SOLUTION KEY: 1443/Semester-1/Math-244/Final Exam

Solution of Question 1:

a) $adj(A) = C^T = \begin{bmatrix} 30 & -12 & 4 \\ -16 & 6 & -2 \\ 18 & -7 & 2 \end{bmatrix}$ (2 marks)

and $|A| = -2.$ (1 mark)

Hence, $A^{-1} = |A|^{-1} adj(A) = \begin{bmatrix} -15 & 6 & -2 \\ 8 & -3 & 1 \\ -9 & 7/2 & -1 \end{bmatrix}$ (1 mark)

b) $det(det(A)B^2A^{-1}) = (det(A))^3(det(B))^2(det(A))^{-1} = (det(A))^2(det(B))^2 = 36$ (2 marks)

c) $|A| = -2$ and $|B| = 0 \Longrightarrow A$ is invertible but B is non-invertible $\Longrightarrow$ A and B are not row equivalent. (2 marks)

Solution of Question 2:

a) $[A:B] \sim \begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 0 & 1 & \alpha-6 & : & -3 \\ 0 & 0 & 8-2\alpha & : & \beta-6 \end{bmatrix}.$ (2 marks)

Hence, the linear system has:

i) no solution if $\alpha = 4$ and $\beta \neq 6$; (1.5 marks)

ii) infinitely many solutions if $\alpha = 4$ and $\beta = 6$. (1.5 marks)

b) $a = 2, b = -3, c = 1, d = 4.$ (3 marks)

Solution of Question 3:

a) i) $[u_1\ u_2\ u_3\ u_4] \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (R.E.F.) (1 marks)

$\Rightarrow F = span\{u_1, u_2, u_3\} \Rightarrow dim(F) = 3.$ (1 marks)

ii) $\{u_1, u_2, u_3, (1,1,0,-1)\}$ is linearly independent and $F = span\{u_1, u_2, u_3\}$. So, $(1,1,0,1) \notin F$ (2 marks)

b) ${}_BP_C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = [\ [u_1]_B\ \ [u_2]_B\ \ [u_3]_B\ ].$ (1 mark)

Hence, $u_1 = 1v_1 + 1v_2 - 1v_3 = (2,2,-2)$. Similarly, $u_2 = (1,1,8)$ and $u_3 = (3,2,7)$. (3 marks)

Solution of Question 4:

a) i) $\theta = cos^{-1}\frac{1}{\sqrt{3}} \approx 0.955\, rad$ (1 mark)

ii) $u_1 = w_1 = (0,0,1),\ u_2 = w_2 - \frac{<w_2, u_1>}{||u_1||^2}u_1 = (0,1,0)$

and $u_3 = w_3 - \frac{<w_3, u_1>}{||u_1||^2}u_1 - \frac{<w_3, u_2>}{||u_2||^2}u_2 = (1,0,0).$ (3 marks)

So, $\{e_1 = u_1 = (0,0,1), e_2 = u_2 = (0,1,0), e_3 = u_3 = (1,0,0)\}$ is an orthonormal basis of $\mathbb{R}^3$.

b) i) $Ker(T) = \{(0,0)\}$ (1 mark)

ii) From Part i), T is one-one and so $\text{Im}(T) = \mathbb{R}^2$. Hence, $dim\ \text{Im}(T) = 2$. (1 mark)

c) $[T]_B^C = [[T(1,0)]_C\ [T(0,1)]_C] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}.$ (2 marks)

Solution of Question 5:

a) Eigenvalues = 1, 1, 1 (1 mark)

$E_1 = span\{(0,0,1)\}$ (1 mark)

So, the algebraic multiplicity of the eigenvalue 1 is 3 which is different from its geometric multiplicity 1.

Hence, the given matrix A is not diagonalizable. (2 mark)

b) $P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, so that $D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Thus, $A^{-1} = PD^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$. (1+1+2 marks)

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