King Saud University
Faculty of Sciences
Department of Mathematics

Midterm2	Math 106	November 2023

Question 1 : (3+3+3)

1. Find $\lim_{x\to 0^+}\left(e^{-2x}-2x\right)^{\frac{1}{x}}$.
2. Compute $\int (x^5+1)\ln x\,dx$.
3. Evaluate the integral $\int \sin^5 x\cos^8 x\,dx$.

Question 2 : (2+3+3)

1. Compute $\int \frac{\cos(2x)}{\sec(5x)}dx$.
2. Find the indefinite integral $\int \frac{3x+1}{(4+x^2)^{\frac{3}{2}}}dx$.
3. Evaluate $\int \frac{x^2}{x^2-x-2}dx$.

Question 3 : (2+3+3)

1. Evaluate the integral $\int \frac{dx}{\sqrt{x^2+6x+13}}$.
2. Compute $\int \frac{dx}{4x^{\frac{2}{3}}+x^{\frac{4}{3}}}$.
3. Does the integral $\int_0^{+\infty}\frac{e^{-x}}{(1+e^{-x})^2}dx$ converge?
   Find its value if it does..

Question 1 :

1.

$$\lim_{x\to 0^+}\left(e^{-2x}-2x\right)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{\ln(e^{-2x}-2x)}{x}} \quad (\mathbf{1})$$

$$= \lim_{x\to 0^+} e^{\frac{-2e^{-2x}-2}{e^{-2x}-2x}} = e^{-4} \quad (\mathbf{2})$$

2. By parts

$$\int (x^5+1)\ln x dx = (\frac{x^6}{6}+x)\ln x - \int (\frac{x^5}{6}+1)dx \quad (\mathbf{2})$$

$$= (\frac{x^6}{6}+x)\ln x - \frac{x^6}{36} - x + c \quad (\mathbf{1})$$

3.

$$\int \sin^5 x \cos^8 x dx \overset{u=\cos x}{=} -\int u^8(1-u^2)^2 du \quad (\mathbf{1.5})$$

$$= -\frac{u^9}{9}+2\frac{u^{11}}{11}-\frac{u^{13}}{13}+c$$

$$= -\frac{\cos^9 x}{9}+2\frac{\cos^{11} x}{11}-\frac{\cos^{13} x}{13}+c. \quad (\mathbf{1.5})$$

Question 2 :

1.

$$\int \frac{\cos(2x)}{\sec(5x)}dx = \int \cos(2x)\cos(5x)dx$$

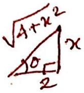
$$= \frac{1}{2}\int (\cos(3x)+\cos(7x))\,dx \quad (\mathbf{1})$$

$$= \frac{\sin(3x)}{6}+\frac{\sin(7x)}{14}+c. \quad (\mathbf{1})$$

$$= \int \frac{6\tan\theta+1}{8\sec^3\theta}\, 2\sec^2\theta\, d\theta = \frac{1}{4}\int \frac{6\tan\theta+1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int (6\tan\theta+1)\cos\theta\, d\theta$$

2.



$$\int \frac{3x+1}{(4+x^2)^{\frac{3}{2}}}dx \overset{x=2\tan\theta}{=} \frac{1}{4}\int (6\sin\theta+\cos\theta)d\theta \quad \textbf{(2)}$$

$$= \frac{-6}{4}\cos\theta + \frac{1}{4}\sin\theta + c$$

$$= -\frac{3}{\sqrt{4+x^2}} + \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + c. \quad \textbf{(1)}$$

3.

$$x^2 \div (x^2-x-2) = 1, \quad x^2 - (x^2-x-2) = x+2$$

$$\frac{x^2}{x^2-x-2} = 1 + \frac{x+2}{(x+1)(x-2)}$$

$$\int \frac{x^2}{x^2-x-2}dx = \int \frac{x^2}{(x+1)(x-2)}dx$$

$$= \int \left(1 - \frac{1}{3(x+1)} + \frac{4}{3(x-2)}\right) dx \quad \textbf{(2)}$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{4}{3}\ln|x-2| + c. \quad \textbf{(1)}$$

Question 3 :

1.

$$\int \frac{dx}{\sqrt{x^2+6x+13}} = \int \frac{dx}{\sqrt{(x+3)^2+4}} \quad \textbf{(1)}$$

$$= \sinh^{-1}\left(\frac{x+3}{2}\right) + c. \quad \textbf{(1)}$$

2.

$$\int \frac{dx}{4x^{\frac{2}{3}} + x^{\frac{4}{3}}} \overset{x=t^3}{=} 3\int \frac{dt}{4+t^2} \quad \textbf{(2)}$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{t}{2}\right) + c = \frac{3}{2}\tan^{-1}\left(\frac{x^{\frac{1}{3}}}{2}\right) + c. \quad \textbf{(1)}$$

3.

$$\int_0^{+\infty} \frac{e^{-x}}{(1+e^{-x})^2}dx \overset{t=e^{-x}}{=} \int_0^1 \frac{dt}{(1+t)^2} = \frac{1}{2}. \quad \textbf{(2)} + \textbf{(1)}$$

(Implicitely $\int_0^c \frac{e^{-x}}{(1+e^{-x})^2}dx = \left[\frac{1}{1+e^{-x}}\right]_0^c \xrightarrow[c\to+\infty]{} \frac{1}{2}$)