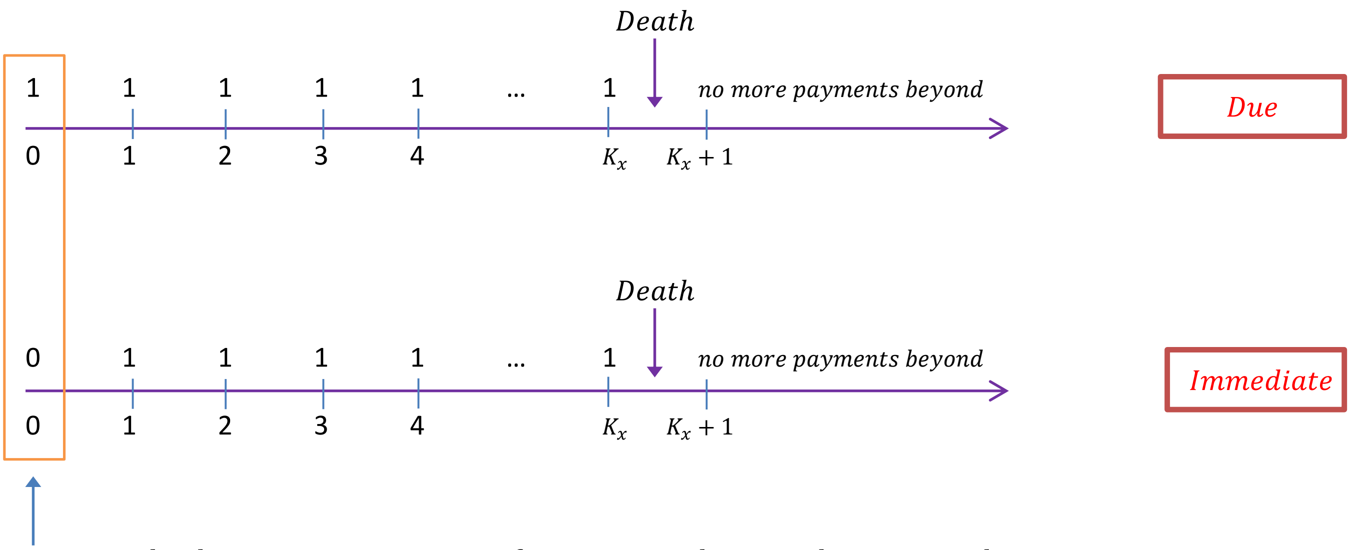
life annuity : Immediate

Remember there is no formula to get the annuity Immediate directly , so we will calculate throught annuity due .

Whole life annuity

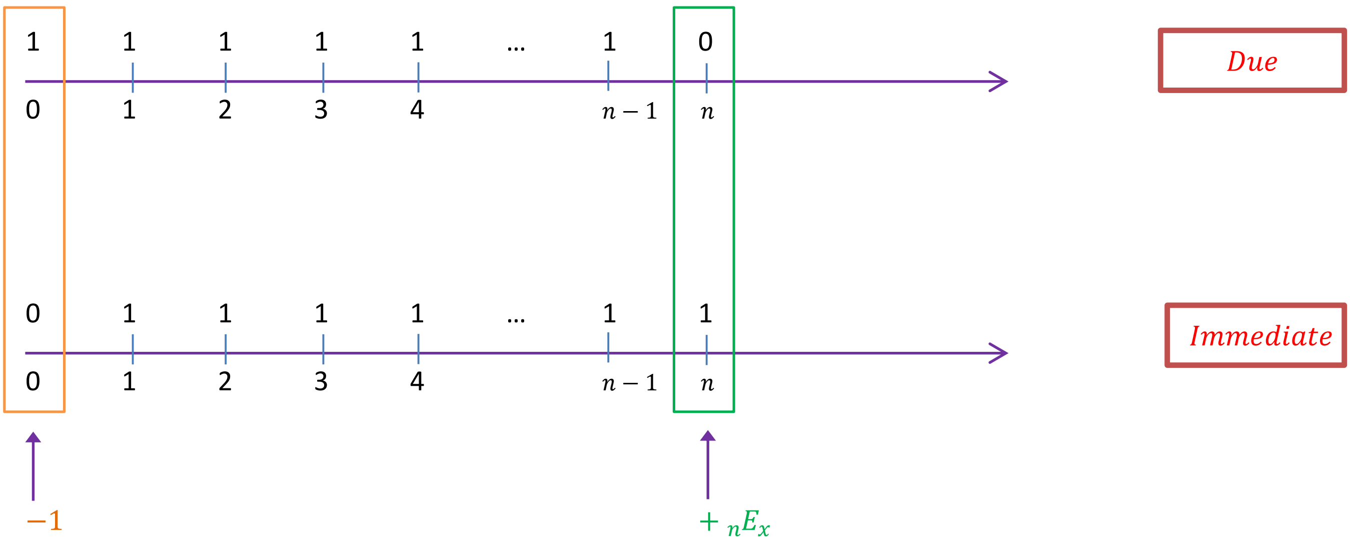


Annuity due has an extra payment of 1SR , to get the immediate we need to remove it :

$$a_x = \ddot{a}_x - 1$$

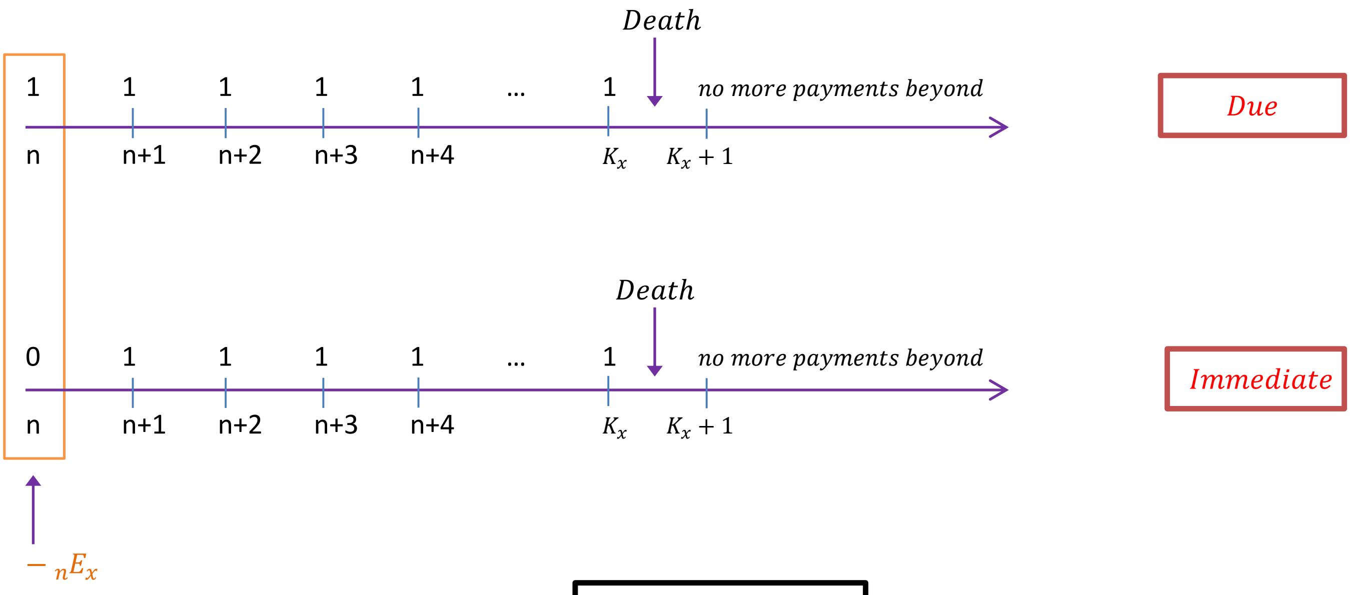
where a_x *is APV of Immediate life annuity of 1SR on* (x).

Temporary life annuity



$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + {}_nE_x$$

Deffered life annuity



$$_{n|}a_x = {}_{n|}\ddot{a}_x - {}_nE_x$$

Example:

- $A_x = 0.28$
- $A_{x+20} = 0.4$
- $A_{\overset{1}{x:\overline{20}|}} = 0.25$
- $i = 0.05$

Find $a_{x:\overline{20}|}$.

solution.

$$A_x = A_{\overset{1}{x:\overline{20}|}} + {}_{20|}A_x$$

$$A_x = A_{\overset{1}{x:\overline{20}|}} + {}_{20}E_x \cdot A_{x+20}$$

$$0.28 = A_{\overset{1}{x:\overline{20}|}} + (0.25) \cdot (0.4)$$

$$\rightarrow A_{\overset{1}{x:\overline{20}|}} = 0.18$$

now ...

$$A_{x:\overline{20}|} = A_{\overset{1}{x:\overline{20}|}} + A_{x:\overset{\ \ 1}{\overline{20}|}} = 0.18 + 0.25 = 0.43$$

Then ...

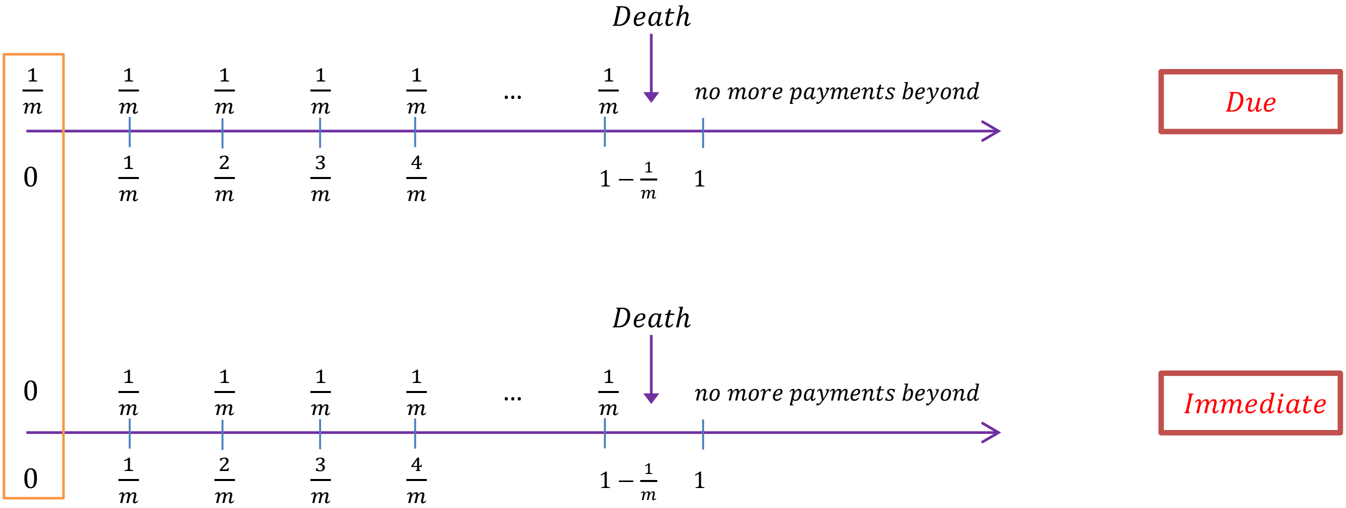
$$\ddot{a}_{x:\overline{20}|} = \frac{1 - A_{x:\overline{20}|}}{d} = \frac{1 - 0.43}{\frac{1}{21}} = 11.97$$

Finally ...

$$a_{x:\overline{n}|} = 11.97 - 1 + 0.25 = 11.22 \blacksquare$$

$m-thly\ annuity$

this annuity makes payments every $\frac{1}{m}$ *of a year and each payment is* $\frac{1}{m}$ *SR .*



hence we can conclude :

$$\ddot{a}_x^{(m)} = \frac{1}{m} + \frac{1}{m} v^{\frac{1}{m}} \cdot {}_{\frac{1}{m}}p_x + \frac{1}{m} v^{\frac{2}{m}} \cdot {}_{\frac{2}{m}}p_x + \frac{1}{m} v^{\frac{3}{m}} \cdot {}_{\frac{3}{m}}p_x + \cdots$$

$$\boxed{\ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \frac{1}{m} \cdot v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x}$$

And for the m − thly annuity Immediate :

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$$

$$a_{x:\overline{n|}}^{(m)} = \ddot{a}_{x:\overline{n|}}^{(m)} - \frac{1}{m} + \frac{{}_nE_x}{m}$$

$$_{n|}a_x^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{{}_nE_x}{m}$$

Let us denote the following :

$$eq1 \qquad \bar{a}_x = \bar{a}_{x:\overline{n|}} + {}_{n|}\bar{a}_x$$

$$eq2 \qquad {}_{n|}\bar{a}_x = v^n\, {}_np_x \cdot \bar{a}_{x+n}$$

Recursions

	$eq1$	$eq2$
Continuous	$\bar{a}_x = \bar{a}_{x:\overline{n\|}} + {}_{n\|}\bar{a}_x$	${}_{n\|}\bar{a}_x = v^n\, {}_np_x \cdot \bar{a}_{x+n}$
Discrete (Due)	$\ddot{a}_x = \ddot{a}_{x:\overline{n\|}} + {}_{n\|}\ddot{a}_x$	${}_{n\|}\ddot{a}_x = v^n\, {}_np_x \cdot \ddot{a}_{x+n}$
Discrete (Immediate)	$a_x = a_{x:\overline{n\|}} + {}_{n\|}a_x$	${}_{n\|}a_x = v^n\, {}_np_x \cdot a_{x+n}$
Discrete m − thly (Due)	$\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n\|}}^{(m)} + {}_{n\|}\ddot{a}_x^{(m)}$	${}_{n\|}\ddot{a}_x^{(m)} = v^n\, {}_np_x \cdot \ddot{a}_{x+n}^{(m)}$
Discrete m − thly (Immediate)	$a_x^{(m)} = a_{x:\overline{n\|}}^{(m)} + {}_{n\|}a_x^{(m)}$	${}_{n\|}a_x^{(m)} = v^n\, {}_np_x \cdot a_{x+n}^{(m)}$

Example:

- $\bar{A}_{30} = 0.6$
- $\bar{A}^{\,1}_{30:\overline{10|}} = 0.1$
- $\delta = 0.02$
- $\bar{A}_{30:\overline{10|}}^{\quad\;\, 1} = 0.7$

find:

1. $\bar{a}_{30:\overline{10|}}$
2. $\bar{a}_{30}$
3. ${}_{10|}\bar{a}_{30}$
4. $\bar{a}_{40}$

solution.

$\bar{A}_{30:\overline{10|}} = \bar{A}^{\,1}_{30:\overline{10|}} + \bar{A}_{30:\overline{10|}}^{\quad\;\, 1} = 0.1 + 0.7 = 0.8$

1. $\bar{a}_{30:\overline{10|}} = \frac{1-0.8}{0.02} = 10\blacksquare$
2. $\bar{a}_{30} = \frac{1-\bar{A}_{30}}{\delta} = \frac{1-0.6}{0.02} = 20\blacksquare$
3. ${}_{10|}\bar{a}_{30} = \bar{a}_{30} - \bar{a}_{30:\overline{10|}} = 20 - 10 = 10\blacksquare$
4. $\bar{a}_{40} = \frac{1}{{}_{10}E_{30}} \cdot \bar{a}_{30} = \frac{1}{0.7} \cdot 10 = 14.28571429\blacksquare$

Recursion formula

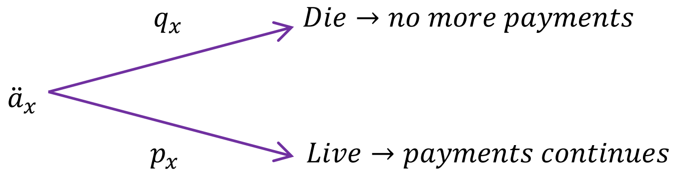
is given by :

$$\ddot{a}_x = 1 + vp_x \cdot \ddot{a}_{x+1}$$

$$\ddot{a}_{x:\overline{n|}} = 1 + vp_x \cdot \ddot{a}_{x+1:\overline{n-1|}}$$

time 0 *time 1*

age x



Proof :

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x = 1 + \sum_{k=1}^{\infty} v^k \cdot {}_kp_x$$

$$= 1 + \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k+1}p_x = 1 + \sum_{k=0}^{\infty} v^{k+1} \cdot p_x\ {}_kp_{x+1}$$

$$= 1 + vp_x \sum_{k=0}^{\infty} v^k \cdot {}_kp_{x+1}$$

$$= 1 + vp_x \cdot \ddot{a}_{x+1} \blacksquare$$

for m − thly for each step :

$$\ddot{a}_x^{(m)} = \frac{1}{m} + v^{\frac{1}{m}} \cdot {}_{\frac{1}{m}}p_x \cdot \ddot{a}_{x+1}^{(m)}$$

Example:

You're given :

- ***for a certain mortality table with $q_{30} = 0.05$ the value $\ddot{a}_{30} = 12.6$***
- ***for a mortality table identical exept that $q_{30} = 0.2$ and $\ddot{a}_{30} = y$.***

compute.

1. ***y.***
2. ***a_{30} on the basis of life table with $q_{30} = 0.2$.***

solution .

1.

$$\ddot{a}_{30} = 1 + vp_{30} \cdot \ddot{a}_{31}$$

$$\rightarrow 12.6 = 1 + 1.06^{-1} \cdot (1 - 0.05) \cdot \ddot{a}_{31}$$

$$\rightarrow \ddot{a}_{31} = 12.94316$$

Then ...

$$y = \ddot{a}_{30} = 1 + vp_{30} \cdot \ddot{a}_{31}$$

$$y = \ddot{a}_{30} = 1 + 1.06^{-1} \cdot (1 - 0.2) \cdot 12.94316$$

$$y = \ddot{a}_{30} = 10.76842 \blacksquare$$

2.

$$a_{30} = \ddot{a}_{30} - 1 = 10.76842 - 1 = 9.76842 \blacksquare$$

Relation between discrete and continuous + m − thly life annuities

Y − Z relation and UDD

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$$

$$= \frac{1-\frac{i}{\delta}A_x}{\delta}$$

$$\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}}$$

$$= \frac{1-\frac{i}{i^{(m)}}A_x}{d^{(m)}}$$

$$= \frac{1-\frac{i}{i^{(m)}}(1-d\ddot{a}_x)}{d^{(m)}}$$

$$= \frac{1-\frac{i}{i^{(m)}}A_x}{d^{(m)}}$$

$$= \frac{id}{i^{(m)}\cdot d^{(m)}}\cdot\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}\cdot d^{(m)}}$$

$UDD \Rightarrow$

$$\bar{A}_x = \frac{i}{\delta}A_x$$

$$A_x^{(m)} = \frac{i}{i^{(m)}}A_x$$

$$d^{(m)} = m[1-(1-d)^{\frac{1}{m}}]$$

$$\beta(m) = \frac{i-i^{(m)}}{i^{(m)}\cdot d^{(m)}} \qquad \alpha(m) = \frac{id}{i^{(m)}\cdot d^{(m)}}$$

$$\ddot{a}_x^{(m)} = \alpha(m)\cdot\ddot{a}_x - \beta(m)$$

If m → ∞

$$\beta(\infty) = \frac{i-\delta}{\delta^2} \qquad \alpha(\infty) = \frac{id}{\delta^2}$$

$$\bar{a}_x = \alpha(\infty)\cdot\ddot{a}_x - \beta(\infty)$$

for n − year deffered life annuity we have :

Proof:

$$_{n|}\ddot{a}_x^{(m)} = {}_nE_x\cdot\ddot{a}_{x+n}^{(m)}$$

$$= {}_nE_x[\alpha(m)\cdot\ddot{a}_{x+n} - \beta(m)]$$

$$= \alpha(m)\,{}_nE_x\cdot\ddot{a}_{x+n} - {}_nE_x\cdot\beta(m)$$

$$= {}_{n|}\ddot{a}_x^{(m)} = \alpha(m)\cdot{}_{n|}\ddot{a}_x - \beta(m)\cdot{}_nE_x$$

for n − year temporary life annuity we have :

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\cdot\ddot{a}_{x:\overline{n}|} - \beta(m)[1-{}_nE_x]$$

$m-thly\ life\ annuities\ formulas:$

$Whole\ life$	$\alpha(m)\cdot\ddot{a}_x-\beta(m)$
$n-year\ deffered$	$\alpha(m)\cdot{}_{n\vert}\ddot{a}_x-\beta(m)\cdot{}_nE_x$
$n-year\ temporary$	$\alpha(m)\cdot\ddot{a}_{x:\overline{n\vert}}-\beta(m)[1-{}_nE_x]$

$Example$:

$For\ annuity\ payable\ semiannually\ you're\ given:$

$Death\ is\ uniformly\ distributed\ over\ each\ year\ of\ age.$

$q_{69}=0.03$

$i=0.06$

$1000\bar{A}_{70}=530$

$Find\ \ddot{a}_{69}^{(2)}.$

$solution.$

$$\bar{A}_{70}=\frac{i}{\delta}A_{70}$$

$$0.530=\frac{0.06}{\ln(1.06)}A_{70}$$

$$\rightarrow A_{70}=0.5147086884$$

$$\ddot{a}_{70}=\frac{1-0.5147086884}{\frac{3}{53}}=8.573479838$$

$$d=\frac{3}{53}$$

$$d^{(2)}=2\cdot\left[1-\left(1-\frac{3}{53}\right)^{\frac{1}{2}}\right]=0.05742827529$$

$$i^{(2)}=0.0591260282$$

$$\ddot{a}_{69}=1+vp_{69}\cdot\ddot{a}_{70}$$

$$\ddot{a}_{69}=8.845542871$$

$$\ddot{a}_{69}^{(2)}=\alpha(2)\cdot\ddot{a}_{69}-\beta(2)$$

$$\alpha(2)=\frac{id}{i^{(2)}\cdot d^{(2)}}=\frac{(0.06)\left(\frac{3}{53}\right)}{(0.0591260282)(0.05742827529)}=1.000212219$$

$$\beta(2)=\frac{i-i^{(2)}}{i^{(2)}\cdot d^{(2)}}=\frac{0.06-0.0591260282}{(0.0591260282)(0.05742827529)}=0.2573907527$$

$$\ddot{a}_{69}^{(2)}=(1.000212219)\cdot(8.845542871)-(0.2573907527)=8.590029311\blacksquare$$

Woolhouse formula

No need to use UDD

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

or also you can use :

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{m-1}{2m}$$

but it's not accurate .

If you don't have μ_x you can approximat μ_x by :

$$\mu_x \cong -\frac{1}{2}[\ln(p_{x-1} + ln(p_x)]$$

Example :

for group of individuals all (x), you're given:

- **30% are smokers and 70% are non − smokers**
- **the constant force of mortality for smokers is 0.06**
- **the constant force of mortality for non − smokers is 0.03**
- **$\delta = 0.08$**

calculate $var(\bar{a}_{T_x})$ for an indivisual chosen at random from this group.

solution.

$$var(\bar{a}_{T_x}) = var(\bar{Y}) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$$

$$\bar{A}_x = \bar{A}_x^s(0.3) + \bar{A}_x^n(0.7)$$

$$\bar{A}_x^s = \frac{\mu^S}{\delta + \mu^S} = \frac{0.06}{0.08 + 0.06} = \frac{6}{14} = \frac{3}{7}$$

$$\bar{A}_x^n = \frac{\mu^n}{\delta + \mu^n} = \frac{0.03}{0.08 + 0.03} = \frac{3}{11}$$

$$\bar{A}_x = \left(\frac{3}{7}\right)(0.3) + \left(\frac{3}{11}\right)(0.7) = \frac{123}{385} = 0.3194805195$$

$${}^2\bar{A}_x = {}^2\bar{A}_x^s(0.3) + {}^2\bar{A}_x^n(0.7)$$

$${}^2\bar{A}_x^s = \frac{\mu^S}{2\delta + \mu^S} = \frac{0.06}{2(0.08) + 0.06} = \frac{0.06}{0.16 + 0.06} = \frac{6}{22} = \frac{3}{11}$$

$${}^2\bar{A}_x^n = \frac{1}{2\delta + \mu^n} = \frac{1}{2(0.08) + 0.03} = \frac{1}{0.16 + 0.03} = \frac{3}{19}$$

$${}^2\bar{A}_x = \left(\frac{3}{11}\right)(0.3) + \left(\frac{3}{19}\right)(0.7) = \frac{201}{1045} = 0.1923444976$$

$$var(\bar{Y}) = \frac{\left(\frac{201}{1045}\right) - \left(\frac{123}{385}\right)^2}{0.08^2} = 14.10573364 \blacksquare$$

Annuity : Exercises

Exercise 1:

you're given:

- $\ddot{a}_{25:\overline{20|}} = 17$
- $\delta = 0.05$
- ${}_{20}p_{25} = 0.8$
- $\mu_{25} = 0.02$
- $\mu_{45} = 0.03$

Find $\ddot{a}^{(2)}_{25:\overline{20|}}$ ***by using woolhouse formula with 3 terms .***

Solution.

Then applying Woolhouse formula for $m = 2$ *and* $x = 25$:

$$\ddot{a}^{(2)}_{25} \cong \ddot{a}_{25} - \frac{2-1}{2(2)} - \frac{2^2-1}{(12)(2)^2}(0.05+0.02)$$

$$\ddot{a}^{(2)}_{25} \cong \ddot{a}_{25} - \frac{1}{4} - \frac{(3)(0.07)}{48} \qquad \boxed{1}$$

$$\ddot{a}^{(2)}_{25:\overline{25|}} = \ddot{a}^{(2)}_{25} - {}_{20|}\ddot{a}^{(2)}_{25} = \ddot{a}^{(2)}_{25} - {}_{20}E_{25} \cdot \ddot{a}^{(2)}_{45}$$

from $\boxed{1}$

$$\ddot{a}^{(2)}_{25:\overline{25|}} \cong \left[\ddot{a}_{25} - \frac{1}{4} - 3\right] - {}_{20}E_{25} \cdot \ddot{a}^{(2)}_{45}$$

Then applying Woolhouse for $m = 2$ *and* $x = 45$:

$$\ddot{a}^{(2)}_{45} \cong \ddot{a}_{45} - \frac{2-1}{2(2)} - \frac{2^2-1}{(12)(2)^2}(0.05+0.03)$$

$$\ddot{a}^{(2)}_{45} \cong \ddot{a}_{45} - \frac{1}{4} - \frac{(3)(0.08)}{48} \qquad \boxed{2}$$

from $\boxed{2}$

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong \left[\ddot{a}_{25} - \frac{1}{4} - \frac{(3)(0.07)}{48}\right] - {}_{20}E_{25} \cdot \left[\ddot{a}_{45} - \frac{1}{4} - \frac{(3)(0.08)}{48}\right]$$

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong \ddot{a}_{25} - \frac{1}{4} - \frac{(3)(0.07)}{48} - {}_{20}E_{25} \cdot \ddot{a}_{45} + {}_{20}E_{25} \cdot \frac{1}{4} + {}_{20}E_{25} \cdot \frac{(3)(0.08)}{48}$$

by regrouping terms :

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong \ddot{a}_{25} - {}_{20}E_{25} \cdot \ddot{a}_{45} - \frac{1}{4} - \frac{(3)(0.07)}{48} + {}_{20}E_{25} \cdot \frac{1}{4} + {}_{20}E_{25} \cdot \frac{(3)(0.08)}{48}$$

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong \ddot{a}_{25} - {}_{20|}\ddot{a}_{25} - \frac{1}{4} - \frac{(3)(0.07)}{48} + {}_{20}E_{25} \cdot \frac{1}{4} + {}_{20}E_{25} \cdot \frac{(3)(0.08)}{48}$$

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong \ddot{a}_{25:\overline{20|}} - \frac{1}{4} - \frac{(3)(0.07)}{48} + {}_{20}E_{25} \cdot \frac{1}{4} + {}_{20}E_{25} \cdot \frac{(3)(0.08)}{48}$$

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong 17 - \frac{1}{4} - \frac{(3)(0.07)}{48} + (0.8)(e^{-(20)(0.05)}) \cdot \frac{1}{4} + (0.8)(e^{-(20)(0.05)}) \cdot \frac{(3)(0.08)}{48}$$

$$\ddot{a}^{(2)}_{25:\overline{20|}} \cong 16.8207 \blacksquare$$

Exercise 2:

you're given:

- $A_x = 0.24$
- $A_{x+25} = 0.4$
- ${}_{20}E_x = 0.3$
- $d = 0.08$

Find $\ddot{a}_{x:\overline{20|}}$.

Solution.

$$\ddot{a}_{x:\overline{20|}} = \frac{1 - A_{x:\overline{20|}}}{d}$$

$$A^{1}_{x:\overline{20|}} = A_x - {}_{25|}A_x = A_x - {}_{20}E_x \cdot A_{x+25}$$

$$A^{1}_{x:\overline{20|}} = 0.24 - (0.3)(0.4) = 0.12$$

$$A_{x:\overline{20|}} = A^{1}_{x:\overline{20|}} + A_{x:\overline{20|}^{\,1}} = 0.12 + 0.3 = 0.42$$

$$\ddot{a}_{x:\overline{20|}} = \frac{1 - A_{x:\overline{20|}}}{d} = \frac{1 - 0.42}{0.08} = 7.25 \blacksquare$$

Exercise 3:

ILT $i = 6\%$.

x	l_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$
65	7,533,964	21.32	9.8969	439.8
66	7,373,338	23.29	9.6362	454.56
67	7,201,635	25.44	9.3726	469.47
68	7,018,432	27.79	9.1066	484.53

Find ${}_{3|}\ddot{a}_{65}$ ***and*** $\ddot{a}_{65:\overline{3|}}$.

Solution.

$${}_{3|}\ddot{a}_{65} = v^3\ {}_{3}p_{65} \cdot \ddot{a}_{68} = 1.06^{-3} \cdot \frac{l_{68}}{l_{65}} \cdot \ddot{a}_{68}$$

$$= 1.06^{-3} \cdot \left(\frac{7{,}018{,}432}{7{,}533{,}964}\right) \cdot (9.1066) = 7.1229 \blacksquare$$

$$\ddot{a}_{65:\overline{3|}} = \ddot{a}_{65} - {}_{3|}\ddot{a}_{65} = 9.8969 - 7.1229 = 2.7740 \blacksquare$$

Exercise 4:

you're given:

- $\boldsymbol{A_{60} = 2A_{40}}$
- $\boldsymbol{\ddot{a}_{40} = 3\ddot{a}_{60}}$

Find $\boldsymbol{A_{40}}$.

Solution.

$$\ddot{a}_{40} = \frac{1 - A_{40}}{d} = 3\ddot{a}_{60} = 3\left(\frac{1 - A_{60}}{d}\right) = 3\left(\frac{1 - 2A_{40}}{d}\right)$$

$$\frac{1 - A_{40}}{d} = 3\left(\frac{1 - 2A_{40}}{d}\right)$$

$$1 - A_{40} = 3 - 6A_{40}$$

$$A_{40} = \frac{2}{5} \blacksquare$$

Exercise 5:

for special whole life annuity on (30) you're given :

- ***The annuity pays 1000 a year for the first 20 years and 2000 a year thereafter.***
- ***All payments are made at the beginning of the year .***
- ***Mortality follows ILT .***
- $\boldsymbol{i = 0.06}$

Find the APV of the annuity .

Solution.

Let Z the present value random variable .

$$E(Z) = 1000 \cdot \ddot{a}_{30:\overline{20|}} + 2000 \cdot {}_{20}E_{30} \cdot \ddot{a}_{50}$$

from ILT

$$\ddot{a}_{30} = 15.8561$$

$$\ddot{a}_{50} = 13.2668$$

$$1000\,{}_{20}E_{30} = 293.74$$

now …

$$\ddot{a}_{30:\overline{20|}} = \ddot{a}_{30} - {}_{20|}\ddot{a}_{30}$$

$$= 15.8561 - (0.29374) \cdot (13.2668) = 11.9591102$$

$$E(Z) = 1000 \cdot (11.9591102) + 2000 \cdot (0.29374) \cdot (13.2668) = 11{,}959.1102 + 7{,}793.979664 = 19753.08986 \blacksquare$$

Annuity : Exercises

Exercise 1 :

For a continuos whole life annuity on (x) you're given :

- **T_x is the future lifetime random variable for (x)**
- **The force of mortality is 0.06 constante for all ages.**
- **The force of interest is 0.04 .**

Calculate $Pr(\bar{a}_{\overline{T_x|}} > \bar{a}_x)$.

Solution.

$$\bar{a}_{\overline{T_x|}} = \frac{1 - v^{Tx}}{\delta} = \frac{1 - e^{-\delta Tx}}{\delta}$$

$$\bar{a}_x = \frac{1}{\delta + \mu} = \frac{1}{0.04 + 0.06} = 10 \ \ , \textit{Becuase we are under CF}.$$

$$Pr(\bar{a}_{\overline{T_x|}} > \bar{a}_x) = Pr\left(\frac{1 - e^{-(0.04)Tx}}{0.04} > 10\right)$$

$$= Pr(1 - e^{-(0.04)Tx} > 0.4) = Pr(e^{-(0.04)Tx} > 0.6)$$

$$= Pr\left(T_x > \frac{ln(0.6)}{-0.04}\right) = Pr(T_x > 12.77064059) \ \ \textit{Which is the survival function .}$$

$$= e^{-0.06(12.77064059)} = 0.4647580015 \ ■$$

Exercise 2 :

Suppose we have :

- **$\mu_{x+t} = 0.01$**
- **$\mu_{x+t} = 0.02$**
- **$\delta = 0.06$**

Remember :
Under CF $f_x(t) = e^{-t(\mu+\delta)} \cdot \mu$

Find $\bar{a}_x$.

Solution .

for $0 \le t \le 5$

$$\bar{A}_{\substack{1\\ x:\overline{5|}}} = \int_0^5 v^t \cdot {}_tp_x \cdot \mu_{x+t} \, dx = \int_0^5 e^{-t(\mu+\delta)} \cdot \mu \, dx = \int_0^5 e^{-t(0.06+0.01)}(0.01)dx = (0.01)\left[\frac{e^{-t(0.07)}}{-0.07}\right]_0^5$$

$$= \frac{0.01}{0.07}\left(1 - e^{-5(0.07)}\right) = 0.04218741575$$

for $t > 5$

$$_{5|}\bar{A}_x = {}_5E_x \cdot \bar{A}_{x+5} = e^{-5(0.01+0.06)} \cdot \frac{0.02}{0.02 + 0.06} = 0.1761720224$$

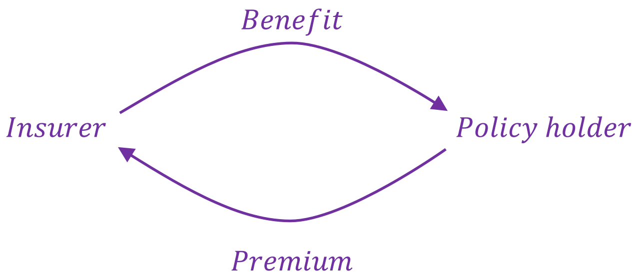
for all t

$$\bar{A}_x = \bar{A}_{\substack{1\\ x:\overline{5|}}} + {}_{5|}\bar{A}_x = 0.2183594382$$

now solving for the annuity ...

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta} = \frac{1 - 0.2183594382}{0.06} = 13.0273427■$$

Premiums: Net premium for standard fully discrete Insurance policies



1) *net premium for n − year term Endowment Insurence of 1SR* :

1 1 1 1 … 1 1 *death benefits* = $A^{\;1}_{x:\overline{n}|}$

1 *survival benefit* = ${}_nE_x$

} *Endowment Insurence*

0 1 2 3 4 $n-1$ n

$P_{x:\overline{n}|}$ $P_{x:\overline{n}|}$ $P_{x:\overline{n}|}$ $P_{x:\overline{n}|}$ $P_{x:\overline{n}|}$ … $P_{x:\overline{n}|}$ *net premium*

Notations :

- *APV of future benefit* : $A_{x:\overline{n}|}$
- *APV of future net premium* : $P_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{n}|}$

In Order to get the premium we use

equivalence principle

$$A_{x:\overline{n}|} = P_{x:\overline{n}|}\ddot{a}_{x:\overline{n}|}$$

$$\rightarrow P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

*If **S** of saudi ryials policy has benefit of ???*

$$\boldsymbol{S} \cdot P_{x:\overline{n}|}$$

Remember :
$$A_{x:\overline{n}|} = 1 - d \cdot \ddot{a}_{x:\overline{n}|}$$

formulas for $P_{x:\overline{n}|}$:

we can represent $P_{x:\overline{n}|}$ *by* $A_{x:\overline{n}|}$:

$$P_{x:\overline{n}|} = \frac{d \cdot A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}$$

Also we can write $P_{x:\overline{n}|}$ *in terms of* $\ddot{a}_{x:\overline{n}|}$:

$$P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d$$

2) Net premium for an m − payment year n − year endowment insurence of 1SR:

$m \le n$

1 1 1 1 1 1 … 1 1 death benefits $= A^{1}_{x:\overline{n}|}$

1 survival benefit $= {}_nE_x$

Endowment Insurence

0 1 2 3 4 $m-1$ m … $n-1$ n

${}_mP_{x:\overline{n}|}$ ${}_mP_{x:\overline{n}|}$ ${}_mP_{x:\overline{n}|}$ ${}_mP_{x:\overline{n}|}$ ${}_mP_{x:\overline{n}|}$ ${}_mP_{x:\overline{n}|}$ ${}_mP_{x:\overline{n}|}$ net premium

Applying equivalence principle

$A_{x:\overline{n}|} = {}_mP_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{m}|}$

$$\rightarrow {}_mP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}$$

formulas for $P_{x:\overline{n}|}$:

We can represent ${}_mP_{x:\overline{n}|}$ by $A_{x:\overline{n}|}$

$${}_mP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}} = \frac{A_{x:\overline{n}|}}{\frac{1-A_{x:\overline{n}|}}{d}}$$

$$\Rightarrow {}_mP_{x:\overline{n}|} = \frac{d \cdot A_{x:\overline{n}|}}{1-A_{x:\overline{n}|}}$$

Also we can write ${}_mP_{x:\overline{n}|}$ in terms of $\ddot{a}_{x:\overline{m}|}$:

$${}_mP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}} = \frac{1-d \cdot \ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}$$

$$\Rightarrow {}_mP_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{m}|}}$$

3) Net premium for an m − payment year n − year term Insurence of 1SR:

$m \le n$

1 1 1 1 1 1 … 1 1 death benefits $= A^{1}_{x:\overline{n}|}$

0 1 2 3 4 $m-1$ m … $n-1$ n

${}_mP^{1}_{x:\overline{n}|}$ ${}_mP^{1}_{x:\overline{n}|}$ ${}_mP^{1}_{x:\overline{n}|}$ ${}_mP^{1}_{x:\overline{n}|}$ ${}_mP^{1}_{x:\overline{n}|}$ ${}_mP^{1}_{x:\overline{n}|}$ ${}_mP^{1}_{x:\overline{n}|}$ net premium

Notations :

- APV of future benefit : $A^{1}_{x:\overline{n}|}$
- APV of future net premium : ${}_mP^{1}_{x:\overline{m}|} \cdot \ddot{a}_{x:\overline{n}|}$

$$A^{1}_{x:\overline{n}|} = {}_mP^{1}_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{m}|} \Rightarrow {}_mP^{1}_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}$$

$m = n$

$$\Rightarrow {}_nP^{1}_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = P^{1}_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

4) *Net premium for fully discrete whole life insurance of 1SR*:

$$P_x = \frac{A_x}{\ddot{a}_x}$$

formulas for $P_{x:\overline{n}|}$:

We can represent ${}_mP_{x:\overline{n}|}$ *by* $A_{x:\overline{n}|}$

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{A_x}{\frac{1-A_x}{d}}$$

$$\Rightarrow P_x = \frac{d \cdot A_x}{1 - A_x}$$

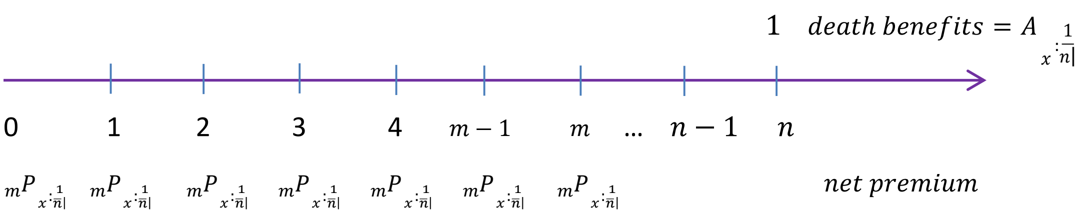
Also we can write ${}_mP_{x:\overline{n}|}$ *in terms of* $\ddot{a}_{x:\overline{m}|}$:

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{1 - d \cdot \ddot{a}_x}{\ddot{a}_x}$$

$$\Rightarrow P_x = \frac{1}{\ddot{a}_x} - \frac{d \cdot \ddot{a}_x}{\ddot{a}_x}$$

4) *Net premium for an m − payment year n − year Pure Endowment of 1SR*:

$m \leq n$



Notations :

- *APV of future benefit* : $A_{x:\overset{1}{\overline{n}|}}$
- *APV of future net premium* : ${}_mP_{x:\overset{1}{\overline{m}|}} \cdot \ddot{a}_{x:\overline{n}|}$

$$A_{\underset{x:\overline{n}|}{1}} = {}_mP_{x:\overset{1}{\overline{n}|}} \cdot \ddot{a}_{x:\overline{m}|} \Rightarrow {}_mP_{x:\overset{1}{\overline{n}|}} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{m}|}}$$

$m = n$

$${}_nP_{x:\overset{1}{\overline{n}|}} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}$$

$$= P_{x:\overset{1}{\overline{n}|}} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}$$

Conclusion :

$m \leq n$

$${}_mP_{x:\overline{n}|} = {}_mP_{\underset{x:\overline{n}|}{1}} + {}_mP_{x:\overset{1}{\overline{n}|}}$$

$m = n$

$$P_{x:\overline{n}|} = P_{\underset{x:\overline{n}|}{1}} + P_{x:\overset{1}{\overline{n}|}}$$

Example :
Suppose that

1. $\mu_x = \mu \ \forall x$.
2. $l_x = \omega - x \quad ; 0 \le x \le \omega \quad ; n \le \omega - x$.

calculate $P_{\substack{1\\x:\overline{n}|}}$ **for both cases .**

solution.

$$P_{\substack{1\\x:\overline{n}|}} = \frac{A_{\substack{1\\x:\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}$$

1.

under CF term insurence is :

$$A_{\substack{1\\x:\overline{n}|}} = \sum_{k=0}^{n-1} v^{k+1} \cdot {}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1} \cdot (p)^k (q) = vq \sum_{k=0}^{n-1} (vp)^k = vq \left[\frac{1-(vp)^n}{1-vp}\right]$$

$$= vq \left[\frac{1-(vp)^n}{1-v(1-q)}\right] = vq \left[\frac{1-(vp)^n}{1-v+vq)}\right] = \frac{vq}{vq+d}[1-(vp)^n]$$

under CF term annuity is :

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_kp_x = \sum_{k=0}^{n-1} v^k \cdot (p)^k = \sum_{k=0}^{n-1} (vp)^k = \left[\frac{1-(vp)^n}{1-vp}\right]$$

$$= \left[\frac{1-(vp)^n}{1-v(1-q)}\right] = \left[\frac{1-(vp)^n}{1-v+vq)}\right] = \frac{1}{vq+d}[1-(vp)^n]$$

Finally ...

$$P_{\substack{1\\x:\overline{n}|}} = \frac{\frac{vq}{vq+d} \cdot [1-(vp)^n]}{\frac{1}{vq+d} \cdot [1-(vp)^n]} = vq \blacksquare$$

2.

$$A_{\substack{1\\x:\overline{n}|}} = \sum_{k=0}^{n-1} v^{k+1} \cdot {}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1} \cdot \frac{l_{x+k} - l_{x+k+1}}{l_x} = v \sum_{k=0}^{n-1} v^k \cdot \frac{\omega - x - k - \omega + x + k + 1}{\omega - x}$$

$$= v \sum_{k=0}^{n-1} v^k \cdot \frac{1}{\omega - x} = \frac{v}{\omega - x} \cdot \frac{1-v^n}{1-v} = \frac{v}{\omega - x} \cdot \frac{1-v^n}{d} = \frac{v}{\omega - x} \cdot \frac{1-v^n}{iv}$$

$$= \frac{1}{\omega - x} \cdot \frac{1-v^n}{i} = \frac{a_{\overline{n}|}}{\omega - x} \blacksquare$$

To find $\ddot{a}_{x:\overline{n}|}$ *we work through* $A_{x:\overline{n}|}$

$$A_{\substack{\ \ 1\\x:\overline{n}|}} = {}_nE_x = v^n \left[1 - \frac{n}{\omega - x}\right]$$

$$A_{x:\overline{n}|} = \frac{1-v^n}{i(\omega - x)} + v^n \left[1 - \frac{n}{\omega - x}\right]$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - \left(\frac{1-v^n}{i(\omega - x)} + v^n \left[1 - \frac{n}{\omega - x}\right]\right)}{d}$$

$$P_{\substack{1\\x:\overline{n}|}} = \frac{\frac{d}{\omega - x} \cdot \frac{1-v^n}{i}}{1 - \left(\frac{1-v^n}{i(\omega - x)} + v^n \left[1 - \frac{n}{\omega - x}\right]\right)} \blacksquare$$

Premiums: Net premium for standard fully continuous Insurance policies

Whole life	$\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_x} = \dfrac{d \cdot \bar{A}_x}{1 - \bar{A}_x} = \dfrac{1}{\bar{a}_x} - d$
$n-$year term	$\bar{P}\left(\bar{A}^{1}_{x:\overline{n}\mid}\right) = \dfrac{\bar{A}^{1}_{x:\overline{n}\mid}}{\bar{a}_{x:\overline{n}\mid}} = \dfrac{d \cdot \bar{A}^{1}_{x:\overline{n}\mid}}{1 - \bar{A}_{x:\overline{n}\mid}}$
$n-$year pure endowment	$\bar{P}\left(\bar{A}_{x:\overline{n}\mid}^{\;\;1}\right) = \dfrac{\bar{A}_{x:\overline{n}\mid}^{\;\;1}}{\bar{a}_{x:\overline{n}\mid}} = \dfrac{d \cdot \bar{A}_{x:\overline{n}\mid}^{\;\;1}}{1 - \bar{A}_{x:\overline{n}\mid}}$
$n-$year endowment	$\bar{P}(\bar{A}_{x:\overline{n}\mid}) = \dfrac{\bar{A}_{x:\overline{n}\mid}}{\bar{a}_{x:\overline{n}\mid}} = \dfrac{d \cdot \bar{A}_{x:\overline{n}\mid}}{1 - \bar{A}_{x:\overline{n}\mid}} = \dfrac{1}{\bar{a}_{x:\overline{n}\mid}} - d$
$m-$payment whole life	${}_m\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_{x:\overline{m}\mid}} = \dfrac{d \cdot \bar{A}_x}{1 - \bar{A}_{x:\overline{m}\mid}}$
$m-$payment $n-$year term	${}_m\bar{P}\left(\bar{A}^{1}_{x:\overline{n}\mid}\right) = \dfrac{\bar{A}^{1}_{x:\overline{n}\mid}}{\bar{a}_{x:\overline{m}\mid}}$
$m-$payment $n-$year endowment	${}_m\bar{P}(\bar{A}_{x:\overline{n}\mid}) = \dfrac{\bar{A}_{x:\overline{n}\mid}}{\bar{a}_{x:\overline{m}\mid}}$