3) Level Continuous annuity

A continuous annuity is an annuity with a finite term and ~~frequence~~ infinite frequency of payments.

Consider an annuity in which a very small payment dt is made at time t. Let r denote the periodic interest rate. Then the total amount paid during each period is

$$\int_{k-1}^{k} dt = [t]_{k-1}^{k} = \$1$$

$\bar{a}_n$: the present value of an annuity payable continuously for n period

$$\bar{a}_{\overline{n|}} = \int_0^n v^t \, dt = \frac{1}{\ln v} v^t \Big]_0^n = \frac{1-v^n}{\delta}$$

where $\delta = -\ln v = \ln(1+$ $\left(v = e^{-\delta}\right)$

δ : force of interest

Since $1-v^n = r\, a_{\overline{n|}}$, we have

$$\bar{a}_{\overline{n|}} = \frac{r}{\delta} \, a_{\overline{n|}} = \frac{d}{\delta} \ddot{a}_{\overline{n|}} = \frac{1-e^{-n\delta}}{\delta}$$

Ex Starting 4 years from today, you will receive payment at the rate of \$1 000 per annum, payable continuously with the payment terminating ~~twelve~~ 12 years from today. Find the present value of this annuity if $\delta = 5\%$

$n = 8$

$v = e^{-\delta}$

$v^4 = e^{-4\delta} = e^{-0.2}$

Let $\bar{s}_{\overline{n}|}$ denote the accumulated value at the end of the term of an annuity payable continuously for n period so 1 is the total amount paid during each period. Then:

$$\bar{s}_{\overline{n}|} = (1+i)^n \, \bar{a}_{\overline{n}|} = \int_0^n (1+r)^{n-t} \, dt$$

$$= \frac{(1+r)^n - 1}{\delta}$$

<u>Note</u> It is easy to see that:

$$\bar{s}_{\overline{n}|} = \frac{e^{n\delta} - 1}{\delta} = \frac{r}{\delta} s_{\overline{n}|} = \frac{d}{\delta} \ddot{s}_{\overline{n}|}$$

<u>Example</u>

Find the force of interest at wich the accumulated value of a continuous payment of 1 every year for 8 years will be equal to four times the accumulated value of a continuous payment of 1 every year for 4 years

<u>Ans</u>

$$\bar{s}_{\overline{8}|} = 4 \, \bar{s}_{\overline{4}|}$$

$$\frac{e^{8\delta} - 1}{\delta} = 4 \, \frac{e^{4\delta} - 1}{\delta}$$

$$\Rightarrow \quad e^{8\delta} - 4e^{4\delta} + 3 = 0$$

$$(e^{4\delta} - 3)(e^{4\delta} - 1) = 0$$

$$\Rightarrow \delta = \frac{\ln 3}{4} \simeq 2.75\% \quad \text{or} \quad e^{4\delta} = 1 \Rightarrow \delta = 0 \ \times$$

A continuously perpetuity is a perpetuity paid continuously. ~~at a rate of 100~~

Since, we have
$$\frac{1}{\bar{a}_{\overline{n}|}} = \frac{1}{\bar{s}_{\overline{n}|}} + \delta$$

the present value of a perpetuity payable continuously with total of 1 per period is given by:

$$\bar{a}_{\infty} = \lim_{n \to \infty} \bar{a}_{\overline{n}|} = \frac{1}{\delta}$$

<u>Example</u> (Hom) A perpetuity paid continuously at a rate of 100 per year has a present value of 800. Calculate the annual ~~effective~~ interest rate used to calculate the present value.

<u>Ans</u>
$$800 = 100\,\bar{a}_{\infty} = 100 \cdot \frac{1}{\delta} = \frac{100}{Ln(1+r)}$$

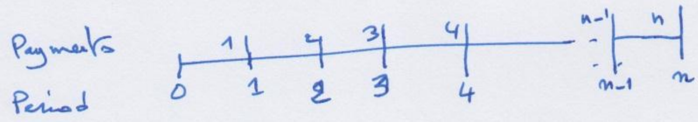
$$\Rightarrow Ln(1+r) = \frac{1}{8}$$
$$\Rightarrow \boxed{r = e^{1/8} - 1} = 13.3\%$$

4) Varying Annuities

In this section, we consider annuities with a varying series of payments.

4-1) Varying Annuity Immediate

a) Payment varying in an Arithmetic Progression



The present value of such annuities is:

$$(Ia)_{\overline{n|}} = v + 2v^2 + 3v^3 + \dots + nv^n$$
$$= v\,(\underbrace{1 + 2v + 3v^2 + \dots nv^{n-1}}_{A})$$

$$\left[\begin{array}{rl} A = & 1 + 2v + 3v^2 + \dots + nv^{n-1} \\ - \quad vA = & \quad v + 2v^2 + \dots + nv^n \\ \hline (1-v)\,A = & \underbrace{1 + v + v^2 + v^3 + \dots + v^{n-1}}_{\ddot{a}_{\overline{n|}}} - nv^n \end{array}\right.$$

$$(Ia)_{\overline{n|}} = \frac{v}{1-v}\left[\ddot{a}_{\overline{n|}} - nv^n\right]$$

$$= \frac{\ddot{a}_{\overline{n|}} - nv^n}{r} = \frac{(1+r)\,a_{\overline{n|}} - nv^n}{r}$$

The accumulated value at $t=n$ of such annuity is:

$$(Is)_{\overline{n}|} = (1+r)^n (I\ddot{a})_{\overline{n}|}$$

$$= \frac{(1+r)^n \ddot{a}_{\overline{n}|} - n(1+r)^n v^n}{r}$$

$$= \frac{\ddot{s}_{\overline{n}|} - n}{r} = \cancel{\frac{s_{\overline{n+1}|} - (n+1)}{r}}$$

$$(s_{\overline{n}|} ?)$$

$$=$$

Example

The following payments ~~will~~ are to be received at the ~~each~~ end of each year:

Year	1	2	3	- - -	16
Payments	500	520	540		800

Using an annual interest rate of 2%

(a) Determine the present values of these payments at time $t=0$

(b) Determine the accumulated value of these payments at the time of last payments

Sol

1 year $\longrightarrow$ $500 + 20 \times 0$

2 year $\longrightarrow$ $500 + 20(2-1)$

3 year $\longrightarrow$ $500 + 20(3-1)$

$\vdots$

n year $\longrightarrow$ $500 + 20(n-1)$

So the total number of payments: $800 = 500 + 20(n-1)$

$\Rightarrow n = 16$