal{S}_{k+1} \circ \mathcal{S}_k(\alpha_k) = \mathcal{S}_{k+1}(\infty) = \alpha_{k+1} = f(\alpha_k)$. Then $f = \mathcal{S}_{k+1} \circ \mathcal{S}_k$.

(2) Let $\mathcal{C}$ be a circle invariant under f. Since the straight line which contain the orbit of ∞ does not cut $\mathcal{C}$, then the closed disc $\mathcal{D}$ with boundary $\mathcal{C}$ is a convex compact subset and $f : \mathcal{D} \to \mathcal{D}$ is continuous. Then from Brouwer's theorem, f has a fixed point in $\mathcal{D}$. The fixed point cannot be on $\mathcal{C}$, then this fixed point in an interior point of $\mathcal{D}$. $\square$

4. Construction of the invariant circles of a 2-cycle

We refer the reader to the paper [2] for some results on 2-cycles.

In this section, f is a 2-cycle such that ∞ is not a fixed point. Let $\alpha \in \mathbb{C}$ such that $f(\alpha) = \infty$. Let a be a fixed point of f, L_1 the straight line (a,α) and L_2 the straight line orthogonal to L_1 and passing through α. (cf Fig. 1) It is obvious that L_1 is invariant under f. Moreover the image of L_2 by f is a straight line passing through α and orthogonal to L_1, then L_2 is invariant under f. Hence L_1 and L_2 are invariant under f.

Proposition 4.1. *Let $\mathcal{S}_1$ be the symmetry with respect to straight line L_1, $\mathcal{S}_2$ the symmetry with respect to straight line L_2 and $b = \mathcal{S}_2(a)$. We denote by $\mathcal{C}$ the circle with center α and passing through a, and $\mathcal{S}$ the symmetry with respect to $\mathcal{C}$. Then*

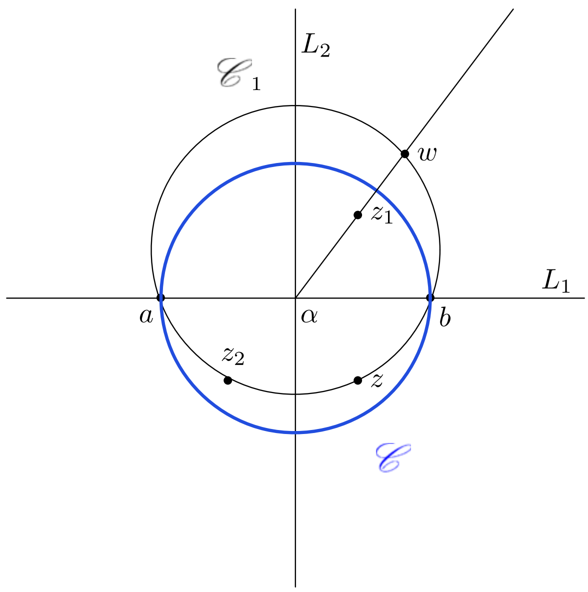

Figure 1 The circles which pass through a and b are invariant.

- *b is a fixed point of f.*
- *Circle $\mathcal{C}$ is invariant under f.*
- *$f = \mathcal{S} \circ \mathcal{S}_1 = \mathcal{S}_1 \circ \mathcal{S}$.*

Proof

- Since f is a Möbius transformation, then $f(b) = f(\mathcal{S}_2(a)) = \mathcal{S}_2(f(a)) = \mathcal{S}_2(a) = b$.
- The image by f of circle $\mathcal{C}$ passes through a and b and it is orthogonal to L_2. Then it is invariant under f.
- $\mathcal{S} \circ \mathcal{S}_1$ and $\mathcal{S}_1 \circ \mathcal{S}$ are Möbius transformations and a and b are fixed points for these two transformations. $\mathcal{S} \circ \mathcal{S}_1(\alpha) = \infty = f(\alpha)$ and $\mathcal{S}_1 \circ \mathcal{S}(\alpha) = \infty = f(\alpha)$. Then $f = \mathcal{S} \circ \mathcal{S}_1 = \mathcal{S}_1 \circ \mathcal{S}$. $\square$

Our goal now is to characterize all circles which are invariant under f.

Theorem 4.2. *Let $\mathbb{O}$ be a circle invariant under f and different from $\mathcal{C}$, L_1 and L_2.*

(1) *$\mathbb{O}$ is a circle and $\mathbb{O} \bigcap \mathcal{C} \neq \emptyset$.*
(2) *Circle $\mathbb{O}$ passes through a and b, or the center of circle $\mathbb{O}$ lies on L_1.*
(3) *If the center of circle $\mathbb{O}$ lies on L_1, then $\mathbb{O}$ is orthogonal to $\mathcal{C}$.*

Proof. Let $D = D(\alpha, |\alpha - a|)$ be the disc of center α and of radius $|\alpha - a|$.

(1) If $\mathbb{O}$ is a line, then $\alpha \in \mathbb{O}$ and $\mathbb{O}$ intersects $\mathcal{C}$ in two points, A and B. Then A, B are fixed points and then $\mathbb{O} = L_1$ or $f(A) = B$ and $f(B) = A$ and then $\mathbb{O} = L_2$. It follows that $\mathbb{O}$ is a circle.Since $f(\alpha) = \infty$ and $\mathcal{C}$ is invariant under f, then if $\mathbb{O} \subset \overline{D}^c$, then $f(\mathbb{O}) \subset D$ and if $\mathbb{O} \subset D$, then $f(\mathbb{O}) \subset \overline{D}^c$. This contradicts that $\mathbb{O}$ is invariant under f. It follows that $\mathbb{O} \bigcap \mathcal{C} \neq \emptyset$.
(2) Let $\{z_1, z_2\} = \mathbb{O} \cap \mathcal{C}$. Since $f = \mathcal{S} \circ \mathcal{S}_1 = \mathcal{S}_1 \circ \mathcal{S}$, then z_1, z_2 are the fixed points of f or $z_1 = \mathcal{S}_1(z_2)$. If $z_1 = \mathcal{S}_1(z_2)$, the center of circle $\mathbb{O}$ lies on L_1 and $\mathbb{O}$ is symmetric with respect to L_1.
(3) If the center of $\mathbb{O}$ is on L_1, we suppose that a, b are not on $\mathbb{O}$. Let $\{z, w\} = \mathbb{O} \cap L_1$. Since $f(\mathbb{O}) = \mathbb{O}$ and $f(L_1) = L_1$, then $f(z) = w$.The power of α with respect to $\mathbb{O}$ is $p(\alpha, \mathbb{O}) = |z - \alpha||\alpha - w|$. Furthermore $f = \mathcal{S} \circ \mathcal{S}_1$, then w is symmetric to z with respect to $\mathcal{C}$, then $|z - \alpha| \; |\alpha - w| = |\alpha - a|^2$. Then the two circles $\mathcal{C}$ and $\mathbb{O}$ are orthogonal. $\square$

Corollary 4.3. *The only lines invariant under f are L_1 and L_2, and the only circles invariant under f are:*

(1) *The circles which pass through a and b.*
(2) *The circles symmetric with respect to L_1 and orthogonal to $\mathcal{C}$.*

Proof

(1) If $\mathcal{C}_1$ is a circle passing through a and b, then the center of the circle lies on L_2, (cf Fig. 1). To prove that this circle is invariant under f, it is sufficient to prove that for all z on $\mathcal{C}_1$, $w = \mathcal{S} \circ \mathcal{S}_1(z)$ lies on $\mathcal{C}_1$.In this situation, L_1 is the radical axis of $\mathcal{C}$ and $\mathcal{C}_1$, then $p(\alpha, \mathcal{C}) = p(\alpha, \mathcal{C}_1)$, where $p(\alpha, \mathcal{C})$ (resp $p(\alpha, \mathcal{C}_1)$) is the power of α with respect to $\mathcal{C}$ (resp with respect to $\mathcal{C}_1$).Let z_1 be symmetric to z with respect to L_1, and let z_2 be symmetric to z with respect to L_2. Since $w = \mathcal{S} \circ \mathcal{S}_1(z)$, we have $(\overline{\alpha - z_1})(\alpha - w) = |\alpha - a|^2$. But $(\alpha - z_1) = -(\alpha - z_2)$, then $(\overline{\alpha - z_2})(\alpha - w) = -|\alpha - a|^2 = p(\alpha, \mathcal{C}) = p(\alpha, \mathcal{C}_1)$. It follows that $w \in \mathcal{C}_1$ and then $\mathcal{C}_1$ is invariant under f.

(2) Let $\mathcal{C}_2$ be a circle symmetric with respect to L_1 and orthogonal to $\mathcal{C}$. Let $\{z, w\} = L_1 \cap \mathcal{C}_2$, (cf Fig. 2). Then $\mathcal{C}_2$ is the circle of diameter $[z, w]$, since $\mathcal{C}$ and $\mathcal{C}_2$ are orthogonal, then $w = \mathcal{S}(z)$, therefore $w \in \mathcal{C}_2$. To prove that circle $\mathcal{C}_2$ is invariant under f, it is sufficient to prove that for all z_1 on this circle; $z_2 = \mathcal{S}(z_1) \in \mathcal{C}_2$. Moreover, $\mathcal{C}$ and $\mathcal{C}_2$ are orthogonal, then $p(\alpha, \mathcal{C}_2) = |\alpha - z_1|\ |\alpha - z_2| = |\alpha - a|^2$, and therefore $z_2 \in \mathcal{C}_2$.Let now $\mathcal{C}_3$ be a circle with center on L_1, and invariant under f. Let $\{z, w\} = L_1 \cap \mathcal{C}_2$, then using the same argument, $\mathcal{C}_3$ is the circle of diameter $[z,w]$ and $f(z) = w$. Thus $f(z) = w = \mathcal{S}(z)$, therefor $\mathcal{C}$ and $\mathcal{C}_3$ are orthogonal. □

From the previous construction, we can deduce a construction of the invariant circles under a Möbius transformation with a 4-cycle. We have the following result:

Let f be a 4-cycle such that ∞ is not a fixed point of f, and let α, β, γ in $\mathbb{C}$ be such that

$$f(\alpha) = \beta, \quad f(\beta) = \gamma, \quad f(\gamma) = \infty, \quad f(\infty) = \alpha. \tag{2}$$

Let L_2 be the straight line orthogonal to L_1 and passing through β, and $\mathcal{C}$ be the circle with diameter (α, γ).

Proposition 4.4.

(1) *β is the mid point of $[\alpha,\gamma]$ or $[\gamma,\alpha]$.*
(2) *The intersection of $\mathcal{C} \cap L_2$ are the fixed points of f.*

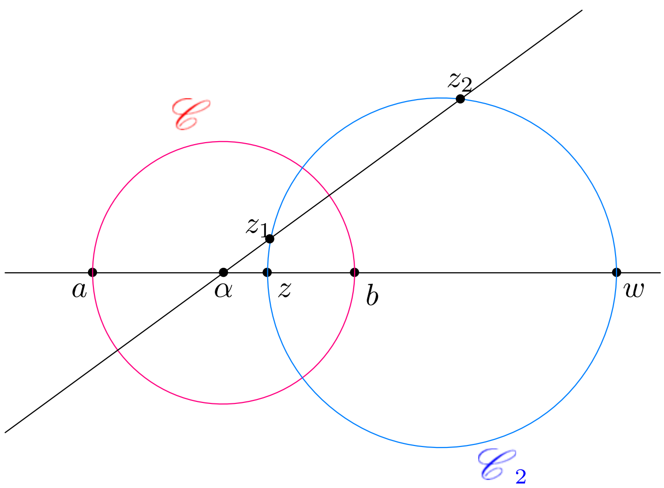

Figure 2 The circles symmetric to L_1 and orthogonal to $\mathcal{C}$ are invariant.

(3) *$f = \mathcal{S} \circ \mathcal{S}_1$, where $\mathcal{S}$ is the symmetry with respect to circle $\mathcal{C}$, and $\mathcal{S}_1$ is the symmetry with respect to circle $\mathcal{C}_1$ of center α and passing through the fixed points a, b of f.*

Proof

(1) Since the cross-ratio is invariant under any Möbius transformation, then $[\alpha,\beta,\gamma,\infty] = [\beta,\gamma,\infty,\alpha]$, which is equivalent to $\frac{\alpha-\gamma}{\alpha-\beta} = \frac{\alpha-\gamma}{\beta-\gamma}$, hence $\beta = \frac{\alpha+\gamma}{2}$, and then β is the mid point of $[\alpha,\gamma]$ or $[\gamma,\alpha]$.
(2) It is straight forward to prove that $f(L_1) = L_1$, $f(\mathcal{C}) = L_2$ and $f(L_2) = \mathcal{C}$. Then the two points a, b of $\mathcal{C} \cap L_2$ are the fixed points of f.
(3) $\mathcal{S} \circ \mathcal{S}_1(\alpha) = \beta = f(\alpha)$ and a, b are the fixed points of f, and $\mathcal{S} \circ \mathcal{S}_1$ is a Möbius transformation, then $f = \mathcal{S} \circ \mathcal{S}_1$. $\square$

5. Construction of the invariant circles and lines by a Möbius transformation with a 3-cycle

Let f be a 3-cycle such that ∞ is not a fixed point. We denote by L the straight line which contain the orbit of ∞, then there exist two different complex numbers α, β such that

$$f(\alpha) = \beta, \quad f(\beta) = \infty, \quad f(\infty) = \alpha. \tag{3}$$

Then $L = (\alpha, \beta)$.

$$f(z) = \frac{\alpha z - (\alpha^2 + \beta^2 - \alpha\beta)}{z - \beta}. \tag{4}$$

Let $\mathcal{C}_1$ be the circle of center β and passing through $\alpha, \mathcal{C}_2$ be the circle of center α and passing through β, a and b the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2, L_1$ the line (α, β), and L_2 the line passing through a and b (see Fig. 3).

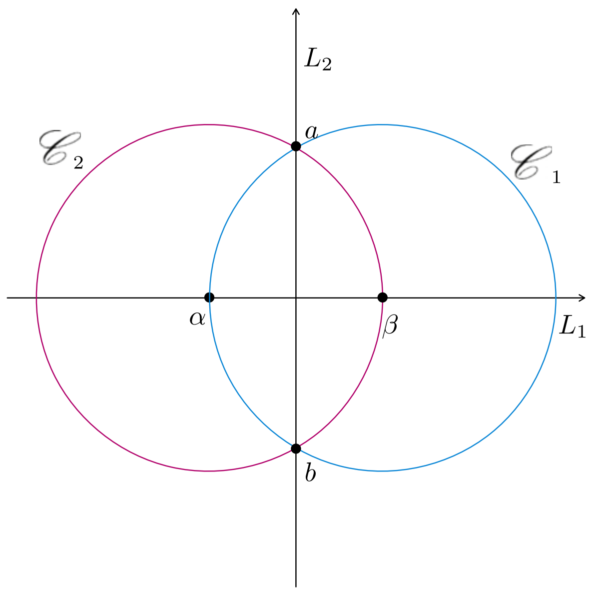

Figure 3 The two fixed points of a 3-cycle.

Proposition 5.1.

(1) *$f(L_1) = L_1$.*
(2) $f(\mathcal{C}_1) = \mathcal{C}_2,\ f(\mathcal{C}_2) = L_2$ and $f(L_2) = \mathcal{C}_1$.
(3) *a, b are the two fixed points of f.*

Proof

(1) Since $f(\alpha) = \beta, f(\beta) = \infty$ and $f(\infty) = \alpha$, then $f(L_1) = L_1$.
(2) $f(\mathcal{C}_1)$ is a circle passing through β. β and ∞ are symmetric with respect to $\mathcal{C}_1$, so $f(\beta) = \infty$ and $f(\infty) = \alpha$ are symmetric with respect to $f(\mathcal{C}_1)$, hence $f(\mathcal{C}_1) = \mathcal{C}_2.\mathcal{C}_2$ is a circle orthogonal to L_1 and $f(\mathcal{C}_2)$ is a straight line, so $f(\mathcal{C}_2)$ is a straight line orthogonal to L_1. α and ∞ are symmetric with respect to $\mathcal{C}_2$, so $f(\alpha) = \beta$ and $f(\infty) = \alpha$ are symmetric with respect to $f(\mathcal{C}_2)$, hence $f(\mathcal{C}_2) = L_2$. L_2 is the bisector of $[\alpha, \beta]$ and passes through a and b.The pole of f is not on L_2, therefore $f(L_2)$ is a circle. Since α and β are symmetric with respect to L_2, so $f(\alpha) = \beta$ and $f(\beta) = \infty$ are symmetric with respect to $f(\mathcal{C}_2)$, hence $f(L_2) = \mathcal{C}_1$.
(3) a and b are symmetric with respect to L_1, and $\{a, b\} = \mathcal{C}_1 \bigcap \mathcal{C}_2 \bigcap L_2$, so $f(a) = a$ or $f(a) = b$, but $f^3 = \mathrm{Id}$, therefore $f(a) = a$ and $f(b) = b$. Hence a and b are the two fixed points of f. $\square$

Theorem 5.2. *Let f be a 3-cycle such that ∞ is not a fixed point, then the set of invariant circles by f which are different from L_1 form a pencil of coaxial circles. L_1 is the radical axis of each two of these circles.*

Proof. We denote by $\mathcal{S}_1$ the symmetry with respect to circle $\mathcal{C}_1$ and $\mathcal{S}_2$ the symmetry with respect to circle $\mathcal{C}_2$. Let c be the middle point of $[\alpha, \beta]$, and d is symmetric to c with respect to $\mathcal{C}_2$. It is easy to prove the following properties:

$f(c) = d$ and $f = \mathcal{S}_1 \circ \mathcal{S}_2$.

Let z be any point on L_2 different from a, b and c (see Fig. 4). Since L_2 is the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$, the power of z with respect to $\mathcal{C}_1$ is

$$p(z, \mathcal{C}_1) = (z - \alpha)\overline{(z - z_1)} = (z - \beta)\overline{(z - z_2)} = |z - \alpha|^2 - R^2,$$

with $R = |\alpha - \beta|$. It follows that $(z - \alpha)\overline{(z_1 - \alpha)} = R^2$, then z_1 is the symmetric of z with respect to $\mathcal{C}_2$, and z_2 is the symmetric of z with respect to $\mathcal{C}_1$. Furthermore, $f = \mathcal{S}_1 \circ \mathcal{S}_2$, then $f(z) = z_1$ and $f(z_1) = \mathcal{S}_1 \circ \mathcal{S}_2(z_1) = \mathcal{S}_1(z) = z_2$.

Let $\mathcal{C}$ be the circle which passes through z, z_1, z_2, and $w \in \mathcal{C}$, then the cross ratio $[w, z, f(z), f^2(z)]$ is real. Since the cross ratio is invariant under any Möbius transformation, $[w, z, f(z), f^2(z)] = [f(w), f(z), f^2(z), z]$, which means that $f(w) \in \mathcal{C}$. Then $\mathcal{C}$ is invariant under f and contains the orbit of z. We remark that the power of α with respect to circle $\mathcal{C}$ is equal to $|z - \alpha||z_1 - \alpha| = R^2$. Furthermore, the power of β with respect to circle $\mathcal{C}$ is also equal to R^2. The straight line L_2 is the radical axis of the invariant circles of f.

It is easy to prove that if $\mathcal{C}$ and $\mathcal{C}'$ are invariant under f, then $\mathcal{C} = \mathcal{C}'$ or $\mathcal{C} \cap \mathcal{C}' = \emptyset$. Then the theorem is proved. $\square$

6. Construction of the circles invariant under n-cycles, $n \geqslant 4$

Lemma 6.1. *Let f be an n-cycle, then there exists a Möbius transformation g and a rotation h such that $f = g^{-1} \circ h \circ g$.*

Proof. We associate with f a matrix $A \in \mathrm{SL}(2, \mathbb{C})$ such that $A^n = \mathrm{Id}$. Then A is diagonalizable. There exist $P, D \in \mathrm{GL}(2, \mathbb{C})$ such that $A = P^{-1}DP$, with $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$ and $\alpha^n = \beta^n = 1$.

We associate with P a Möbius transformation g and with D the rotation h defined by: $h(z) = \frac{\alpha}{\beta} z$. $\square$

Theorem 6.2. *Let f be an n-cycle such that ∞ is not a fixed point and let L be the straight line which contains the orbit of ∞. Then the general circles invariant under f which are different from L form a pencil of coaxial circles and L is their radical axis (i.e. each two of these circles have L as radical axis). If f is an n-cycle and ∞ is a fixed point, then $f(z) = wz + a$, with $w \neq 1$ and w a n-root of the unity. In this case f has only one finite fixed point α and the circles of center α are the only fixed circles of f.*

Proof. Let g and h be as in Lemma 6.1. The invariant circles under h are the circles of center 0.

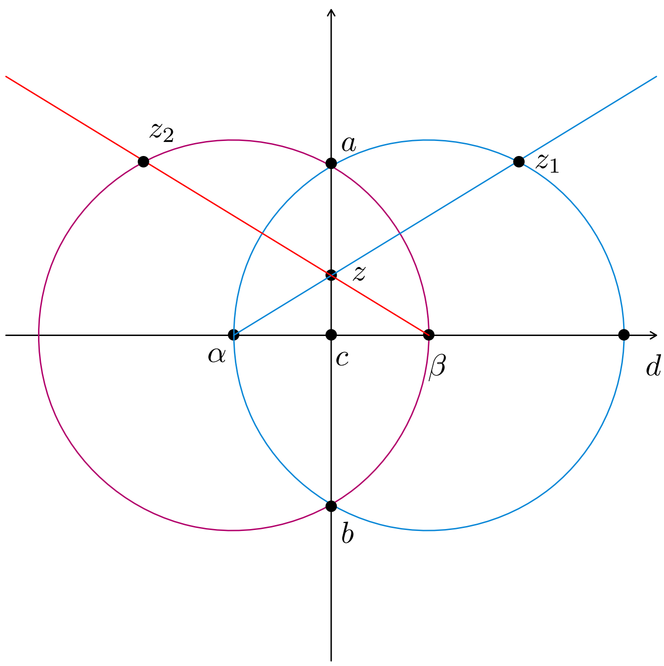

Figure 4 The invariant circles of a 3-cycles.

We define

- $a, b, c \in \mathbb{C}$ such that $g(a) = \infty$, $g(b) = 0$ and $g(\infty) = c$,
- $\mathcal{C}$ the circle of center 0 passing through c,
- straight line $\mathcal{L} = g^{-1}(\mathcal{C})$,
- $\alpha \in \mathbb{C}$ the intersection of $\mathcal{L}$ and the straight line (a, b).Then
- Since 0 and ∞ are the fixed points of h, then a and b are fixed points of f.
- 0 and ∞ are symmetric with respect to $\mathcal{C}$, then a and b are symmetric with respect to $\mathcal{L}$.
- $-c$ is the intersection of $\mathcal{C}$ and $(0, c)$, then α is the intersection of $\mathcal{L} = g^{-1}(\mathcal{C})$. Furthermore the straight line $(a, b) = g^{-1}(0, c)$, then $\alpha = g^{-1}(-c)$.
- The invariant circles under f are the image of the circles of center 0 by g^{-1}. Except the circle $\mathcal{L}$, these circles have their center on the line (a, b) and are disjoint.Now we will prove that the circles which are different from $\mathcal{L}$ form a pencil of coaxial circles.Let $\mathcal{C}_1$ be an orbit of h, ($\mathcal{C}_1$ is a circle of center 0 and radius r). We denote by z and w the intersection of $g^{-1}(\mathcal{C}_1)$ with the straight line (a, b). Then $g(z)$ and $-g(z)$ are the intersection of $\mathcal{C}_1$ with the straight line $(0, c)$ (cf Figs. 6 and 5).To prove the theorem, it suffices to prove that the power of α with respect to $g^{-1}(\mathcal{C}_1)$ is independent of $\mathcal{C}_1$, which is equivalent to proving that $(\alpha - z)(\overline{\alpha - w})$ is independent of z and w.Since the cross-ratio is invariant under Möbius transformation

$$\frac{\alpha - z}{a - z} = (\alpha, a, z, \infty) = (g(\alpha), g(a), g(z), g(\infty)) = (-c, \infty, g(z), c) = \frac{g(z) + c}{2c}.$$

$$\frac{a - z}{a - b} = (a, \infty, z, b) = (g(a), g(\infty), g(z), g(b)) = (\infty, c, g(z), 0) = \frac{c}{c - g(z)}.$$

$$\frac{\alpha - w}{a - w} = \frac{c - g(z)}{2c}, \quad \frac{a - w}{a - b} = \frac{c}{c + g(z)}.$$

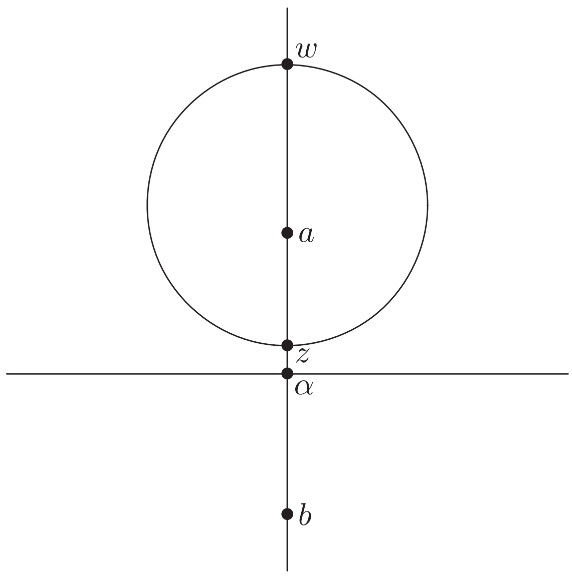

Figure 5 $g^{-1}(\mathcal{C})$ intersects the line (a,b) in z and w.

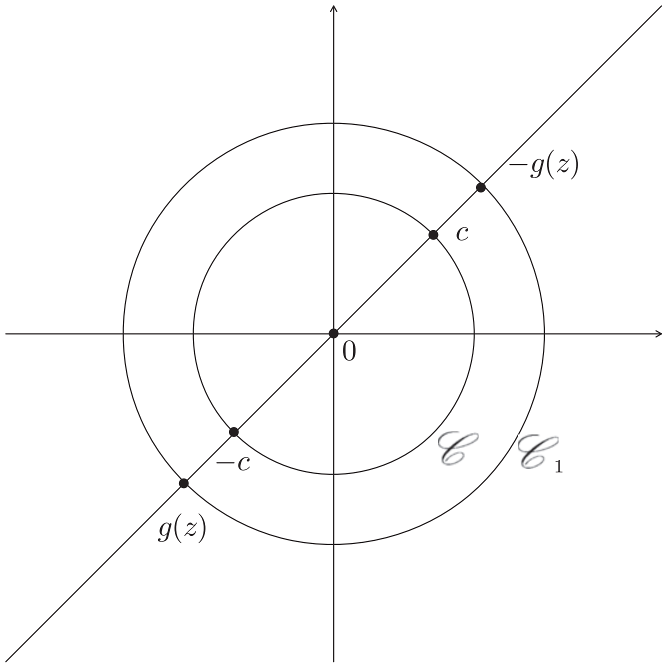

Figure 6 $\mathscr{C}_1$ intersects the line $(0,c)$ in $g(z)$ and $-g(z)$.

Then

$$(\alpha - z)(\overline{\alpha - w}) = \frac{(a-b)c(g(z)+c)\overline{(a-b)}\overline{c}(\overline{c} - \overline{g(z)})}{4c\overline{c}(c - g(z))(\overline{c} + \overline{g(z)})} = \frac{|a-b|^2}{4}.$$

This completes the proof. $\square$

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