Instructor's Manual to Accompany

Introduction to Probability Models

Ninth Edition

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Chapter 1

1. $S = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),(B,R),(B,G),(B,B)\}$.

   The probability of each point in S is $1/9$.

2. $S = \{(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}$.

3. $S = \{(e_1, e_2, \ldots, e_n),\ n \geq 2\}$ where $e_i \in$ (heads, tails$\}$. In addition, $e_n = e_{n-1} =$ heads and for $i = 1, \ldots, n-2$ if $e_i =$ heads, then $e_{i+1} =$ tails.

   $$\begin{aligned} P\{4 \text{ tosses}\} &= P\{(t,t,h,h)\} + P\{(h,t,h,h)\} \\ &= 2\left[\frac{1}{2}\right]^4 = \frac{1}{8}. \end{aligned}$$

4. (a) $F(E \cup G)^c = FE^cG^c$.

   (b) EFG^c.

   (c) $E \cup F \cup G$.

   (d) $EF \cup EG \cup FG$.

   (e) EFG.

   (f) $(E \cup F \cup G)^c = E^cF^cG^c$.

   (g) $(EF)^c(EG)^c(FG)^c$.

   (h) $(EFG)^c$.

5. $\frac{3}{4}$. If he wins, he only wins \$1, while if he loses, he loses \$3.

6. If $E(F \cup G)$ occurs, then E occurs and either F or G occur; therefore, either EF or EG occurs and so

   $E(F \cup G) \subset EF \cup EG$.

   Similarly, if $EF \cup EG$ occurs, then either EF or EG occur. Thus, E occurs and either F or G occurs; and so $E(F \cup G)$ occurs. Hence,

   $EF \cup EG \subset E(F \cup G)$,

   which together with the reverse inequality proves the result.

7. If $(E \cup F)^c$ occurs, then $E \cup F$ does not occur, and so E does not occur (and so E^c does); F does not occur (and so F^c does) and thus E^c and F^c both occur. Hence,

   $(E \cup F)^c \subset E^cF^c$.

   If E^cF^c occurs, then E^c occurs (and so E does not), and F^c occurs (and so F does not). Hence, neither E or F occur and thus $(E \cup F)^c$ does. Thus,

   $E^cF^c \subset (E \cup F)^c$

   and the result follows.

8. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$.

9. $F = E \cup FE^c$, implying since E and FE^c are disjoint that $P(F) = P(E) + P(FE)^c$.

10. Either by induction or use

    $$\overset{n}{\underset{1}{\cup}} E_i = E_1 \cup E_1^cE_2 \cup E_1^cE_2^cE_3 \cup \cdots \cup E_1^c \cdots E_{n-1}^cE_n,$$

    and as each of the terms on the right side are mutually exclusive:

    $$\begin{aligned} P(\underset{i}{\cup} E_i) &= P(E_1) + P(E_1^cE_2) + P(E_1^cE_2^cE_3) + \cdots \\ &\quad + P(E_1^c \cdots E_{n-1}^cE_n) \\ &\leq P(E_1) + P(E_2) + \cdots + P(E_n). \quad \text{(why?)} \end{aligned}$$

11. $$P\{\text{sum is } i\} = \begin{cases} \dfrac{i-1}{36}, & i = 2, \ldots, 7 \\ \dfrac{13-i}{36}, & i = 8, \ldots, 12. \end{cases}$$

12. Either use hint or condition on initial outcome as:

    $$\begin{aligned} &P\{E \text{ before } F\} \\ &= P\{E \text{ before } F \mid \text{initial outcome is } E\}P(E) \\ &\quad + P\{E \text{ before } F \mid \text{initial outcome is } F\}P(F) \\ &\quad + P\{E \text{ before } F \mid \text{initial outcome neither E} \\ &\qquad \text{or } F\}[1 - P(E) - P(F)] \end{aligned}$$

$$= 1 \cdot P(E) + 0 \cdot P(F) + P\{E \text{ before } F\}$$
$$= [1 - P(E) - P(F)].$$

Therefore, $P\{E \text{ before } F\} = \dfrac{P(E)}{P(E)+P(F)}$.

13. Condition an initial toss

$$P\{\text{win}\} = \sum_{i=2}^{12} P\{\text{win} \mid \text{throw } i\} P\{\text{throw } i\}.$$

Now,

$$P\{\text{win}| \text{ throw } i\} = P\{i \text{ before } 7\}$$
$$= \begin{cases} 0 & i = 2, 12 \\ \dfrac{i-1}{5+1} & i = 3, \ldots, 6 \\ 1 & i = 7, 11 \\ \dfrac{13-i}{19-1} & i = 8, \ldots, 10, \end{cases}$$

where above is obtained by using Problems 11 and 12.

$P\{\text{win}\} \approx .49$.

14. $$P\{A \text{ wins}\} = \sum_{n=0}^{\infty} P\{A \text{ wins on } (2n+1)\text{st toss}\}$$
$$= \sum_{n=0}^{\infty} (1-P)^{2n} P$$
$$= P \sum_{n=0}^{\infty} [(1-P)^2]^n$$
$$= P \frac{1}{1-(1-P)^2}$$
$$= \frac{P}{2P - P^2}$$
$$= \frac{1}{2-P}.$$
$$P\{B \text{ wins}\} = 1 - P\{A \text{ wins}\}$$
$$= \frac{1-P}{2-P}.$$

16. $$P(E \cup F) = P(E \cup FE^c)$$
$$= P(E) + P(FE^c)$$

since E and FE^c are disjoint. Also,

$$P(F) = P(FE \cup FE^c)$$
$$= P(FE) + P(FE^c) \text{ by disjointness.}$$

Hence,

$$P(E \cup F) = P(E) + P(F) - P(EF).$$

17. $$\text{Prob}\{\text{end}\} = 1 - \text{Prob}\{\text{continue}\}$$
$$= 1 - P(\{H,H,H\} \cup \{T,T,T\})$$
$$= 1 - [\text{Prob}(H,H,H) + \text{Prob}(T,T,T)].$$

$$\text{Fair coin: Prob}\{\text{end}\} = 1 - \left[\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right]$$
$$= \frac{3}{4}.$$

$$\text{Biased coin: } P\{\text{end}\} = 1 - \left[\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right]$$
$$= \frac{9}{16}.$$

18. Let B = event both are girls; E = event oldest is girl; L = event at least one is a girl.

(a) $P(B|E) = \dfrac{P(BE)}{P(E)} = \dfrac{P(B)}{P(E)} = \dfrac{1/4}{1/2} = \dfrac{1}{2}.$

(b) $P(L) = 1 - P(\text{no girls}) = 1 - \dfrac{1}{4} = \dfrac{3}{4},$

$$P(B|L) = \frac{P(BL)}{P(L)} = \frac{P(B)}{P(L)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

19. E = event at least 1 six $P(E)$

$$= \frac{\text{number of ways to get } E}{\text{number of sample pts}} = \frac{11}{36}$$

D = event two faces are different $P(D)$

$$= 1 - \text{Prob(two faces the same)}$$
$$= 1 - \frac{6}{36} = \frac{5}{6} P(E|D) = \frac{P(ED)}{P(D)} = \frac{10/36}{5/6} = \frac{1}{3}.$$

20. Let E = event same number on exactly two of the dice; S = event all 3 numbers are the same; D = event all 3 numbers are different. These 3 events are mutually exclusive and define the whole sample space. Thus, $1 = P(D) + P(S) + P(E)$, $P(S) = 6/216 = 1/36$; for D have 6 possible values for first die, 5 for second, and 4 for third.

$\therefore$ Number of ways to get $D = 6 \cdot 5 \cdot 4 = 120$.

$$P(D) = 120/216 = 20/36$$
$$\therefore P(E) = 1 - P(D) - P(S)$$
$$= 1 - \frac{20}{36} - \frac{1}{36} = \frac{5}{12}.$$

21. Let C = event person is color blind.

$P(\text{Male}|C)$

$$= \frac{P(C|\text{Male})\,P(\text{Male})}{P(C|\text{Male}\,P(\text{Male}) + P(C|\text{Female})\,P(\text{Female})}$$

$$= \frac{.05 \times .5}{.05 \times .5 + .0025 \times .5}$$

$$= \frac{2500}{2625} = \frac{20}{21}.$$

22. Let trial 1 consist of the first two points; trial 2 the next two points, and so on. The probability that each player wins one point in a trial is $2p(1-p)$. Now a total of $2n$ points are played if the first $(a-1)$ trials all result in each player winning one of the points in that trial and the n^{th} trial results in one of the players winning both points. By independence, we obtain that

$P\{2n \text{ points are needed}\}$
$= (2p(1-p))^{n-1}(p^2 + (1-p)^2), \quad n \geq 1.$

The probability that A wins on trial n is $(2p(1-p))^{n-1}p^2$ and so

$$P\{A \text{ wins}\} = p^2 \sum_{n=1}^{\infty} (2p(1-p))^{n-1}$$
$$= \frac{p^2}{1-2p(1-p)}.$$

23. $P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$

$$= P(E_1)\frac{P(E_1E_2)}{P(E_1)}\frac{P(E_1E_2E_3)}{P(E_1E_2)}\cdots\frac{P(E_1\cdots E_n)}{P(E_1\cdots E_{n-1})}$$

$$= P(E_1\cdots E_n).$$

24. Let a signify a vote for A and b one for B.

(a) $P_{2,1} = P\{a,a,b\} = 1/3.$

(b) $P_{3,1} = P\{a,a\} = (3/4)(2/3) = 1/2.$

(c) $P_{3,2} = P\{a,a,a\} + P\{a,a,b,a\}$
$= (3/5)(2/4)[1/3 + (2/3)(1/2)] = 1/5.$

(d) $P_{4,1} = P\{a,a\} = (4/5)(3/4) = 3/5.$

(e) $P_{4,2} = P\{a,a,a\} + P\{a,a,b,a\}$
$= (4/6)(3/5)[2/4 + (2/4)(2/3)] = 1/3.$

(f) $P_{4,3} = P\{\text{always ahead}|a,a\}(4/7)(3/6)$
$= (2/7)[1 - P\{a,a,a,b,b,b|a,a\}$
$- P\{a,a,b,b|a,a\} - P\{a,a,b,a,b,b|a,a\}]$
$= (2/7)[1 - (2/5)(3/4)(2/3)(1/2)$
$- (3/5)(2/4) - (3/5)(2/4)(2/3)(1/2)]$
$= 1/7.$

(g) $P_{5,1} = P\{a,a\} = (5/6)(4/5) = 2/3.$

(h) $P_{5,2} = P\{a,a,a\} + P\{a,a,b,a\}$
$= (5/7)(4/6)[(3/5) + (2/5)(3/4)] = 3/7.$

By the same reasoning we have that

(i) $P_{5,3} = 1/4,$

(j) $P_{5,4} = 1/9.$

(k) In all the cases above, $P_{n,m} = \dfrac{n-n}{n+n}.$

25. (a) $P\{pair\} = P\{\text{second card is same denomination as first}\}$
$= 3/51.$

(b) $P\{\text{pair}|\text{different suits}\}$
$$= \frac{P\{\text{pair, different suits}\}}{P\{\text{different suits}\}}$$
$= P\{\text{pair}\}/P\{\text{different suits}\}$
$$= \frac{3/51}{39/51} = 1/13.$$

26. $P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} = \dfrac{39\cdot 38\cdot 37}{51\cdot 50\cdot 49}.$

$P(E_2|E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} = \dfrac{26\cdot 25}{38\cdot 37}.$

$P(E_3|E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13} = 13/25.$

$P(E_4|E_1E_2E_3) = 1.$

$$P(E_1E_2E_3E_4) = \frac{39\cdot 26\cdot 13}{51\cdot 50\cdot 49}.$$

27. $P(E_1) = 1$
$P(E_2|E_1) = 39/51$, since 12 cards are in the ace of spades pile and 39 are not.

$P(E_3|E_1E_2) = 26/50$, since 24 cards are in the piles of the two aces and 26 are in the other two piles.

$P(E_4|E_1E_2E_3) = 13/49.$

So

$P\{\text{ each pile has an ace}\} = (39/51)(26/50)(13/49).$

28. Yes. $P(A|B) > P(A)$ is equivalent to $P(AB) > P(A)P(B)$ which is equivalent to $P(B|A) > P(B)$.

29. (a) $P(E|F) = 0$.

(b) $P(E|F) = P(EF)/P(F) = P(E)/P(F) \geq P(E) = .6$.

(c) $P(E|F) = P(EF)/P(F) = P(F)/P(F) = 1$.

30. (a) $P\{\text{George}|\text{exactly 1 hit}\}$

$$= \frac{P\{\text{George, not Bill}\}}{P\{\text{exactly 1}\}}$$
$$= \frac{P\{G, \text{ not } B\}}{P\{G, \text{ not } B\} + P\{B, \text{ not } G)\}}$$
$$= \frac{(.4)(.3)}{(.4)(.3) + (.7)(.6)}$$
$$= 2/9.$$

(b) $P\{G|\text{hit}\}$

$$= P\{G, \text{hit}\}/P\{\text{hit}\}$$
$$= P\{G\}/P\{\text{hit}\} = .4/[1 - (.3)(.6)]$$
$$= 20/41.$$

31. Let S = event sum of dice is 7; F = event first die is 6.

$$P(S) = \frac{1}{6} P(FS) = \frac{1}{36} P(F|S) = \frac{P(F|S)}{P(S)}$$
$$= \frac{1/36}{1/6} = \frac{1}{6}.$$

32. Let E_i = event person i selects own hat.
P (no one selects own hat)

$$= 1 - P(E_1 \cup E_2 \cup \cdots \cup E_n)$$
$$= 1 - \left[\sum_{i_1} P(Ei_1) - \sum_{i_1<i_2} P(Ei_1Ei_2) + \cdots \right.$$
$$\left. +(-1)^{n+1} P(E_1E_2E_n) \right]$$
$$= 1 - \sum_{i_1} P(Ei_1) - \sum_{i_1<i_2} P(Ei_1Ei_2)$$
$$- \sum_{i_1<i_2<i_3} P(Ei_1Ei_2Ei_3) + \cdots$$
$$+ (-1)^n P(E_1E_2E_n).$$

Let $k \in \{1, 2, \ldots, n\}$. $P(Ei_1EI_2Ei_k)$ = number of ways k specific men can select own hats $\div$ total number of ways hats can be arranged $= (n-k)!/n!$. Number of terms in summation $\Sigma_{i_1<i_2<\cdots<i_k}$ = number of ways to choose k variables out of n variables $= \begin{bmatrix} n \\ k \end{bmatrix} = n!/k!(n-k)!$.

Thus,

$$\sum_{i_1<\cdots<i_k} P(Ei_1Ei_2 \cdots Ei_k)$$
$$= \sum_{i_1<\cdots<i_k} \frac{(n-k)!}{n!}$$
$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(n-k)!}{n!} = \frac{1}{k!}.$$

$\therefore P$(no one selects own hat)

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$
$$= \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}.$$

33. Let S = event student is sophomore; F = event student is freshman; B = event student is boy; G = event student is girl. Let x = number of sophomore girls; total number of students = $16 + x$.

$$P(F) = \frac{10}{16+x} P(B) = \frac{10}{16+x} P(FB) = \frac{4}{16+x}$$
$$\frac{4}{16+x} = P(FB) = P(F)P(B) = \frac{10}{16+x}.$$
$$\frac{10}{16+x} \Rightarrow x = 9.$$

34. Not a good system. The successive spins are independent and so

$$P\{11^{\text{th}} \text{ is red}|\text{1st 10 black}\} = P\ \{11^{\text{th}} \text{ is red}\}$$
$$= P\left[= \frac{18}{38}\right].$$

35. (a) 1/16

(b) 1/16

(c) 15/16, since the only way in which the pattern H, H, H, H can appear before the pattern T, H, H, H is if the first 4 flips all land heads.

36. Let B = event marble is black; B_i = event that box i is chosen. Now

$$B = BB_1 \cup BB_2 P(B) = P(BB_1) + P(BB_2)$$
$$= P(B|B_1)P(B_1) + P(B|B_2)P(B_2)$$
$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{12}.$$

37. Let W = event marble is white.

$$P(B_1|W) = \frac{P(W|B_1)P(B_1)}{P(W|B_1)P(B_1)+P(W|B_2)P(B_2)}$$
$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}.$$

38. Let T_W = event transfer is white; T_B = event transfer is black; W = event white ball is drawn from urn 2.

$$P(T_W|W) = \frac{P(W|T_W)P(T_W)}{P(W|T_W)P(T_W)+P(W|T_B)P(T_B)}$$
$$= \frac{\frac{2}{7}\cdot\frac{2}{3}}{\frac{2}{7}\cdot\frac{2}{3}+\frac{1}{7}\cdot\frac{1}{3}} = \frac{\frac{4}{21}}{\frac{5}{21}} = \frac{4}{5}.$$

39. Let W = event woman resigns; A, B, C are events the person resigning works in store A, B, C, respectively.

$$P(C|W)$$
$$= \frac{P(W|C)P(C)}{P(W|C)P(C)+P(W|B)P(B)+P(W|A)P(A)}$$
$$= \frac{.70\times\frac{100}{225}}{.70\times\frac{100}{225}+.60\times\frac{75}{225}+.50\,\frac{50}{225}}$$
$$= \frac{70}{225}\Big/\frac{140}{225} = \frac{1}{2}.$$

40. (a) F = event fair coin flipped; U = event two-headed coin flipped.

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F)+P(H|U)P(U)}$$
$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

(b) $$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH|F)P(F)+P(HH|U)P(U)}$$
$$= \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5}.$$

(c) $$P(F|HHT)$$
$$= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F)+P(HHT|U)P(U)}$$
$$= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F)+0} = 1,$$

since the fair coin is the only one that can show tails.

41. Note first that since the rat has black parents and a brown sibling, we know that both its parents are hybrids with one black and one brown gene (for if either were a pure black then all their offspring would be black). Hence, both of their offspring's genes are equally likely to be either black or brown.

(a) $P(\text{2 black genes} \mid \text{at least one black gene})$
$$= \frac{P(\text{2 black genes})}{P(\text{at least one black gene})}$$
$$= \frac{1/4}{3/4} = 1/3$$

(b) Using the result from part (a) yields the following:

$$P(\text{2 black genes} \mid \text{5 black offspring})$$
$$= \frac{P(\text{2 black genes})}{P(\text{5 black offspring})}$$
$$= \frac{1/3}{1(1/3)+(1/2)^5(2/3)}$$
$$= 16/17$$

where P(5 black offspring) was computed by conditioning on whether the rat had 2 black genes.

42. Let B = event biased coin was flipped; F & U (same as above).

$$P(U|H)$$
$$= \frac{P(H|U)P(U)}{P(H|U)P(U)+P(H|B)P(B)+P(H|F)P(F)}$$
$$= \frac{1\cdot\frac{1}{3}}{1\cdot\frac{1}{3}+\frac{3}{4}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{9}{12}} = \frac{4}{9}.$$

43. Let i = event coin was selected; $P(H|i) = \frac{i}{10}$.

$$P(5|H) = \frac{P(H|5)P(5)}{\sum_{i=1}^{10} P(H|i)P(i)} = \frac{\frac{5}{10}\cdot\frac{1}{10}}{\sum_{i=1}^{10}\frac{1}{10}\cdot\frac{1}{10}}$$
$$= \frac{5}{\sum_{i=1}^{10} i} = \frac{1}{11}.$$

44. Let $W =$ event white ball selected.

$$P(T|W) = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|H)P(H)}$$
$$= \frac{\frac{1}{5}\cdot\frac{1}{2}}{\frac{1}{5}\cdot\frac{1}{2} + \frac{5}{12}\cdot\frac{1}{2}} = \frac{12}{37}.$$

45. Let $B_i =$ event i^{th} ball is black; $R_i =$ event i^{th} ball is red.

$$P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)}$$
$$= \frac{\frac{r}{b+r+c}\cdot\frac{b}{b+r}}{\frac{r}{b+r+c}\cdot\frac{b}{b+r} + \frac{r+c}{b+r+c}\cdot\frac{r}{b+r}}$$
$$= \frac{rb}{rb + (r+c)r}$$
$$= \frac{b}{b+r+c}.$$

46. Let $X(=B \text{ or } =C)$ denote the jailer's answer to prison A. Now for instance,

$$P\{A \text{ to be executed}|X = B\}$$
$$= \frac{P\{A \text{ to be executed, } X = B\}}{P\{X = B\}}$$
$$= \frac{P\{A\text{to be executed}\}P\{X = B|A \text{ to be executed}}{P\{X = B\}}$$
$$= \frac{(1/3)P\{X = B|A \text{ to be executed}\}}{1/2}.$$

Now it is reasonable to suppose that if A is to be executed, then the jailer is equally likely to answer either B or C. That is,

$$P\{X = B|\, A \text{ to be executed}\} = \frac{1}{2}$$

and so,

$$P\{A \text{ to be executed}|X = B\} = \frac{1}{3}$$

Similarly,

$$P\{A \text{ to be executed}|X = C\} = \frac{1}{3}$$

and thus the jailer's reasoning is invalid. (It is true that if the jailer were to answer B, then A knows that the condemned is either himself or C, but it is twice as likely to be C.)

47. 1. $0 \le P(A|B) \le 1$

2. $P(S|B) = \dfrac{P(SB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

3. For disjoint events A and D

$$P(A \cup D|B) = \frac{P((A \cup D)B)}{P(B)}$$
$$= \frac{P(AB \cup DB)}{P(B)}$$
$$= \frac{P(AB) + P(DB)}{P(B)}$$
$$= P(A|B) + P(D|B)$$

Direct verification is as follows:

$$P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$$
$$= \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)}\frac{P(BC^c)}{P(B)}$$
$$= \frac{P(ABC)}{P(B)} + \frac{P(ABC^c)}{P(B)}$$
$$= \frac{P(AB)}{P(B)}$$
$$= P(A|B)$$

Chapter 2

1. $P\{X=0\} = \left[\begin{matrix}7\\2\end{matrix}\right] \Big/ \left[\begin{matrix}10\\2\end{matrix}\right] = \dfrac{14}{30}$.

2. $-n, -n+2, -n+4, \ldots, n-2, n$.

3. $P\{X=-2\} = \dfrac{1}{4} = P\{X=2\}$

   $P\{X=0\} = \dfrac{1}{2}$.

4. (i) $1, 2, 3, 4, 5, 6$.
   (ii) $1, 2, 3, 4, 5, 6$.
   (iii) $2, 3, \ldots, 11, 12$.
   (iv) $-5, -4, \ldots, 4, 5$.

5. $P\{\max = 6\} = \dfrac{11}{36} = P\{\min = 1\}$

   $P\{\max = 5\} = \dfrac{1}{4} = P\{\min = 2\}$

   $P\{\max = 4\} = \dfrac{7}{36} = P\{\min = 3\}$

   $P\{\max = 3\} = \dfrac{5}{36} = P\{\min = 4\}$

   $P\{\max = 2\} = \dfrac{1}{12} = P\{\min = 5\}$

   $P\{\max = 1\} = \dfrac{1}{36} = P\{\min = 6\}$.

6. (H, H, H, H, H), p^5 if $p = P\,\{\text{heads}\}$.

7. $p(0) = (.3)^3 = .027$

   $p(1) = 3(.3)^2(.7) = .189$

   $p(2) = 3(.3)(.7)^2 = .441$

   $p(3) = (.7)^3 = .343$.

8. $p(0) = \dfrac{1}{2}, \quad p(1) = \dfrac{1}{2}$.

9. $p(0) = \dfrac{1}{2}, \quad p(1) = \dfrac{1}{10}, \quad p(2) = \dfrac{1}{5},$

   $p(3) = \dfrac{1}{10}, \quad p(3.5) = \dfrac{1}{10}$.

10. $1 - \left[\begin{matrix}3\\2\end{matrix}\right]\left[\dfrac{1}{6}\right]^2\left[\dfrac{5}{6}\right] - \left[\begin{matrix}3\\3\end{matrix}\right]\left[\dfrac{1}{6}\right]^3 = \dfrac{200}{216}$.

11. $\dfrac{3}{8}$.

12. $\left[\begin{matrix}5\\4\end{matrix}\right]\left[\dfrac{1}{3}\right]^4\left[\dfrac{2}{3}\right] + \left[\begin{matrix}5\\5\end{matrix}\right]\left[\dfrac{1}{3}\right]^5 = \dfrac{10+1}{243} = \dfrac{11}{243}$.

13. $\sum\limits_{i=7}^{10}\left[\begin{matrix}10\\1\end{matrix}\right]\left[\dfrac{1}{2}\right]^{10}$.

14. $P\{X=0\} = P\{X=6\} = \left[\dfrac{1}{2}\right]^6 = \dfrac{1}{64}$

    $P\{X=1\} = P\{X=5\} = 6\left[\dfrac{1}{2}\right]^6 = \dfrac{6}{64}$

    $P\{X=2\} = P\{X=4\} = \left[\begin{matrix}6\\2\end{matrix}\right]\left[\dfrac{1}{2}\right]^6 = \dfrac{15}{64}$

    $P\{X=3\} = \left[\begin{matrix}6\\3\end{matrix}\right]\left[\dfrac{1}{2}\right]^6 = \dfrac{20}{64}$.

15. $$\frac{P\{X=k\}}{P\{X=k-1\}} = \frac{\dfrac{n!}{(n-k)!\,k!}p^k(1-p)^{n-k}}{\dfrac{n!}{(n-k+1)!(k-1)!}p^{k-1}(1-p)^{n-k+1}} = \frac{n-k+1}{k}\,\frac{p}{1-p}.$$

    Hence,

    $$\frac{P\,\{X=k\}}{P\,\{X=k-1\}} \geq 1 \leftrightarrow (n-k+1)p > k(1-p) \leftrightarrow (n+1)p \geq k.$$

    The result follows.

16. $1 - (.95)^{52} - 52(.95)^{51}(.05)$.

17. Follows since there are $\dfrac{n!}{x_1! \cdots x_r!}$ permutations of n objects of which x_1 are alike, x_2 are alike, $\ldots$, x_r are alike.

18. Follows immediately.

19. $P\{X_1 + \cdots + X_k = m\}$

$$= \begin{bmatrix} n \\ m \end{bmatrix} (p_1 + \cdots + p_k)^m (p_{k+1} + \cdots + p_r)^{n-m}.$$

20. $\dfrac{5!}{2!1!2!} \left[\dfrac{1}{5}\right]^2 \left[\dfrac{3}{10}\right]^2 \left[\dfrac{1}{2}\right]^1 = .054.$

21. $1 - \left[\dfrac{3}{10}\right]^5 - 5\left[\dfrac{3}{10}\right]^4 \left[\dfrac{7}{10}\right] - \left[\dfrac{5}{2}\right] \left[\dfrac{3}{10}\right]^3 \left[\dfrac{7}{10}\right]^2.$

22. $\dfrac{1}{32}.$

23. In order for X to equal n, the first $n-1$ flips must have $r-1$ heads, and then n^{th} flip must land heads. By independence the desired probability is thus

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix} p^{r-1}(1-p)^{n-r} x p.$$

24. It is the number of tails before heads appears for the r^{th} time.

25. A total of 7 games will be played if the first 6 result in 3 wins and 3 losses. Thus,

$$P\{7 \text{ games}\} = \binom{6}{3} p^3(1-p)^3.$$

Differentiation yields that

$$\frac{d}{dp}P\{7\} = 20\left[3p^2(1-p)^3 - p^3 3(1-p)^2\right]$$
$$= 60p^2(1-p)^2[1 - 2p].$$

Thus, the derivative is zero when $p = 1/2$. Taking the second derivative shows that the maximum is attained at this value.

26. Let X denote the number of games played.

(a) $P\{X = 2\} = p^2 + (1-p)^2$
$P\{X = 3\} = 2p(1-p)$
$E[X] = 2\left\{p^2 + (1-p)^2\right\} + 6p(1-p)$
$= 2 + 2p(1-p).$
Since $p(1-p)$ is maximized when $p = 1/2$, we see that $E[X]$ is maximized at that value of p.

(b) $P\{X = 3\} = p^3 + (1-p)^3.$
$P\{X = 4\}$
$= P\{X = 4, \text{ I has 2 wins in first 3 games}\}$
$+P\{X = 4, \text{ II has 2 wins in first 3 games}\}$
$= 3p^2(1-p)p + 3p(1-p)^2(1-p).$
$P\{X = 5\}$
$= P\{\text{each player has 2 wins in the first 4 games}\}$
$= 6p^2(1-p)^2.$
$E[X] = 3\left[p^3 + (1-p)^3\right] + 12p(1-p)\left[p^2 + (1-p)^2\right] + 30p^2(1-p)^2.$
Differentiating and setting equal to 0 shows that the maximum is attained when $p = 1/2$

27. $P\{\text{same number of heads}\} = \sum_i P\{A = i, B = i\}$

$$= \sum_i \binom{k}{i}(1/2)^k \binom{n-k}{i}(1/2)^{n-k}$$
$$= \sum_i \binom{k}{i}\binom{n-k}{i}(1/2)^n$$
$$= \sum_i \binom{k}{k-i}\binom{n-k}{i}(1/2)^n$$
$$= \binom{n}{k}(1/2)^n.$$

Another argument is as follows.

$P\{\#\text{ heads of } A = \#\text{ heads of } B\}$
$= P\{\#\text{ tails of } A = \#\text{ heads of } B\}$

since coin is fair

$= P\{k - \#\text{ heads of } A = \#\text{ heads of } B\}$
$= P\{k = \text{total \# heads}\}.$

28. (a) Consider the first time that the two coins give different results. Then

$$P\{X = 0\} = P\{(t,h) | (t,h) \text{ or } (h,t)\}$$
$$= \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

(b) No, with this procedure

$$P\{X = 0\} = P\{\text{first flip is a tail}\} = 1 - p.$$

29. Each flip after the first will, independently, result in a changeover with probability $1/2$. Therefore,

$$P\{k \text{ changeovers}\} = \binom{n-1}{k}(1/2)^{n-1}.$$

30. $$\frac{P\{X=i\}}{P\{X=i-1\}} = \frac{e^{-\lambda}\lambda^i/i!}{e^{-\lambda}\lambda^{i-1}/(i-1)!} = \lambda/i.$$

Hence, $P\{X=i\}$ is increasing for $\lambda \geq i$ and decreasing for $\lambda < i$.

32. (a) .394 (b) .303 (c) .091.

33. $$c\int_{-1}^{1}\left(1-x^2\right)dx = 1$$
$$c\left[x-\frac{x^3}{3}\right]\Bigg|_{-1}^{1} = 1$$
$$c = \frac{3}{4}.$$
$$F(y) = \frac{3}{4}\int_{-1}^{1}(1-x^2)dx$$
$$= \frac{3}{4}\left[y-\frac{y^3}{3}+\frac{2}{3}\right], \qquad -1 < y < 1.$$

34. $$c\int_0^2\left(4x-2x^2\right)dx = 1$$
$$c(2x^2-2x^3/3) = 1$$
$$8c/3 = 1$$
$$c = \frac{3}{8}.$$
$$P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{3}{8}\int_{1/2}^{3/2}\left(4x-2x^2\right)dx$$
$$= \frac{11}{16}.$$

35. $$P\{X > 20\} = \int_{20}^{\infty}\frac{10}{x^2}dx = \frac{1}{2}.$$

36. $$P\{D \leq x\} = \frac{\text{area of disk of radius } x}{\text{area of disk of radius } 1}$$
$$= \frac{\pi x^2}{\pi} = x^2$$

37. $$P\{M \leq x\} = P\{\max(X_1,\ldots,X_n) \leq x\}$$
$$= P\{X_1 \leq x,\ldots,X_n \leq x\}$$
$$= \prod_{i=1}^{n} P\{X_i \leq x\}$$
$$= x^n.$$
$$f_M(x) = \frac{d}{dx}P\{M \leq x\} = nx^{n-1}.$$

38. $c = 2$.

39. $E[X] = \dfrac{31}{6}$.

40. Let X denote the number of games played.
$$P\{X=4\} = p^4 + (1-p)^4$$
$$P\{X=5\} = P\{X=5, \text{ I wins 3 of first 4}\}$$
$$+ P\{X=5, \text{ II wins 3 of first 4}\}$$
$$= 4p^3(1-p)p + 4(1-p)^3p(1-p)$$
$$P\{X=6\} = P\{X=6, \text{ I wins 3 of first 5}\}$$
$$+ P\{X=6, \text{ II wins 3 of first 5}\}$$
$$= 10p^3(1-p)^2p + 10p^2(1-p)^3$$
$$(1-p)$$
$$P\{X=7\} = P\{\text{first 6 games are split}\}$$
$$= 20p^3(1-p)^3.$$
$$E[X] = \sum_{i=4}^{7} iP\{X=i\}.$$
When $p = 1/2$, $\quad E[X] = 93/16 = 5.8125$.

41. Let X_i equal 1 if a changeover results from the i^{th} flip and let it be 0 otherwise. Then

$$\text{Number of changeovers} = \sum_{i=2}^{n} X_i.$$

As,
$$E[X_i] = P\{X_i = 1\} = P\{\text{flip } i-1 \neq \text{flip } i\}$$
$$= 2p(1-p)$$
we see that
$$E[\text{ Number of changeovers}] = \sum_{i=2}^{n} E[X_i]$$
$$= 2(n-1)p(1-p).$$

42. Suppose the coupon collector has i different types. Let X_i denote the number of additional coupons collected until the collector has $i+1$ types. It is easy to see that the X_i are independent geometric random variables with respective parameters $(n-i)/n$, $i = 0, 1, \ldots, n-1$. Therefore,
$$\Sigma\left[\sum_{i=0}^{n-1} X_i\right] = \sum_{i=0}^{n-1}\Sigma[X_i] = \sum_{i=0}^{n-1} n/(n-i)$$
$$= n\sum_{j=1}^{n} 1/j.$$

43. (a) $X = \sum_{i=1}^{n} X_i$.

(b) $E[X_i] = P\{X_i = 1\}$
$= P\{$red ball i is chosen before all n black balls$\}$
$= 1/(n+1)$ since each of these $n+1$ balls is equally likely to be the one chosen earliest

Therefore,

$$E[X] = \sum_{i=1}^{n} E[X_i] = n/(n+1).$$

44. (a) Let Y_i equal 1 if red ball i is chosen after the first but before the second black ball, $i = 1, \ldots, n$. Then
$Y = \sum_{i=1}^{n} Y_i$.

(b) $E[Y_i] = P\{Y_i = 1\}$
$= P\{$red ball i is the second chosen from a set of $n+1$ balls$\}$
$= 1/(n+1)$ since each of the n+1 is equally likely to be the second one chosen.

Therefore,

$$E[Y] = n/(n+1).$$

(c) Answer is the same as in Problem 41.

(d) We can let the outcome of this experiment be the vector $(R_1, R_2, \ldots, R_n)$ where R_i is the number of red balls chosen after the $(i-1)^{st}$ but before the i^{th} black ball. Since all orderings of the $n+m$ balls are equally likely it follows that all different orderings of $R_1, \ldots, R_n$ will have the same probability distribution.

For instance,

$$P\{R_1 = a, R_2 = b\} = P\{R_2 = a, R_1 = b\}.$$

From this it follows that all the R_i have the same distribution and thus the same mean.

45. Let N_i denote the number of keys in box i, $i = 1, \ldots, k$. Then, with X equal to the number of collisions we have that $X = \sum_{i=1}^{k} (N_i - 1)^+ = \sum_{i=1}^{k} (N_i - 1 + I\{N_i = 0\})$ where $I\{N_i = 0\}$ is equal to 1 if $N_i = 0$ and is equal to 0 otherwise. Hence,

$$E[X] = \sum_{i=1}^{k} (rp_i - 1 + (1-p_i)^r) = r - k + \sum_{i=1}^{k} (1-p_i)^r$$

Another way to solve this problem is to let Y denote the number of boxes having at least one key, and then use the identity $X = r - Y$, which is true since only the first key put in each box does not result in a collision. Writing $Y = \sum_{i=1}^{k} I\{N_i > 0\}$ and taking expectations yields that

$$E[X] = r - E[Y] = r - \sum_{i=1}^{k} [1 - (1-p_i)^r] = r - k + \sum_{i=1}^{k} (1-p_i)^r$$

46. Using that $X = \sum_{n=1}^{\infty} I_n$, we obtain

$$E[X] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P\{X \geq n\}$$

Making the change of variables $m = n - 1$ gives

$$E[X] = \sum_{m=0}^{\infty} P\{X \geq m+1\} = \sum_{m=0}^{\infty} P\{X > m\}$$

47. Let X_i be 1 if trial i is a success and 0 otherwise.

(a) The largest value is .6. If $X_1 = X_2 = X_3$, then
$1.8 = E[X] = 3E[X_1] = 3P\{X_1 = 1\}$;

and so

$P\{X = 3\} = P\{X_1 = 1\} = .6.$

That this is the largest value is seen by Markov's inequality which yields that

$P\{X \geq 3\} \leq E[X]/3 = .6.$

(b) The smallest value is 0. To construct a probability scenario for which $P\{X = 3\} = 0$ let U be a uniform random variable on (0, 1), and define

$$X_1 = \begin{cases} 1 & \text{if } U \leq .6 \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{if } U \geq .4 \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if either } U \leq .3 \text{ or } U \geq .7 \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to see that

$P\{X_1 = X_2 = X_3 = 1\} = 0.$

48. $\dfrac{1}{3}$.

49. $E[X^2] - (E[X])^2 = Var(X) = E(X - E[X])^2 \geq 0$. Equality when $Var(X) = 0$, that is, when X is constant.

50. $$\begin{aligned} Var(cX) &= E[(cX - E[cX])^2] \\ &= E[c^2(X - E(X))^2] \\ &= c^2 Var(X). \end{aligned}$$
$$\begin{aligned} Var(c + X) &= E[(c + X - E[c + X])^2] \\ &= E[(X - E[X])^2] \\ &= Var(X). \end{aligned}$$

51. $N = \sum_{i=1}^{r} X_j$ where X_i is the number of flips between the $(i-1)^{st}$ and i^{th} head. Hence, X_i is geometric with mean $1/p$. Thus,
$$E[N] = \sum_{i=1}^{r} E[X_i] = \frac{r}{p}.$$

52. (a) $\dfrac{n}{n+1}$.
(b) 0.
(c) 1.

53. $\dfrac{1}{n+1}, \quad \dfrac{1}{2n+1} - \left[\dfrac{1}{n+1}\right]^2.$

54. (a) Using the fact that $E[X + Y] = 0$ we see that
$0 = 2p(1, 1) - 2p(-1, -1)$
which gives the result.
(b) This follow since
$0 = E[X - Y] = 2p(1, -1) - 2p(-1, 1).$
(c) $Var(X) = E[X^2] = 1$.
(d) $Var(Y) = E[Y^2] = 1$.
(e) Since
$$\begin{aligned} 1 &= p(1, 1) + p(-1, 1) + p(1, -1) + p(-1, 1) \\ &= 2p(1, 1) + 2p(1, -1) \end{aligned}$$
we see that if $p = 2p(1, 1)$ then $1 - p = 2p(1, -1)$.
Now,
$$\begin{aligned} Cov(X, Y) &= E[XY] \\ &= p(1, 1) + p(-1, -1) \\ &\quad - p(1, -1) - p(-1, 1) \\ &= p - (1 - p) = 2p - 1. \end{aligned}$$

55. $$\begin{aligned} 1 &= \int_0^a f(x)dx + \int_a^\infty f(x)dx \\ &\leq \int_0^a c dx + P\{X > a\} \\ &\leq ac + P\{X > a\}. \end{aligned}$$

56. Let X_j equal 1 if there is a type i coupon in the collection, and let it be 0 otherwise. The number of distinct types is $X = \sum_{i=1}^{n} X_i$.
$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{X_i = 1\} = \sum_{i=1}^{n} (1 - p_i)^k$$
To compute $Cov(X_i, X_j)$ when $i \neq j$, note that $X_i X_j$ is either equal to 1 or 0 if either X_i or X_j is equal to 0, and that it will equal 0 if there is either no type i or type j coupon in the collection. Therefore,
$$\begin{aligned} P\{X_i X_j = 0\} &= P\{X_i = 0\} + P\{X_j = 0\} \\ &\quad - P\{X_i = X_j = 0\} \\ &= (1 - p_i)^k + (1 - p_j)^k \\ &\quad - (1 - p_i - p_j)^k \end{aligned}$$
Consequently, for $i \neq j$
$$\begin{aligned} Cov(X_i, X_j) &= P\{X_i X_j = 1\} - E[X_i]E[X_j] \\ &= 1 - [(1 - p_i)^k + (1 - p_j)^k \\ &\quad - (1 - p_i - p_j)^k] - (1 - p_i)^k \\ &\quad (1 - p_j)^k \end{aligned}$$
Because $Var(X_i) = (1 - p_i)^k[1 - (1 - p_i)^k]$ we obtain
$$\begin{aligned} Var(X) &= \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j) \\ &= \sum_{i=1}^{n} (1 - p_i)^k[1 - (1 - p_i)^k] \\ &\quad + 2\sum_j \sum_{i<j} [1 - [(1 - p_i)^k \\ &\quad + (1 - p_j)^k - (1 - p_i - p_j)^k] \\ &\quad - (1 - p_i)^k(1 - p_j)^k. \end{aligned}$$

57. It is the number of successes in $n + m$ independent p-trials.

58. Let X_i equal 1 if both balls of the i^{th} withdrawn pair are red, and let it equal 0 otherwise. Because
$$E[X_i] = P\{X_i = 1\} = \frac{r(r-1)}{2n(2n-1)}$$

we have

$$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{r(r-1)}{(4n-2)}$$

Because

$$E[X_iX_j] = \frac{r(r-1)(r-2)(r-3)}{2n(2n-1)(2n-2)2n-3)}$$

For $Var(X)$ use

$$Var(X) = \sum_i Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) = n\, Var(X_1) + n(n-1)\, Cov(X_1, X_2)$$

where

$$Var(X_1) = E[X_1](1 - E[X_1])$$

$$Cov(X_1, X_2) = \frac{r(r-1)(r-2)(r-3)}{2n(2n-1)(2n-2)(2n-3)} - (E[X_1])^2$$

59. (a) Use the fact that $F(X_i)$ is a uniform $(0,1)$ random variable to obtain that

$$p = P\{F(X_1) < F(X_2) > F(X_3) < F(X_4)\} = P\{U_1 < U_2 > U_3 < U_4\}$$

where the $U_i, i = 1,2,3,4$, are independent uniform $(0,1)$ random variables.

(b)
$$p = \int_0^1 \int_{x_1}^1 \int_0^{x_2} \int_{x_3}^1 dx_4 dx_3 dx_2 dx_1$$
$$= \int_0^1 \int_{x_1}^1 \int_0^{x_2} (1-x_3) dx_3 dx_2 dx_1$$
$$= \int_0^1 \int_{x_1}^1 (x_2 - x_2^2/2) dx_2 dx_1$$
$$= \int_0^1 (1/3 - x_1^2/2 + x_1^3/6) dx_1$$
$$= 1/3 - 1/6 + 1/24 = 5/24$$

(c) There are 5 (of the 24 possible) orderings such that $X_1 < X_2 > X_3 < X_4$. They are as follows:

$X_2 > X_4 > X_3 > X_1$
$X_2 > X_4 > X_1 > X_3$
$X_2 > X_1 > X_4 > X_3$
$X_4 > X_2 > X_3 > X_1$
$X_4 > X_2 > X_1 > X_3$

60.
$$E[e^{tX}] = \int_0^1 e^{tx} dx = \frac{e^t - 1}{t}$$
$$\frac{d}{dt} E[e^{tX}] = \frac{te^t - e^t + 1}{t^2}$$
$$\frac{d^2}{dt^2} E[e^{tX}] = \frac{[t^2(te^2 + e^t - e^t) - 2t(te^t - e^t + 1)]}{t^4} = \frac{t^2e^t - 2(te^t - e^t + 1)}{t^3}.$$

To evaluate at $t = 0$, we must apply l'Hospital's rule.

This yields

$$E[X] = \lim_{t=0} \frac{te^t + e^t - e^t}{2t} = \lim_{t=0} \frac{e^t}{2} = \frac{1}{2}$$
$$E[X^2] = \lim_{t=0} \frac{2te^t + t^2e^t - 2te^t - 2e^t + 2e^t}{3t^2} = \lim_{t=0} \frac{e^t}{3} = \frac{1}{3}.$$

Hence, $Var(X) = \frac{1}{3} - \left[\frac{1}{2}\right]^2 = \frac{1}{12}$.

61. $\phi(t) = \frac{1}{3}(e^t + e^{2t} + e^{3t})$.

62. $E[e^{\alpha\lambda X}] = \int e^{\alpha\lambda x} \lambda e^{-\lambda x} dx = \frac{1}{1-\alpha}$

Therefore,

$$P = -\frac{1}{\alpha\lambda} \ln(1-\alpha).$$

The inequality $\ln(1-x) \leq -x$ shows that $P \geq 1/\lambda$.

63.
$$\phi(t) = \sum_{n=1}^{\infty} e^{tn}(1-p)^{n-1}p = pe^t \sum_{n=1}^{\infty} ((1-p)e^t)^{n-1} = \frac{pe^t}{1-(1-p)e^t}.$$

64. (See Section 2.3 of Chapter 5.)

65.
$$Cov(X_i, X_j) = Cov(\mu_i + \sum_{k=1}^{n} a_{ik}Z_k, \mu_j + \sum_{t=1}^{n} a_{jt}Z_t) = \sum_{t=1}^{n}\sum_{k=1}^{n} Cov(a_{jk}Z_k, a_{jt}Z_t)$$

$$= \sum_{t=1}^{n}\sum_{k=1}^{n} a_{ik}a_{jt}Cov(Z_k, Z_t)$$
$$= \sum_{k=1}^{n} a_{ik}a_{jk}$$

where the last equality follows since

$$Cov(Z_k, Z_t) = \begin{matrix} 1 & \text{if } k = t \\ 0 & \text{if } k \neq t \end{matrix}.$$

66. $P\left\{\left|\dfrac{X_1 + \cdots + X_n - n\mu}{n}\right| > \in\right\}$

$$= P\{|X_1 + \cdots + X_n - n\mu| > n \in\}$$
$$\leq Var\{X_1 + \cdots + X_n\}/n^2 \in^2$$
$$= n\sigma^2/n^2 \in^2$$
$$\to 0 \text{ as } n \to \infty.$$

67. $P\{5 < X < 15\} \geq \dfrac{2}{5}$.

68. (i) $P\{X_1 + \cdots + X_{10} > 15\} \leq \dfrac{2}{3}$.

(ii) $P\{X_1 + \cdots + X_{10} > 15\} \approx 1 - \Phi\left[\dfrac{5}{\sqrt{10}}\right]$.

69. $\Phi(1) - \Phi\left[\dfrac{1}{2}\right] = .1498$.

70. Let X_i be Poisson with mean 1. Then

$$P\left\{\sum_{1}^{n} X_i \leq n\right\} = e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}.$$

But for n large $\sum_{1}^{n} x_i - n$ has approximately a normal distribution with mean 0, and so the result follows.

71. (i) $P\{X = i\} = \begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} m \\ k-i \end{bmatrix}\Big/\begin{bmatrix} n+m \\ k \end{bmatrix}$

$$i = 0, 1, \ldots, min(k, n).$$

(ii) $X = \sum_{i=1}^{k} X_i$

$$E[X] = \sum_{i=1}^{K} E[X_i] = \frac{kn}{n+m}$$

since the i^{th} ball is equally likely to be either of the $n + m$ balls, and so $E[X_i] = P\{X_i = 1\} = \dfrac{n}{n+m}$

$$X = \sum_{i=1}^{n} Y_i$$
$$E[X] = \sum_{i=1}^{n} E[Y_i]$$
$$= \sum_{i=1}^{n} P\{i^{th} \text{ white ball is selected}\}$$
$$= \sum_{i=1}^{n}\frac{k}{n+m} = \frac{nk}{n+m}.$$

72. For the matching problem, letting

$X = X_1 + \cdots + X_N$

where

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ man selects his own hat} \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$Var(X) = \sum_{i=1}^{N} Var(X_i) + 2\sum_{i<j}\sum Cov(X_i, X_j).$$

Since $P\{X_i = 1\} = 1/N$, we see

$$Var(X_i) = \frac{1}{N}\left[1 - \frac{1}{N}\right] = \frac{N-1}{N^2}.$$

Also

$$Cov(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j].$$

Now,

$$X_iX_j = \begin{cases} 1 & \text{if the } i^{th} \text{ and } j^{th} \text{ men both select their own hats} \\ 0 & \text{otherwise,} \end{cases}$$

and thus

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\}$$
$$= P\{X_i = 1\}P\{X_j = 1|X_i = 1\}$$
$$= \frac{1}{N}\frac{1}{N-1}.$$

Hence,

$$Cov(X_i, X_j) = \frac{1}{N(N-1)} - \left[\frac{1}{N}\right]^2 = \frac{1}{N^2(N-1)}$$

and

$$Var(X) = \frac{N-1}{N} + 2\begin{bmatrix} N \\ 2 \end{bmatrix}\frac{1}{N^2(N-1)}$$
$$= \frac{N-1}{N} + \frac{1}{N}$$
$$= 1.$$

73. As N_i is a binomial random variable with parameters (n, P_i), we have (i) $E[N_i] = nP_{ji}$ (ii) $Var(X_i) = nP_i = (1 - P_i)$; (iii) for $i \neq j$, the covariance of N_i and N_j can be computed as:

$$Cov(N_i, N_j) = Cov\left[\sum_k X_k, \sum_k Y_k\right],$$

where $X_k(Y_k)$ is 1 or 0, depending upon whether or not outcome k is type $i(j)$. Hence,

$$Cov(N_i, N_j) = \sum_k \sum_\ell Cov(X_k, Y_\ell).$$

Now for $k \neq \ell, Cov(X_k, Y_\ell) = 0$ by independence of trials and so

$$\begin{aligned} Cov(N_i, N_j) &= \sum_k Cov(X_k, Y_k) \\ &= \sum_k (E[X_k Y_k] - E[X_k]E[Y_k]) \\ &= -\sum_k E[X_k]E[Y_k] \text{ (since } X_k Y_k = 0) \\ &= -\sum_k P_i P_j \\ &= -nP_i P_j. \end{aligned}$$

(iv) Letting

$$Y_i = \begin{cases} 1, & \text{if no type } i\text{'s occur} \\ 0, & \text{otherwise,} \end{cases}$$

we have that the number of outcomes that never occur is equal to $\sum_1^r Y_i$ and thus,

$$\begin{aligned} E\left[\sum_1^r Y_i\right] &= \sum_1^r E[Y_i] \\ &= \sum_1^r P\{\text{outcomes } i \text{ does not occur}\} \\ &= \sum_1^r (1 - P_i)^n. \end{aligned}$$

74. (i) As the random variables are independent, identically distributed, and continuous, it follows that, with probability 1, they will all have different values. Hence the largest of $X_1, \ldots, X_n$ is equally likely to be either X_1 or X_2 ... or X_n. Hence, as there is a record at time n when X_n is the largest value, it follows that

$$P\{\text{a record occurs at } n\} = \frac{1}{n}.$$

(ii) Let $I_j = \begin{cases} 1, & \text{if a record occurs at } j \\ 0, & \text{otherwise.} \end{cases}$

Then

$$E\left[\sum_1^n I_j\right] = \sum_1^n E[I_j] = \sum_1^n \frac{1}{j}.$$

(iii) It is easy to see that the random variables $I_1, I_2, \ldots, I_n$ are independent. For instance, for $j < k$

$$P\{I_j = 1/I_k = 1\} = P\{I_j = 1\},$$

since knowing that X_k is the largest of $X_1, \ldots, X_j, \ldots, X_k$ clearly tells us nothing about whether or not X_j is the largest of $X_1, \ldots, X_j$. Hence,s

$$Var \sum_1^n I_j = \sum_1^n Var(I_j) = \sum_{j=1}^n \left[\frac{1}{j}\right]\left[\frac{j-1}{j}\right].$$

(iv) $P\{N > n\}$

$$= P\{X_1 \text{ is the largest of } X_1, \ldots, X_n\} = \frac{1}{n}.$$

Hence,

$$E[N] = \sum_{n=1}^\infty P\{N > n\} = \sum_{n=1}^\infty \frac{1}{n} = \infty.$$

75. (i) Knowing the values of $N_1, \ldots, N_j$ is equivalent to knowing the relative ordering of the elements $a_1, \ldots, a_1$. For instance, if $N_1 = 0$, $N_2 = 1, N_3 = 1$ then in the random permutation a_2 is before a_3, which is before a_1. The independence result follows for clearly the number of $a_1, \ldots, a_1$ that follow a_{i+1} does not probabilistically depend on the relative ordering of $a_1, \ldots, a_i$.

(ii) $P\{N_i = k\} = \frac{1}{i}, \quad k = 0, 1, \ldots, i-1$

which follows since of the elements $a_1, \ldots, a_{i+1}$ the element a_{i+1} is equally likely to be first or second or ... or $(i+1)^{st}$.

(iii) $E[N_i] = \frac{1}{i}\sum_{k=0}^{i-1} k = \frac{i-1}{2}$

$$E[N_i^2] = \frac{1}{i}\sum_{k=0}^{i-1} k^2 = \frac{(i-1)(2i-1)}{6}$$

and so

$$\begin{aligned} Var(N_i) &= \frac{(i-1)(2i-1)}{6} - \frac{(i-1)^2}{4} \\ &= \frac{i^2 - 1}{12}. \end{aligned}$$

76. $E[XY] = \mu_x \mu_y$

$$E[(XY)^2] = (\mu_x^2 + \sigma_x^2)(\mu_y^2 + \sigma_y^2)$$

$$Var(XY) = E[(XY)^2] - (E[XY])^2$$

77. If $g_1(x, y) = x + y,\ g_2(x, y) = x - y$, then

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = 2$$

Hence, if $U = X + Y,\ V = X - Y$, then

$$f_{U,V}(u, v) = \frac{1}{2} f_{X,\,Y}\left[\frac{u+v}{2}, \frac{u-v}{2}\right].$$

$$= \frac{2}{4\tau\sigma^2}\, exp\left[-\frac{1}{2\sigma^2}\left[\left[\frac{u+v}{2} - \mu\right]^2\right.\right.$$

$$\left.\left.+ \left[\frac{u-v}{2} - \mu\right]^2\right]\right]$$

$$= \frac{e{-}\mu^2/\sigma^2}{4\tau\sigma^2}\, exp\left[\frac{u\mu}{\sigma^2} - \frac{u^2}{4\sigma^2}\right]$$

$$exp\left\{-\frac{v^2}{4\sigma^2}\right\}$$

78. (a) $\phi_{x_i}(t_i) = \phi(0, 0 \ldots 0, 1, 0 \ldots 0)$ with the 1 in the i^{th} place.

(b) If independent, then

$$E\left[e^{\Sigma\, t_i x_i}\right] = \underset{i}{\pi}\ \left[e^{t_i x_i}\right]$$

On the other hand, if the above is satisfied, then the joint moment generating function is that of the sum of n independent random variables the i^{th} of which has the same distribution as x_i. As the joint moment generating function uniquely determines the joint distribution, the result follows.

79. $$E\left[e^{tz^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx^2} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\{-x^2(1-2t)/2\} dx$$

$$= (1-2t)^{-1/2}$$

where the last equality follows since with $\sigma^{-2} = 1/(1-2t)$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}/2\ \overset{-2}{\sigma} = \bar{\sigma}$$

Hence,

$$E\left[e^{tX}\right] = (1-2t)^{-n/2}$$

Chapter 3

1. $\sum_x p_{X|Y^{(x|y)}} = \frac{\sum_x p(x,y)}{p_{Y(y)}} = \frac{p_{Y(y)}}{p_{Y(y)}} = 1.$

2. Intuitively it would seen that the first head would be equally likely to occur on either of trials $1, \ldots, n-1$. That is, it is intuitive that

$$P\{X_1 = i | X_1 + X_2 = n\} = 1/(n-1),$$
$$i = 1, \ldots, n-1.$$

Formally,

$$\begin{aligned} P\{X_1 = i | X_1 + X_2 = n\} &= \frac{P\{X_1 = i, X_1 + X_2 = n\}}{P\{X_1 + X_2 = n\}} \\ &= \frac{P\{X_1 = i, X_2 = n-i\}}{P\{X_1 + X_2 = n\}} \\ &= \frac{p(1-p)^{i-1} p(1-p)^{n-i-1}}{\binom{n-1}{1} p(1-p)^{n-2} p} \\ &= 1/(n-1). \end{aligned}$$

In the above, the next to last equality uses the independence of X_1 and X_2 to evaluate the numerator and the fact that $X_1 + X_2$ has a negative binomial distribution to evaluate the denominator.

3. $E[X|Y=1] = 2$

$$E[X|Y=2] = \frac{5}{3}$$
$$E[X|Y=3] = \frac{12}{5}.$$

4. No.

5. (a) $P\{X = i | Y = 3\} = P\{i$ white balls selected when choosing 3 balls from 3 white and 6 red$\}$

$$= \frac{\begin{bmatrix}3\\i\end{bmatrix}\begin{bmatrix}6\\3-i\end{bmatrix}}{\begin{bmatrix}9\\3\end{bmatrix}}, \quad i = 0,1,2,3.$$

(b) By same reasoning as in (a), if $Y = 1$, then X has the same distribution as the number of white balls chosen when 5 balls are chosen from 3 white and 6 red. Hence,

$$E[X|Y] = 5\frac{3}{9} = \frac{5}{3}.$$

6. $p_{X|Y}(1|3) = P\{X = 1, Y = 3\}/P\{Y = 3\}$

$$\begin{aligned} &= P\{1 \text{ white, 3 black, 2 red}\} \\ &\quad /P\{3 \text{ black}\} \\ &= \frac{6!}{1!3!2!}\left[\frac{3}{14}\right]^1\left[\frac{5}{14}\right]^3\left[\frac{6}{14}\right]^2 \\ &\quad \bigg/ \frac{6!}{3!3!}\left[\frac{5}{14}\right]^3\left[\frac{9}{14}\right]^3 \\ &= \frac{4}{9} \end{aligned}$$

$$p_{X|Y}(0|3) = \frac{8}{27}$$
$$p_{X|Y}(2|3) = \frac{2}{9}$$
$$p_{X|Y}(3|3) = \frac{1}{27}$$
$$E[X|Y=1] = \frac{5}{3}.$$

7. Given $Y = 2$, the conditional distribution of X and Z is

$$P\{(X,Z) = (1,1)|Y = 2\} = \frac{1}{5}$$
$$P\{(1,2)|Y = 2\} = 0$$
$$P\{(2,1)|Y = 2\} = 0$$
$$P\{(2,2)|Y = 2\} = \frac{4}{5}.$$

So

$$E[X|Y=2] = \frac{1}{5} + \frac{8}{5} = \frac{9}{5}$$
$$E[X|Y=2, Z=1] = 1.$$

8. (a) $E[X] = E[X|\text{first roll is 6}]\dfrac{1}{6}$

$$+ E[X|\text{first roll is not 6}]\frac{5}{6}$$
$$= \frac{1}{6} + (1 + E[X])\frac{5}{6}$$

implying that $E[X] = 6$.

(b) $E[X|Y = 1] = 1 + E[X] = 7$.

(c) $E[X|Y = 5]$

$$= 1\left[\frac{1}{5}\right] + 2\left[\frac{4}{5}\right]\left[\frac{1}{5}\right] + 3\left[\frac{4}{5}\right]^2\left[\frac{1}{5}\right]$$
$$+ 4\left[\frac{4}{5}\right]^3\left[\frac{1}{5}\right] + 6\left[\frac{4}{5}\right]^4\left[\frac{1}{6}\right]$$
$$+ 7\left[\frac{4}{5}\right]^4\left[\frac{5}{6}\right]\left[\frac{1}{6}\right] + \cdots.$$

9. $E[X|Y = y] = \sum_x xP\{X = x|Y = y\}$

$$= \sum_x xP\{X = x\} \quad \text{by independence}$$
$$= E[X].$$

10. (Same as in Problem 8.)

11. $E[X|Y = y] = C\displaystyle\int_{-y}^{y} x(y^2 - x^2)dx = 0$

12. $$f_{X|Y}(x|y) = \frac{\frac{1}{y}\exp^{-x/y}\exp^{-y}}{\exp^{-y}\int \frac{1}{y}\exp^{-x/y}dx} = \frac{1}{y}\exp^{-x/y}$$

Hence, given $Y = y$, X is exponential with mean y.

13. The conditional density of X given that $X > 1$ is

$$f_{X|X>1}(x) = \frac{f(x)}{P\{X > 1\}} = \frac{\lambda\exp^{-\lambda x}}{\exp^{-\lambda}} \text{ when } x > 1$$

$$E[X|X > 1] = \exp^{\lambda}\int_1^{\infty} x\lambda\exp^{-\lambda x}\,dx = 1 + 1/\lambda$$

by integration by parts.

14. $$f_{X|X<\frac{1}{2}}(x) = \frac{f(x)}{P\{X < 1\}}, \quad x < \frac{1}{2}$$
$$= \frac{1}{1/2} = 2.$$

Hence, $\quad E\left[X|X < \dfrac{1}{2}\right] = \displaystyle\int_0^{1/2} 2x\,dx = \dfrac{1}{4}.$

15. $$f_{X|Y=y}(x|y) = \frac{\frac{1}{y}\exp^{-y}}{f_y(y)} = \frac{\frac{1}{y}\exp^{-y}}{\int_0^y \frac{1}{y}\exp^{-y}\,dx}$$
$$= \frac{1}{y}, \quad 0 < x < y$$

$$E[X^2|Y = y] = \frac{1}{y}\int_0^y x^2dx = \frac{y^2}{3}.$$

17. With $K = 1/P\{X = i\}$, we have that

$$f_{Y|X}(y|i) = KP\{X = i|Y = y\}f_Y(y)$$
$$= K_1e^{-y}y^ie^{-\alpha y}y^{a-1}$$
$$= K_1e^{-(1+\alpha)y}y^{a+i-1}$$

where K_1 does not depend on y. But as the preceding is the density function of a gamma random variable with parameters $(s + i, 1 + \alpha)$ the result follows.

18. In the following $t = \sum_{i=1}^{n} x_i$, and C does not depend on θ. For (a) use that T is normal with mean $n\theta$ and variance n; in (b) use that T is gamma with parameters (n, θ); in (c) use that T is binomial with parameters (n, θ); in (d) use that T is Poisson with mean $n\theta$.

(a) $f(x_1, ..., x_n|T = t)$

$$= \frac{f(x_1, ..., x_n,\ T = t)}{f_T(t)}$$
$$= \frac{f(x_1, ..., x_n)}{f_T(t)}$$
$$= C\frac{\exp\{-\sum(x_i - \theta)^2/2\}}{\exp\{-(t - n\theta)^2/2n\}}$$
$$= C\exp\{(t - n\theta)^2/2n - \sum(x_i - \theta)^2/2\}$$
$$= C\exp\{t^2/2n - \theta t + n\theta^2/2 - \sum x_i^2/2 + \theta t - n\theta^2/2\}$$
$$= C\exp\{(\sum x_i)^2/2n - \sum x_i^2/2\}$$

(b) $$f(x_1, \ldots, x_n|T = t) = \frac{f(x_1, \ldots, x_n)}{f_T(t)}$$
$$= \frac{\theta^n e^{-\theta\sum x_i}}{\theta e^{-\theta t}(\theta t)^{n-1}/(n-1)!}$$
$$= (n-1)!t^{1-n}$$

parts (c) and (d) are similar.

19. $$\int E[X|Y=y]f_Y(y)dy$$
$$= \int\int x f_{X|Y}(x|y)dx f_Y(Y)dy$$
$$= \int\int x\frac{f(x,y)}{f_Y(y)}dx f_Y(y)dy$$
$$= \int x \int f(x\cdot y)dydx$$
$$= \int x f_X(x)dx$$
$$= E[X].$$

20. (a) $$f(x|\text{disease}) = \frac{P\{\text{disease}|x\}f(x)}{\int P\{\text{disease}|x\}f(x)dx} = \frac{P(x)f(x)}{\int P(x)f(x)dx}$$

(b) $$f(x|\text{no disease}) = \frac{[1-P(x)]f(x)}{\int [1-P(x)]f(x)dx}$$

(c) $$\frac{f(x|\text{disease})}{f(x|\text{no disease})} = C\frac{P(x)}{1-P(x)}$$
where C does not depend on x.

21. (a) $X = \sum_{i=1}^{N} T_i$.

(b) Clearly N is geometric with parameter $1/3$; thus, $E[N] = 3$.

(c) Since T_N is the travel time corresponding to the choice leading to freedom it follows that $T_N = 2$, and so $E[T_N] = 2$.

(d) Given that $N = n$, the travel times $T_i i = 1, \ldots, n-1$ are each equally likely to be either 3 or 5 (since we know that a door leading back to the nine is selected), whereas T_n is equal to 2 (since that choice led to safety). Hence,
$$E\left[\sum_{i=1}^{N} T_i|N=n\right] = E\left[\sum_{i=1}^{n-1} T_i|N=n\right] + E[T_n|N=n] = 4(n-1)+2.$$

(e) Since part (d) is equivalent to the equation
$$E\left[\sum_{i=1}^{N} T_i|N\right] = 4N-2,$$
we see from parts (a) and (b) that
$$E[X] = 4E[N]-2 = 10.$$

22. Letting N_i denote the time until the same outcome occurs i consecutive times we obtain, upon conditioning N_{i-1}, that
$$E[N_i] = E[E[N_i|N_{i-1}]].$$
Now,
$$E[N_i|N_{i-1}] = N_{i-1} + \begin{cases} 1 & \text{with probability } 1/n \\ E[N_i] & \text{with probability}(n-1)/n \end{cases}$$

The above follows because after a run of $i-1$ either a run of i is attained if the next trial is the same type as those in the run or else if the next trial is different then it is exactly as if we were starting all over at that point.

From the above equation we obtain that
$$E[N_i] = E[N_{i-1}] + 1/n + E[N_i](n-1)/n.$$
Solving for $E[N_i]$ gives
$$E[N_i] = 1 + nE[N_{i-1}].$$
Solving recursively now yields
$$\begin{aligned} E[N_i] &= 1 + n\{1 + nE[N_{i-2}]\} \\ &= 1 + n + n^2E[N_{i-2}] \\ &\;\;\vdots \\ &= 1 + n + \cdots + n^{k-1}E[N_1] \\ &= 1 + n + \cdots + n^{k-1}. \end{aligned}$$

23. Let X denote the first time a head appears. Let us obtain an equation for $E[N|X]$ by conditioning on the next two flips after X. This gives
$$E[N|X] = E[N|X,h,h]p^2 + E[N|X,h,t]pq + E[N|X,t,h]pq + E[N|X,t,t]q^2$$
where $q = 1-p$. Now
$$E[N|X,h,h] = X+1,\ E[N|X,h,t] = X+1$$
$$E[N|X,t,h] = X+2,\ E[N|X,t,t] = X+2+E[N].$$
Substituting back gives
$$E[N|X] = (X+1)(p^2+pq) + (X+2)pq + (X+2+E[N])q^2$$
Taking expectations, and using the fact that X is geometric with mean $1/p$, we obtain
$$E[N] = 1 + p + q + 2pq + q^2/p + 2q^2 + q^2E[N]$$
Solving for $E[N]$ yields
$$E[N] = \frac{2+2q+q^2/p}{1-q^2}.$$

24. In all parts, let X denote the random variable whose expectation is desired, and start by conditioning on the result of the first flip. Also, h stands for heads and t for tails.

(a) $$E[X] = E[X|h]p + E[X|t](1-p)$$
$$= \left(1 + \frac{1}{1-p}\right)p + \left(1 + \frac{1}{p}\right)(1-p)$$
$$= 1 + p/(1-p) + (1-p)/p$$

(b) $$E[X] = (1 + E[\text{number of heads before first tail}])p + 1(1-p)$$
$$= 1 + p(1/(1-p) - 1)$$
$$= 1 + p/(1-p) - p$$

(c) Interchanging p and $1-p$ in (b) gives result: $1 + (1-p)/p - (1-p)$

(d) $$E[X] = (1 + \text{answer from } (a))p + (1 + 2/p)(1-p)$$
$$= (2 + p/(1-p) + (1-p)/p)p + (1 + 2/p)(1-p)$$

25. Let W denote the number of wins.

(a) $$E[W] = E[E[W|X]] = E[X + Xp] = (1+p)E[X] = (1+p)np$$

(b) $$E[W] = E[E[W|Y]] = E[1 + Yp] = 1 + p/p = 2$$
since Y is geometric with mean $1/p$.

26. Let N_A and N_B denote the number of games needed given that you start with A and given that you start with B. Conditioning on the outcome of the first game gives

$$E[N_A] = E[N_A|w]p_A + E[N_A|l](1-p_A)$$

Conditioning on the outcome of the next game gives

$$E[N_A|w] = E[N_A|ww]p_B + E[N_A|wl](1-p_B)$$
$$= 2p_B + (2 + E[N_A])(1-p_B)$$
$$= 2 + (1-p_B)E[N_A]$$

As, $E[N_A|l] = 1 + E[N_B]$, we obtain that

$$E[N_A] = (2 + (1-p_B)E[N_A])p_A + (1 + E[N_B])(1-p_A)$$
$$= 1 + p_A + p_A(1-p_B)E[N_A] + (1-p_A)E[N_B]$$

Similarly,

$$E[N_B] = 1 + p_B + p_B(1-p_A)E[N_B] + (1-p_B)E[N_A]$$

Subtracting gives

$$E[N_A] - E[N_B] = p_A - p_B + (p_A - 1)(1-p_B)E[N_A] + (1-p_B)(1-p_A)E[N_B]$$

or

$$[1 + (1-p_A)(1-p_B)](E[N_A] - E[N_B]) = p_A - p_B$$

Hence, if $p_B > p_A$ then $E[N_A] - E[N_B] < 0$, showing that playing A first is better.

27. Condition on the outcome of the first flip to obtain

$$E[X] = E[X|H]p + E[X|T](1-p)$$
$$= (1 + E[X])p + E[X|T](1-p)$$

Conditioning on the next flip gives

$$E[X|T] = E[X|TH]p + E[X|TT](1-p)$$
$$= (2 + E[X])p + (2 + 1/p)(1-p)$$

where the final equality follows since given that the first two flips are tails the number of additional flips is just the number of flips needed to obtain a head. Putting the preceding together yields

$$E[X] = (1 + E[X])p + (2 + E[X])p(1-p) + (2 + 1/p)(1-p)^2$$

or

$$E[X] = \frac{1}{p(1-p)^2}$$

28. Let Y_i equal 1 if selection i is red, and let it equal 0 otherwise. Then

$$E[X_k] = \sum_{i=1}^{k} E[Y_i]$$
$$E[Y_1] = \frac{r}{r+b}$$
$$E[X_1] = \frac{r}{r+b}$$

$$E[Y_2] = E[E[Y_2|X_1]]$$
$$= E\left[\frac{r + mX_1}{r + b + m}\right]$$
$$= \frac{r + m\frac{r}{r+b}}{r + b + m}$$
$$= \frac{r}{r + b + m} + \frac{m}{r + b + m}\frac{r}{r + b}$$
$$= \frac{r}{r + b + m}\left(1 + \frac{m}{r + b}\right)$$
$$= \frac{r}{r + b}$$
$$E[X_2] = 2\frac{r}{r + b}$$

To prove by induction that $E[Y_k] = \frac{r}{r+b}$, assume that for all $i < k$, $E[Y_i] = \frac{r}{r+b}$.

Then

$$E[Y_k] = E[E[Y_k|X_{k-1}]]$$
$$= E\left[\frac{r + mX_{k-1}}{r + b + (k-1)m}\right]$$
$$= \frac{r + mE[\sum_{i<k} Y_i]}{r + b + (k-1)m}$$
$$= \frac{r + m(k-1)\frac{r}{r+b}}{r + b + (k-1)m}$$
$$= \frac{r}{r + b}$$

The intuitive argument follows because each selection is equally likely to be any of the $r + b$ types.

29. Let $q_i = 1 - p_i$, $i = 1.2$. Also, let h stand for hit and m for miss.

(a) $\mu_1 = E[N|h]p_1 + E[N|m]q_1$

$$= p_1(E[N|h,h]p_2 + E[N|h,m]q_2) + (1 + \mu_2)q_1$$
$$= 2p_1p_2 + (2 + \mu_1)p_1q_2 + (1 + \mu_2)q_1$$

The preceding equation simplifies to

$$\mu_1(1 - p_1q_2) = 1 + p_1 + \mu_2q_1$$

Similarly, we have that

$$\mu_2(1 - p_2q_1) = 1 + p_2 + \mu_1q_2$$

Solving these equations gives the solution.

$$h_1 = E[H|h]p_1 + E[H|m]q_1$$
$$= p_1(E[H|h,h]p_2 + E[H|h,m]q_2) + h_2q_1$$
$$= 2p_1p_2 + (1 + h_1)p_1q_2 + h_2q_1$$

Similarly, we have that

$$h_2 = 2p_1p_2 + (1 + h_2)p_2q_1 + h_1q_2$$

and we solve these equations to find h_1 and h_2.

30. $E[N] = \sum_{j=1}^{m} E[N|X_o = j]p(j) = \sum_{j=1}^{m} \frac{1}{p(j)}p(j) = m$

31. Let L_i denote the length of run i. Conditioning on X, the initial value gives

$$E[L_1] = E[L_1|X = 1]p + E[L_1|X = 0](1 - p)$$
$$= \frac{1}{1 - p}p + \frac{1}{p}(1 - p)$$
$$= \frac{p}{1 - p} + \frac{1 - p}{p}$$

and

$$E[L_2] = E[L_2|X = 1]p + E[L_2|X = 0](1 - p)$$
$$= \frac{1}{p}p + \frac{1}{1 - p}(1 - p)$$
$$= 2$$

32. Let T be the number of trials needed for both at least n successes and m failures. Condition on N, the number of successes in the first $n + m$ trials to obtain

$$E[T] = \sum_{i=0}^{n+m} E[T|N = i]\binom{n + m}{i}p^i(1 - p)^{n+m-i}$$

Now use

$$E[T|N = i] = n + m + \frac{n - i}{p}, \quad i \le n$$
$$E[T|N = i] = n + m + \frac{i - n}{1 - p}, \quad i > n$$

Let S be the number of trials needed for n successes, and let F be the number needed for m failures. Then $T = \max(S, F)$. Taking expectations of the identity

$$\min(S, F) + \max(S, F) = S + F$$

yields the result

$$E[\min(S, F)] = \frac{n}{p} + \frac{m}{1 - p} - E[T]$$

33. Let $I(A)$ equal 1 if the event A occurs and let it equal 0 otherwise.

$$\begin{aligned}E\left[\sum_{i=1}^{T} R_i\right] &= E\left[\sum_{i=1}^{\infty} I(T \geq i)R_i\right] \\ &= \sum_{i=1}^{\infty} E[I\,(T \geq i)\,R_i] \\ &= \sum_{i=1}^{\infty} E[I(T \geq i)]E[R_i] \\ &= \sum_{i=1}^{\infty} P\{T \geq i\}E[R_i] \\ &= \sum_{i=1}^{\infty} \beta^{i-1}E[R_i] \\ &= E\left[\sum_{i=1}^{\infty} \beta^{i-1}R_i\right]\end{aligned}$$

34. Let X denote the number of the dice that land on six on the first roll.

(a)
$$\begin{aligned}m_n &= \sum_{i=0}^{n} E[N|X=i]\binom{n}{i}(1/6)^i(5/6)^{n-i} \\ &= \sum_{i=0}^{n}(1+m_{n-i})\binom{n}{i}(1/6)^i(5/6)^{n-i} \\ &= 1 + m_n(5/6)^n + \sum_{i=1}^{n-1} m_{n-i}\binom{n}{i}(1/6)^i \\ &\quad (5/6)^{n-i}\end{aligned}$$

implying that

$$m_n = \frac{1+\sum_{i=1}^{n-1} m_{n-i}\binom{n}{i}(1/6)^i(5/6)^{n-i}}{1-(5/6)^n}$$

Starting with $m_0 = 0$ we see that

$$m_1 = \frac{1}{1-5/6} = 6$$

$$m_2 = \frac{1+m_1(2)(1/6)(5/6)}{1-(5/6)^2} = 96/11$$

and so on.

(b) Since each die rolled will land on six with probability 1/6, the total number of dice rolled will equal the number of times one must roll a die until six appears n times. Therefore, $E\left[\sum_{i=1}^{N} X_i\right] = 6n$.

35.
$$\begin{aligned}np_1 &= E[X_1] \\ &= E[X_1|X_2=0](1-p_2)^n \\ &\quad + E[X_1|X_2>0][1-(1-p_2)^n] \\ &= n\frac{p_1}{1-p_2}(1-p_2)^n \\ &\quad + E[X_1|X_2>0][1-(1-p_2)^n]\end{aligned}$$

yielding the result

$$E[X_1|X_2>0] = \frac{np_1(1-(1-p_2)^{n-1})}{1-(1-p_2)^n}$$

36. $E[X] = E[X|X \neq 0](1-p_0) + E[X|X=0]p_0$

yielding that

$$E[X|X \neq 0] = \frac{E[X]}{1-p_0}$$

Similarly,

$$E[X^2] = E[X^2|X \neq 0](1-p_0) + E[X^2|X=0]p_0$$

yielding that

$$E[X^2|X \neq 0] = \frac{E[X^2]}{1-p_0}$$

Hence,

$$\begin{aligned}Var\,(X|X \neq 0) &= \frac{E[X^2]}{1-p_0} - \frac{E^2[X]}{(1-p_0)^2} \\ &= \frac{\mu^2+\sigma^2}{1-p_0} - \frac{\mu^2}{(1-p_0)^2}\end{aligned}$$

37. (a) $E[X] = (2.6+3+3.4)/3 = 3$

(b) $E[X^2] = [2.6+2.6^2+3+9+3.4+3.4^2]/3$
$= 12.1067$, and $Var(X) = 3.1067$

38. Let X be the number of successes in the n trials. Now, given that $U = u$, X is binomial with parameters (n, u). As a result,

$$E[X|U] = nU$$

$$E[X^2|U] = n^2U^2 + nU(1-U) = nU + (n^2-n)U^2$$

Hence,

$$\begin{aligned}E[X] &= nE[U] \\ &= E[X^2] = E[nU + (n^2-n)U^2] \\ &= n/2 + (n^2-n)[(1/2)^2 + 1/12] \\ &= n/6 + n^2/3\end{aligned}$$

Hence,

$$Var(X) = n/6 + n^2/12$$

39. Let N denote the number of cycles, and let X be the position of card 1.

(a) $$m_n = \frac{1}{n}\sum_{i=1}^{n} E[N|X=i] = \frac{1}{n}\sum_{i=1}^{n}(1+m_{n-1})$$
$$= 1 + \frac{1}{n}\sum_{j=1}^{n-1} m_j$$

(b) $$m_1 = 1$$
$$m_2 = 1 + \frac{1}{2} = 3/2$$
$$m_3 = 1 + \frac{1}{3}(1+3/2) = 1 + 1/2 + 1/3$$
$$= 11/6$$
$$m_4 = 1 + \frac{1}{4}(1+3/2+11/6) = 25/12$$

(c) $m_n = 1 + 1/2 + 1/3 + \cdots + 1/n$

(d) Using the recursion and the induction hypothesis gives that
$$m_n = 1 + \frac{1}{n}\sum_{j=1}^{n-1}(1+\cdots+1/j)$$
$$= 1 + \frac{1}{n}(n-1+(n-2)/2+(n-3)/3$$
$$+\cdots+1/(n-1))$$
$$= 1 + \frac{1}{n}[n + n/2 + \cdots + n/(n-1)$$
$$-(n-1)]$$
$$= 1 + 1/2 + \cdots + 1/n$$

(e) $N = \sum_{i=1}^{n} X_i$

(f) $$m_n = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{i \text{ is last of } 1,\ldots,i\}$$
$$= \sum_{i=1}^{n} 1/i$$

(g) Yes, knowing for instance that $i+1$ is the last of all the cards $1,\ldots,i+1$ to be seen, tells us nothing about whether i is the last of $1,\ldots,i$.

(h) $Var(N) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n}(1/i)(1-1/i)$

40. Let X denote the number of door chosen, and let N be the total number of days spent in jail.

(a) Conditioning on X, we get
$$E[N] = \sum_{i=1}^{3} E\{N|X=i\}P\{X=1\}.$$
The process restarts each time the prisoner returns to his cell. Therefore,
$$E(N|X=1) = 2 + E(N)$$
$$E(N|X=2) = 3 + E(N)$$
$$E(N|X=3) = 0.$$
and
$$E(N) = (.5)(2+E(N)) + (.3)(3+E(N))$$
$$+(.2)(0),$$
or
$$E(N) = 9.5 \text{ days}.$$

(b) Let N_i denote the number of additional days the prisoner spends after having initially chosen cell i.
$$E[N] = \frac{1}{3}(2+E[N_1]) + \frac{1}{3}(3+E[N_2]) + \frac{1}{3}(0)$$
$$= \frac{5}{3} + \frac{1}{3}(E[N_1]+E[N_2]).$$
Now,
$$E[N_1] = \frac{1}{2}(3) + \frac{1}{2}(0) = \frac{3}{2}$$
$$E[N_2] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$$
and so,
$$E[N] = \frac{5}{3} + \frac{1}{3}\frac{5}{2} = \frac{5}{2}.$$

41. Let N denote the number of minutes in the maze. If L is the event the rat chooses its left, and R the event it chooses its right, we have by conditioning on the first direction chosen:
$$E(N) = \frac{1}{2}E(N|L) + \frac{1}{2}E(N|R)$$
$$= \frac{1}{2}\left[\frac{1}{3}(2) + \frac{2}{3}(5+E(N))\right] + \frac{1}{2}[3+E(N)]$$
$$= \frac{5}{6}E(N) + \frac{21}{6}$$
$$= 21.$$

42. Let X be the number of people who arrive before you. Because you are equally likely to be the first, or second, or third, ..., or eleventh arrival
$$P\{X=i\} = \frac{1}{10}, \quad i = 0,\ldots,10$$
Therefore,
$$E[X] = \frac{1}{10}\sum_{i=1}^{10} i = 5.5$$
and

$$E[X^2] = \frac{1}{10}\sum_{i=1}^{10} i^2 = \frac{1}{10}\frac{10(11)(21)}{6} = 38.5$$

giving that

$$Var(X) = 38.5 - 30.25 = 8.25$$

43. Using Examples 4d and 4e, mean $= \mu_1\mu_2$, variance $= \mu_1\sigma_2^2 + \mu_2^2\sigma_1^2$.

44. From Examples 4d and 4e, mean $= 500$, variance $= E[N]Var(X) + E^2(X)Var(N)$

$$= \frac{10(100)^2}{12} + (50)^2(10)$$
$$= 33,333.$$

45. Now

$$E[X_n|X_{n-1}] = 0, \quad Var(X_n|X_{n-1}) = \beta X_{n-1}^2.$$

(a) From the above we see that

$$E[X_n] = 0.$$

(b) From (a) we have that $Var(x_n) = E[X_n^2]$. Now

$$\begin{aligned} E[X_n^2] &= E\{E[X_n^2|X_{n-1}]\} \\ &= E[\beta X_{n-1}^2] \\ &= \beta E[X_{n-1}^2] \\ &= \beta^2 E[X_{n-2}^2] \\ &\;\cdot \\ &= \beta^n X_0^2. \end{aligned}$$

46. (a) This follows from the identity $Cov(U, V) = E[UV] - E[U]E[V]$ upon noting that

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]],$$
$$E[Y] = E[E[Y|X]]$$

(b) From part (a) we obtain

$$\begin{aligned} Cov(X, Y) &= Cov(a + bX, X) \\ &= b\ Var(X). \end{aligned}$$

47. $$\begin{aligned} E[X^2Y^2|X] &= X^2E[Y^2|X] \\ &\geq X^2(E[Y|X])^2 = X^2 \end{aligned}$$

The inequality following since for any random variable U. $E[U^2] \geq (E[U])^2$ and this remains true when conditioning on some other random variable X. Taking expectations of the above shows that

$$E[(XY)^2] \geq E[X^2].$$

As

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]] = E[X]$$

the result follows.

48. Using the hint, we see that

$$\begin{aligned} E\left[\sum_1^N X_i\right] &= \phi'(0) \\ &= E[NE[X]] \text{ since } \phi(0) = 1,\ \phi'(0) = E[X] \\ &= E[N]E[X]. \end{aligned}$$

$$\begin{aligned} E\left[\left[\sum_1^N X_i\right]^2\right] &= \phi''(0) \\ &= E[N(N-1)E^2[X] + NE[X^2]] \\ &= E[N^2]E^2[X] - E[N]E^2[X] \\ &\quad + E[N]E[X^2]. \end{aligned}$$

Hence,

$$\begin{aligned} Var\left[\sum_1^N X_i\right] &= E^2[X](E|N^2] - E^2[N] \\ &\quad + E[N](E[X^2] - E^2[X]) \\ &= E^2[X]Var[N] + E[N]Var(X) \end{aligned}$$

49. Let A be the event that A is the overall winner, and let X be the number of games played. Let Y equal to number of wins for A in the first two games.

$$\begin{aligned} P(A) &= P(A|Y = 0)P(Y = 0) \\ &\quad + P(A|Y = 1)P(Y = 1) \\ &\quad + P(A|Y = 2)P(Y = 2) \\ &= 0 + P(A)2p(1 - p) + p^2 \end{aligned}$$

Thus,

$$P(A) = \frac{p^2}{1 - 2p(1 - p)}$$

$$\begin{aligned} E[X] &= E[X|Y = 0]P(Y = 0) \\ &\quad + E[X|Y = 1]P(Y = 1) \\ &\quad + E[X|Y = 2]P(Y = 2) \\ &= 2(1 - p)^2 + (2 + E[X])2p(1 - p) + 2p^2 \\ &= 2 + E[X]2p(1 - p) \end{aligned}$$

Thus,

$$E[X] = \frac{2}{1 - 2p(1 - p)}$$

50. $$P\{N=n\}=\frac{1}{3}\left[\begin{bmatrix}10\\n\end{bmatrix}(.3)^n(.7)^{10-n}\right.$$
$$+\begin{bmatrix}10\\n\end{bmatrix}(.5)^n(.5)^{10-n}$$
$$\left.+\begin{bmatrix}10\\n\end{bmatrix}(.7)^n(.3)^{10-n}\right].$$

N is not binomial.

$$E[N]=3\left[\frac{1}{3}\right]+5\left[\frac{1}{3}\right]+7\left[\frac{1}{3}\right]=5.$$

51. Yes.

52. $$\begin{aligned}P\{X+Y<x\}&=\int P\{X+Y<x|X=s\}f_X(s)ds\\&=\int P\{X+Y<x|X=s\}f_X(s)ds\\&=\int P\{Y<x-s|X=s\}f_X(s)ds\\&=\int P\{Y<x-s\}f_X(s)ds\\&=\int F_Y\{x-s\}f_X(s)ds\end{aligned}$$

53. $$\begin{aligned}P\{X=n\}&=\int_0^\infty P\{X=n|\lambda\}e^{-\lambda}d\lambda\\&=\int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!}e^{-\lambda}d\lambda\\&=\int_0^\infty e^{-2\lambda}\lambda^n\frac{d\lambda}{n!}\\&=\int_0^\infty e^{-t}t^n\frac{dt}{n!}\left[\frac{1}{2}\right]^{n+1}\end{aligned}$$

The result follows since

$$\int_0^\infty e^{-t}t^n dt=\Gamma(n+1)=n!$$

54. $$P\{N=k\}=\sum_{n=1}^{10}\left[\frac{10-n}{10}\right]^{k-1}\frac{n}{10}\frac{1}{10}.$$

N is not geometric. It would be if the coin was reselected after each flip.

55. Suppose in Exercise 42 that, aside from yourself, the number of other people who are invited is a Poisson random variable with mean 10.

(a) Find the expected number of people who arrive before you.

(b) Find the probability that you are the n^{th} person to arrive.

56. Let $Y=1$ if it rains tomorrow, and let $Y=0$ otherwise.

$$\begin{aligned}E[X]&=E[X|Y=1]P\{Y=1\}\\&\quad+E[X|Y=0]P\{Y=0\}\\&=9(.6)+3(.4)=6.6\end{aligned}$$

$$\begin{aligned}P\{X=0\}&=P\{X=0|Y=1\}P\{Y=1\}\\&\quad+P\{X=0|Y=0\}P\{Y=0\}\\&=.6e^{-9}+.4e^{-3}\end{aligned}$$

$$\begin{aligned}E[X^2]&=E[X^2|Y=1]P\{Y=1\}\\&\quad+E[X^2|Y=0]P\{Y=0\}\\&=(81+9)(.6)+(9+3)(.4)=58.8\end{aligned}$$

Therefore,

$$Var(X)=58.8-(6.6)^2=15.24$$

57. Let X be the number of storms.

$$\begin{aligned}P\{X\geq 3\}&=1-P\{X\leq 2\}\\&=1-\int_0^5 P\{X\leq 2|\Lambda=x\}\frac{1}{5}dx\\&=1-\int_0^5[e^{-x}+xe^{-x}+e^{-x}x^2/2]\frac{1}{5}dx\end{aligned}$$

58. Conditioning on whether the total number of flips, excluding the j^{th} one, is odd or even shows that the desired probability is 1/2.

59. Conditioning on the outcome of the initial replication yields

$$\begin{aligned}P\{A\text{ first}\}&=P\{A\text{ first}|\text{outcome in }A\}P\{A\}\\&\quad+P\{A\text{ first}|\text{in }B\}P\{B\}\\&\quad+P\{A\text{ first}|\text{neither in }A\text{ nor }B\}\\&\quad[1-P(A)-P(B)]\\&=P\{A\}+P\{A\text{ first}\}\\&\quad[1-P(A)-P(B)].\end{aligned}$$

Solving this yields that

$$P\{\text{A first}\}=\frac{P(A)}{P(A)+P(B)}.$$

60. (a) Intuitive that $f(p)$ is increasing in p, since the larger p is the greater is the advantage of going first.

 (b) 1

 (c) $1/2$ since the advantage of going first becomes nil.

 (d) Condition on the outcome of the first flip:

$$f(p) = P\{\text{I wins}|h\}p + P\{\text{I wins}|t\}(1-p)$$
$$= p + [1 - f(p)](1-p).$$

Therefore,

$$f(p) = \frac{1}{2-p}.$$

61. (a) $m_1 = E[X|h]p_1 + E[H|m]q_1 = p_1 + (1 + m_2)q_1 = 1 + m_2q_1$. Similarly, $m_2 = 1 + m_1q_2$. Solving these equations gives that

$$m_1 = \frac{1+q_1}{1-q_1q_2}, \quad m_2 = \frac{1+q_2}{1-q_1q_2}$$

 (b) $P_1 = p_1 + q_1P_2$

$P_2 = q_2P_1$

implying that

$$P_1 = \frac{p_1}{1-q_1q_2} \quad P_2 = \frac{p_1q_2}{1-q_1q_2}$$

 (c) Let f_i denote the probability that the final hit was by 1 when i shoots first. Conditioning on the outcome of the first shot gives:

$$f_1 = p_1P_2 + q_1f_2 \quad \text{and} \quad f_2 = p_2P_1 + q_2f_1$$

Solving these equations gives

$$f_1 = \frac{p_1P_2 + q_1p_2P_1}{1-q_1q_2}$$

 (d) and (e) Let B_i denote the event that both hits were by i. Condition on the outcome of the first two shots to obtain:

$$P(B_1) = p_1q_2P_1 + q_1q_2P(B_1) \to P(B_1) = \frac{p_1q_2P_1}{1-q_1q_2}$$

Also,

$$P(B_2) = q_1p_2(1-P_1) + q_1q_2P(B_2) \to P(B_2) = \frac{q_1p_2(1-P_1)}{1-q_1q_2}$$

 (f) $E[N] = 2p_1p_2 + p_1q_2(2+m_1) + q_1p_2(2+m_1) + q_1q_2(2+E[N])$

implying that

$$E[N] = \frac{2 + m_1p_1q_2 + m_1q_1p_2}{1-q_1q_2}$$

62. Let W and L stand for the events that player A wins a game and loses a game, respectively. Let $P(A)$ be the probability that A wins, and let $P(C)$ be the probability that C wins, and note that this is equal to the conditional probability that a player about to compete against the person who won the last round is the overall winner.

$$\begin{aligned}P(A) &= (1/2)P(A|W) + (1/2)P(A|L)\\ &= (1/2)[1/2 + (1/2)P(A|WL)] + (1/2)(1/2)P(C)\\ &= 1/4 + (1/4)(1/2)P(C) + (1/4)P(C) = 1/4 + (3/8)P(C)\end{aligned}$$

Also,

$$P(C) = (1/2)P(A|W) = 1/4 + (1/8)P(C)$$

and so

$$P(C) = 2/7, \quad P(A) = 5/14,$$
$$P(B) = P(A) = 5/14$$

63. Let S_i be the event there is only one type i in the final set.

$$\begin{aligned}P\{S_i = 1\} &= \sum_{j=0}^{n-1} P\{S_i = 1|T = j\}P\{T = j\}\\ &= \frac{1}{n}\sum_{j=0}^{n-1} P\{S_i = 1|T = j\}\\ &= \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{n-j}\end{aligned}$$

The final equality follows because given that there are still $n-j-1$ uncollected type when the first type i is obtained, the probability starting at that point that it will be the last of the set of $n-j$ types consisting of type i along with the $n-j-1$ yet uncollected types to be obtained is, by symmetry, $1/(n-j)$. Hence,

$$E\left[\sum_{i=1}^{n} S_i\right] = nE[S_i] = \sum_{k=1}^{n}\frac{1}{k}$$

64. (a) $P(A) = 5/36 + (31/36)(5/6)P(A)$
$\rightarrow P(A) = 30/61$

(b) $E[X] = 5/36 + (31/36)[1 + 1/6 + (5/6)$
$(1 + E[X])] \rightarrow E[X] = 402/61$

(c) Let Y equal 1 if A wins on her first attempt, let it equal 2 if B wins on his first attempt, and let it equal 3 otherwise. Then

$$Var(X|Y = 1) = 0, \quad Var(X|Y = 2) = 0,$$
$$Var(X|Y = 3) = Var(X)$$

Hence,

$$E[Var(X|Y)] = (155/216)Var(X)$$

Also,

$$E[X|Y = 1] = 1, \quad E[X|Y = 2] = 2,$$
$$E[X|Y = 3] = 2 + E[X] = 524/61$$

and so

$$\begin{aligned} Var(E[X|Y]) = 1^2(5/36) + 2^2(31/216) \\ + (524/61)^2(155/216) \\ - (402/61)^2 \approx 10.2345 \end{aligned}$$

Hence, from the conditional variance formula we see that

$$\begin{aligned} Var(X) \approx z(155/216)Var(X) + 10.2345 \\ \rightarrow Var(X) \approx 36.24 \end{aligned}$$

65. (a) $P\{Y_n = j\} = 1/(n+1), \quad j = 0, \ldots, n$

(b) For $j = 0, \ldots,\ n-1$

$$\begin{aligned} P\{Y_{n-1} = j\} &= \sum_{i=0}^{n} \frac{1}{n+1} P\{Y_{n-1} = j|Y_n = i\} \\ &= \frac{1}{n+1}(P\{Y_{n-1} = j|Y_n = j\} \\ &\quad + P\{Y_{n-1} = j|Y_n = j+1\}) \\ &= \frac{1}{n+1}(P(\text{last is nonred}|\, j \text{ red}) \\ &\quad + P(\text{last is red}|\, j+1 \text{ red}) \\ &= \frac{1}{n+1}\left(\frac{n-j}{n} + \frac{j+1}{n}\right) = 1/n \end{aligned}$$

(c) $P\{Y_k = j\} = 1/(k+1), \quad j = 0, \ldots, k$

(d) For $j = 0, \ldots, k-1$

$$\begin{aligned} P\{Y_{k-1} = j\} &= \sum_{i=0}^{k} P\{Y_{k-1} = j|Y_k = i\} \\ &\quad P\{Y_k = i\} \\ &= \frac{1}{k+1}(P\{Y_{k-1} = j|Y_k = j\} \\ &\quad + P\{Y_{k-1} = j|Y_k = j+1\}) \\ &= \frac{1}{k+1}\left(\frac{k-j}{k} + \frac{j+1}{k}\right) = 1/k \end{aligned}$$

where the second equality follows from the induction hypothesis.

66. (a) $E[G_1 + G_2] = E[G_1] + E[G_2]$
$= (.6)2 + (.4)3 + (.3)2 + (.7)3 = 5.1$

(b) Conditioning on the types and using that the sum of independent Poissons is Poisson gives the solution

$$\begin{aligned} P\{5\} = (.18)e^{-4}4^5/5! + (.54)e^{-5}5^5/5! \\ + (.28)e^{-6}6^5/5! \end{aligned}$$

67. A run of j successive heads can occur in the following mutually exclusive ways: (i) either there is a run of j in the first $n-1$ flips, or (ii) there is no j-run in the first $n-j-1$ flips, flip $n-j$ is a tail, and the next j flips are all heads. Consequently, (a) follows. Condition on the time of the first tail:

$$P_j(n) = \sum_{k=1}^{j} P_j(n-k)p^{k-1}(.1-p) + p^j, \qquad j \le n$$

68. (a) p^n

(b) After the pairings have been made there are 2^{k-1} players that I could meet in round k. Hence, the probability that players 1 and 2 are scheduled to meet in round k is $2^{k-1}/(2^n-1)$. Therefore, conditioning on the event R, that player I reaches round k, gives

$$\begin{aligned} P\{W_2\} &= P\{W_2|\, R\}p^{k-1} \\ &\quad + P\{W_2|\, R^c\}(1-p^{k-1}) \\ &= p^{n-1}(1-p)p^{k-1} + p^n(1-p^{k-1}) \end{aligned}$$

69. (a) Let $I(i,j)$ equal 1 if i and j are a pair and 0 otherwise. Then

$$E\left[\sum_{i<j} I(i,j)\right] = \binom{n}{2}\frac{1}{n}\frac{1}{n-1} = 1/2$$

Let X be the size of the cycle containing person 1. Then

$$Q_n = \sum_{i=1}^{n} P\{\text{no pairs}|X = i\}1/n = \frac{1}{n}\sum_{i \neq 2} Q_{n-i}$$

70. (a) Condition on X, the size of the cycle containing person 1, to obtain

$$M_n = \sum_{i=1}^{n} \frac{1}{n}(1 + M_{n-i}) = 1 + \frac{1}{n}\sum_{j=1}^{n-1} M_j$$

(b) Any cycle containing, say, r people is counted only once in the sum since each of the r people contributes $1/r$ to the sum. The identity gives

$$E[C] = nE[1/C_1] = n\sum_{i=1}^{n}(1/i)(1/n) = \sum_{i=1}^{n} 1/i$$

(c) Let p be the desired probability.
Condition on X

$$p = \frac{1}{n}\sum_{i=k}^{n}\frac{\binom{n-k}{i-k}}{\binom{n-1}{i-1}}$$

(d) $\dfrac{(n-k)!}{n!}$

72. For $n \geq 2$

$$\begin{aligned} &P\{N > n | U_1 = y\} \\ &\quad = P\{y \geq U_2 \geq U_3 \geq \cdots \geq U_n\} \\ &\quad = P\{U_i \leq y, i = 2, \ldots, n\} \\ &\qquad P\{U_2 \geq U_3 \geq \cdots \boldsymbol{geq} U_n | \\ &\qquad U_i \leq y, i = 2, \ldots, n\} \\ &\quad = y^{n-1}/(n-1)! \end{aligned}$$

$$\begin{aligned} E[N|U_1 = y] &= \sum_{n=0}^{\infty} P\{N > n|U_1 = y\} \\ &= 2 + \sum_{n=2}^{\infty} y^{n-1}/(n-1)! = 1 + e^y \end{aligned}$$

Also,

$$\begin{aligned} P\{M > n|U_1 = 1 - y\} &= P\{M(y) > n - 1\} \\ &= y^{n-1}/(n-1)! \end{aligned}$$

73. Condition on the value of the sum prior to going over 100. In all cases the most likely value is 101. (For instance, if this sum is 98 then the final sum is equally likely to be either 101, 102, 103, or 104. If the sum prior to going over is 95 then the final sum is 101 with certainty.)

74. Condition on whether or not component 3 works. Now

$$\begin{aligned} &P\{\text{system works}|3 \text{ works}\} \\ &\quad = P\{\text{either 1 or 2 works}\}P\{\text{either 4 or 5 works}\} \\ &\quad = (p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5). \end{aligned}$$

Also,

$$\begin{aligned} &P\{\text{system works}|\text{ 3 is failed}\} \\ &\quad = P\{\text{1 and 4 both work, or 2 and 5 both work}\} \\ &\quad = p_1p_4 - p_2p_5 - p_1p_4p_2p_5. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} &P\{\text{system works}\} \\ &\quad = p_3(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) \\ &\qquad + (1 - p_3)(p_1p_4 + p_2p_5 - p_1p_4p_2p_5). \end{aligned}$$

75. (a) Since A receives more votes than B (since $a > a$) it follows that if A is not always leading then they will be tied at some point.

(b) Consider any outcome in which A receives the first vote and they are eventually tied, say $a, a, b, a, b, a, b, b \ldots$ We can correspond this sequence to one which takes the part of the sequence until they are tied in the reverse order. That is, we correspond the above to the sequence $b, b, a, b, a, b, a, a \ldots$ where the remainder of the sequence is exactly as in the original. Note that this latter sequence is one in which B is initially ahead and then they are tied. As it is easy to see that this correspondence is one to one, part (b) follows.

(c) Now,
$P\{B$ receives first vote and they are eventually tied$\}$
$= P\{B$ receives first vote$\}= n/(n + m)$.
Therefore, by part (b) we see that
$P\{$eventually tied$\}= 2n/(n + m)$
and the result follows from part (a).

76. By the formula given in the text after the ballot problem we have that the desired probability is

$$\frac{1}{3}\binom{15}{5}(18/38)^{10}(20/38)^5.$$

77. We will prove it when X and Y are discrete.

(a) This part follows from (b) by taking $g(x, y) = xy$.

(b) $E[g(X,Y)|Y=\overline{y}] = \sum_y \sum_x g(x,y)$
$P\{X=x, Y=y|Y=\overline{y}\}$.

Now,

$$P\{X=x, Y=y|Y=\overline{y}\} = \begin{cases} 0, & \text{if } y \neq \overline{y} \\ P\{X=x,\ Y=\overline{y}\}, & \text{if } y=\overline{y}. \end{cases}$$

So,

$$E[g(X,Y)|Y=\overline{y}] = \sum_k g(x,\overline{y})P\{X=x|Y=\overline{y}\}.$$
$$= E[g(x,\ \overline{y})|Y=\overline{y}.$$

(c) $E[XY] = E[E[XY|Y]]$
$= E[YE[X|Y]]$ by (a).

78. Let $Q_{n,m}$ denote the probability that A is never behind, and $P_{n,m}$ the probability that A is always ahead. Computing $P_{n,m}$ by conditioning on the first vote received yields

$$P_{n,m} = \frac{n}{n+m}Q_{n-1,m}.$$

But as $P_{n,m} = \dfrac{n-m}{n+m}$, we have

$$Q_{n-1,m} = \frac{n+m}{n}\frac{n-m}{n+m} = \frac{n-m}{n},$$

and so the desired probability is

$$Q_{n,m} = \frac{n+1-m}{n+1}.$$

This also can be solved by conditioning on who obtains the last vote. This results in the recursion

$$Q_{n,m} = \frac{n}{n+m}Q_{n-1,m} + \frac{m}{n+m}Q_{n,m} - 1,$$

which can be solved to yield

$$Q_{n,m} = \frac{n+1-m}{n+1}.$$

79. Let us suppose we take a picture of the urn before each removal of a ball. If at the end of the experiment we look at these pictures in reverse order (i.e., look at the last taken picture first), we will see a set of balls increasing at each picture. The set of balls seen in this fashion always will have more white balls than black balls if and only if in the original experiment there were always more white than black balls left in the urn. Therefore, these two events must have same probability, i.e., $n-m/n+m$ by the ballot problem.

80. Condition on the total number of heads and then use the result of the ballot problem. Let p denote the desired probability, and let j be the smallest integer that is at least $n/2$.

$$p = \sum_{i=j}^{n} \binom{n}{i} p^i(1-p)^{n-i}\frac{2i-n}{n}$$

81. (a) $f(x) = E[N] = \int_0^1 E[N|X_1=y]dy,$

$$E[N|X_1=y] = \begin{cases} 1 & \text{if } y<x, \\ 1+f(y) & \text{if } y>x. \end{cases}$$

Hence,

$$f(x) = 1 + \int_x^1 f(y)dy.$$

(b) $f'(x) = -f(x)$.

(c) $f(x) = ce^{-x}$. Since $f(1)=1$, we obtain that $c=e$, and so $f(x)=e^{1-x}$.

(d) $P\{N>n\} = P\{x<X_1<X_2<\cdots<X_n\} = (1-x)^n/n!$ since in order for the above event to occur all of the n random variables must exceed x [and the probability of this is $(1-x)^n$], and then among all of the $n!$ equally likely orderings of this variables the one in which they are increasing must occur.

(e) $E[N] = \sum_{n=0}^{\infty} P\{N>n\}$
$= \sum_n (1-x)^n/n! = e^{1-x}$.

82. (a) Let A_i denote the event that X_i is the k^{th} largest of $X_1,\ldots,X_i$. It is easy to see that these are independent events and $P(A_i)=1/i$.

$$P\{N_k=n\} = P(A_k^c A_{k+1}^c \cdots A_{n-1}^c A_n)$$
$$= \frac{k-1}{k}\frac{k}{k+1}\cdots\frac{n-2}{n-1}\frac{1}{n}$$
$$= \frac{k-1}{n(n-1)}$$

(b) Since knowledge of the set of values $\{X_1,\ldots,X_n\}$ gives us no information about the order of these random variables it follows that given $N_k=n$, the conditional distribution of X_{N_k} is the same as the distribution of the k^{th} largest of n random variables having distribution F. Hence,

$$f_{X_{N_k}}(x) = \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)}\frac{n!}{(n-k)!(k-1)!}$$
$$\times (F(x))^{n-k}(\overline{F}(x))^{k-1}f(x)$$

Now make the change of variable $i = n - k$. (c) follow the hint (d) It follows from (b) and (c) that $f_{X_{N_k}}(x) = f(x)$.

83. Let I_j equal 1 if ball j is drawn before ball i and let it equal 0 otherwise. Then the random variable of interest is $\sum_{j \neq i} I_j$. Now, by considering the first time that either i or j is withdrawn we see that $P\{j \text{ before } i\} = w_j/(w_i + w_j)$. Hence,

$$E\left[\sum_{j\neq i} I_j\right] = \sum_{j\neq i} \frac{w_j}{w_i + w_j}$$

84. We have

E[Position of element requested at time t]

$$= \sum_{i=i}^{n} E[\text{Position at time } t \mid e_i \text{ selected}]P_i$$
$$= \sum_{i=1}^{n} E[\text{Position of } e_i \text{ at time } t]P_i,$$

with $I_j = \begin{cases} 1, & \text{if } e_j \text{ precedes } e_i \text{ at time } t \\ 0, & \text{otherwise.} \end{cases}$

We have

Position of e_i at time $t = 1 + \sum_{j\neq i} I_j$

and so,

E[Position of e_i at time t]

$$= 1 + \sum_{j\neq i} E(I_j)$$
$$= 1 + \sum_{j\neq i} P\{e_j \text{ precedes } e_i \text{ at time } t\}.$$

Given that a request has been made for either e_i or e_j, the probability that the most recent one was for e_j is $P_j/(P_i + P_j)$. Therefore,

$$P\{e_j \text{ precedes } e_i \text{ at time } t | e_i \text{ or } e_j \text{ was requested}\} = \frac{P_j}{P_i + P_j}.$$

On the other hand,

$P\{e_j$ precedes e_i at time t | neither was ever requested$\}$

$$= \frac{1}{2}.$$

As

$$P\{\text{Neither } e_i \text{ or } e_j \text{ was ever requested by time } t\} = (1 - P_i - P_j)^{t-1}.$$

we have

E[Position of e_i at time t]

$$= 1 + \sum_{j\neq i} \left[\frac{1}{2}(1 - P_i - P_j)^{t-1} + \frac{P_j}{P_j + P_i}\,(1 - (1 - P_i - P_j)^{t-1})\right]$$

and

$$E[\text{Position of element requested at } t] = \sum P_j E[\text{Position of } e_i \text{ at time } t].$$

85. Consider the following ordering:

$e_1, e_2, \ldots, e_{l-1}, i, j, e_{l+1}, \ldots, e_n$ where $P_i < P_j$.

We will show that we can do better by interchanging the order of i and j, i.e., by taking $e_1, e_2, \ldots, e_{l-1}, j, i, e_{l+2}, \ldots, e_n$. For the first ordering, the expected position of the element requested is

$$E_{i,\,j} = P_{e_1} + 2P_{e_2} + \cdots + (l-1)P_{e_{l-1}} + lp_i + (l+1)P_j + (l+2)P_{e_{l+2}} + \cdots .$$

Therefore,

$$E_{i,j} - E_{j,i} = l(P_i - P_j) + (l+1)(P_j - P_i) = P_j - P_i > 0,$$

and so the second ordering is better. This shows that every ordering for which the probabilities are not in decreasing order is not optimal in the sense that we can do better. Since there are only a finite number of possible orderings, the ordering for which $p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n$ is optimum.

87. (i) This can be proved by induction on m. It is obvious when $m = 1$ and then by fixing the value of x_1 and using the induction hypothesis, we see that there are $\sum_{i=0}^{n} \begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix}$ such solutions. As $\begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix}$ equals the number of ways of choosing $m-1$ items from a set of size $n+m-1$ under the constraint that the lowest numbered item selected is number $i+1$ (that is, none of $1, \ldots, i$ are selected where $i+1$ is), we see that

$$\sum_{i=0}^{n}\begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix} = \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}.$$

It also can be proven by noting that each solution corresponds in a one-to-one fashion with a permutation of n ones and $(m-1)$ zeros. The correspondence being that x_1 equals the number of ones to the left of the first zero, x_2 the number of ones between the first and second zeros, and so on. As there are $(n+m-1)!/n!(m-1)!$ such permutations, the result follows.

(ii) The number of positive solutions of $x_1+\cdots+x_m=n$ is equal to the number of nonnegative solutions of $y_1+\cdots+y_m=n-m$, and thus there are $\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$ such solutions.

(iii) If we fix a set of k of the x_i and require them to be the only zeros, then there are by (ii) (with m replaced by $m-k$) $\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix}$ such solutions. Hence, there are $\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix}$ outcomes such that exactly k of the X_i are equal to zero, and so the desired probability is $\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix} \Big/ \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$.

88. (a) Since the random variables $U, X_1, \ldots, X_n$ are all independent and identically distributed it follows that U is equally likely to be the i^{th} smallest for each $i+1,\ldots,n+1$. Therefore,

$$P\{X=i\} = P\{U \text{ is the } (i+1)^{st} \text{ smallest}\}$$
$$= 1/(n+1).$$

(b) Given U, each X_i is less than U with probability U, and so X is binomial with parameters n, U. That is, given that $U < p$ X is binomial with parameters n, p. Since U is uniform on $(0,1)$ this is exactly the scenario in Section 6.3.

89. Condition on the value of I_n This gives

$$P_n(K) = P\{\sum_{j=1}^{n} jI_j \le K | I_n = 1\}1/2$$
$$+ P\{\sum_{j=1}^{n} jI_j \le K | I_n = 0\}1/2$$
$$= P\{\sum_{j=1}^{n-1} jI_j + n \le K\}1/2$$
$$+ P\{\sum_{j=1}^{n-1} jI_j \le K\}1/2$$
$$= [P_{n-1}(k-n) + P_{n-1}(K)]/2.$$

90. (a) $\dfrac{1}{e^{-5}5^2/2! \cdot 5e^{-5} \cdot e^{-5}}$

(b) $\dfrac{1}{e^{-5}5^2/2! \cdot 5e^{-5} \cdot e^{-5} \cdot e^{-5}5^2/2!} + \dfrac{1}{e^{-5}5^2/2!}$

91. $\dfrac{1}{p^5(1-p)^3} + \dfrac{1}{p^2(1-p)} + \dfrac{1}{p}$

92. Let X denote the amount of money Josh picks up when he spots a coin. Then

$$E[X] = (5+10+25)/4 = 10,$$
$$E[X^2] = (25+100+625)/4 = 750/4$$

Therefore, the amount he picks up on his way to work is a compound Poisson random variable with mean $10 \cdot 6 = 60$ and variance $6 \cdot 750/4 = 1125$. Because the number of pickup coins that Josh spots is Poisson with mean $6(3/4) = 4.5$, we can also view the amount picked up as a compound Poisson random variable $S = \sum_{i=1}^{N} X_i$ where N is Poisson with mean 4.5, and (with 5 cents as the unit of measurement) the X_i are equally likely to be 1, 2, 3. Either use the recursion developed in the text or condition on the number of pickups to determine $P(S=5)$. Using the latter approach, with $P(N=i) = e^{-4.5}(4.5)^i/i!$, gives

$$P(S=5) = (1/3)P(N=1) + 3(1/3)^3P(N=3)$$
$$+ 4(1/3)^4P(N=4) + 5(1/3)^5P(N=5)$$

94. Using that $E[N] = rw/(w+b)$ yields

$$P\{M-1=n\}$$
$$= \frac{(n+1)P\{N=n+1\}}{E[N]}$$
$$= \frac{(n+1)\binom{w}{n+1}\binom{b}{r-n-1}(w+b)}{rw\binom{w+b}{r}}$$

Using that

$$\frac{(n+1)\binom{w}{n+1}}{w} = \binom{w-1}{n}\frac{w+b}{r\binom{w+b}{r}}$$

$$= \frac{1}{\binom{w+b-1}{r-1}}$$

shows that

$$P\{M-1=n\} = \frac{\binom{w-1}{n}\binom{b}{r-n-1}}{\binom{w+b-1}{r-1}}$$

$$P_{w,r}(k) = \frac{rw}{k(w+b)}\sum_{i=1}^{k} i\alpha_i P_{w-1,r-1}(k-i)$$

When $k=1$

$$P_{w,r}(1) = \frac{rw}{w+b}\alpha_1\frac{\binom{b}{r-1}}{\binom{w+b-1}{r-1}}$$

Chapter 4

1. $P_{01} = 1, \quad P_{10} = \frac{1}{9}, \quad P_{21} = \frac{4}{9}, \quad P_{32} = 1$

$$P_{11} = \frac{4}{9}, \quad P_{22} = \frac{4}{9}$$

$$P_{12} = \frac{4}{9}, \quad P_{23} = \frac{1}{9}.$$

2, 3.

$$P = \begin{array}{c} \\ (RRR) \\ (RRD) \\ (RDR) \\ (RDD) \\ (DRR) \\ (DRD) \\ (DDR) \\ (DDD) \end{array} \begin{array}{c} \begin{array}{cccccccc} (RRR) & (RRD) & (RDR) & (RDD) & (DRR) & (DRD) & (DDR) & (DDD) \end{array} \\ \left\| \begin{array}{cccccccc} .8 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & .4 & .6 & & & & \\ & & & & .6 & .4 & & \\ & & & & & & .4 & .6 \\ .6 & .4 & & & & & & \\ & & .4 & .6 & & & & \\ & & & & .6 & .4 & & \\ & & & & & & .2 & .8 \end{array} \right\| \end{array}$$

where $D =$ dry and $R =$ rain. For instance, (DDR) means that it is raining today, was dry yesterday, and was dry the day before yesterday.

4. Let the state space be $S = \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$, where state $i(\bar{i})$ signifies that the present value is i, and the present day is even (odd).

5. Cubing the transition probability matrix, we obtain P^3:

$$\begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$$

Thus,

$$E[X_3] = \frac{1}{4}47/108 + \frac{1}{4}11/27 + \frac{1}{2}13/36$$

6. It is immediate for $n = 1$, so assume for n. Now use induction.

7. $P_{30}^2 + P_{31}^2 = P_{31}P_{10} + P_{33}P_{11} + P_{33}P_{31}$

$$= (.2)(.5) + (.8)(0) + (.2)(0) + (.8)(.2)$$

$$= .26.$$

8. Let the state on any day be the number of the coin that is flipped on that day.

$$\underline{P} = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

and so,

$$\underline{P}^2 = \begin{bmatrix} .67 & .33 \\ .66 & .34 \end{bmatrix}$$

and

$$\underline{P}^3 = \begin{bmatrix} .667 & .333 \\ .666 & .334 \end{bmatrix}$$

Hence,

$$\frac{1}{2}\left[P_{11}^3 + P_{21}^3\right] \equiv .6665.$$

9. If we let the state be 0 when the most recent flip lands heads and let it equal 1 when it lands tails, then the sequence of states is a Markov chain with transition probability matrix

$$\begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

The desired probability is $P_{0,0}^4 = .6667$

10. The answer is $1 - P_{0,2}^3$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ 0 & 0 & 1 \end{bmatrix}$$

11. The answer is $\dfrac{P_{2,2}^4}{1 - P_{2,0}^4}$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

12. The result is not true. For instance, suppose that $P_{0,1} = P_{0,2} = 1/2, P_{1,0} = 1, P_{2,3} = 1$. Given $X_0 = 0$ and that state 3 has not been entered by time 2, the equality implies that X_1 is equally likely to be 1 or 2, which is not true because, given the information, X_1 is equal to 1 with certainty.

13. $P_{ij}^n = \sum_k P_{ik}^{n-r} P_{kj}^r > 0$

14. (i) $\{0,1,2\}$ recurrent.
    (ii) $\{0,1,2,3\}$ recurrent.
    (iii) $\{0,2\}$ recurrent, $\{1\}$ transient, $\{3,4\}$ recurrent.
    (iv) $\{0,1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

15. Consider any path of states $i_0 = i, i_1, i_2, \ldots, i_n = j$ such that $P_{i_k i_{k+1}} > 0$. Call this a path from i to j. If j can be reached from i, then there must be a path from i to j. Let $i_0, \ldots, i_n$ be such a path. If all of the values $i_0, \ldots, i_n$ are not distinct, then there is a subpath from i to j having fewer elements (for instance, if $i, 1, 2, 4, 1, 3, j$ is a path, then so is $i, 1, 3, j$). Hence, if a path exists, there must be one with all distinct states.

16. If P_{ij} were (strictly) positive, then P_{ji}^n would be 0 for all n (otherwise, i and j would communicate). But then the process, starting in i, has a positive probability of at least P_{ij} of never returning to i. This contradicts the recurrence of i. Hence $P_{ij} = 0$.

17. $\sum_{i=1}^{n} Y_j/n \to E[Y]$ by the Strong law of large numbers. Now $E[Y] = 2p - 1$. Hence, if $p > 1/2$, then $E[Y] > 0$, and so the average of the Y_i's converges in this case to a positive number, which implies that $\sum_1^n Y_i \to \infty$ as $n \to \infty$. Hence, state 0 can be visited only a finite number of times and so must be transient. Similarly, if $p < 1/2$, then $E[Y] < 0$, and so $\lim \sum_1^n Y_i = -\infty$, and the argument is similar.

18. If the state at time n is the n^{th} coin to be flipped then sequence of consecutive states constitute a two state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, \quad P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\pi_1 = .6\pi_1 + .5\pi_2$$
$$\pi_1 + \pi_2 = 1$$

Solving yields that $\pi_1 = 5/9, \pi_2 = 4/9$. So the proportion of flips that use coin 1 is $5/9$.

(b) $P_{1,2}^4 = .44440$

19. The limiting probabilities are obtained from

$$r_0 = .7r_0 + .5r_1$$
$$r_1 = .4r_2 + .2r_3$$
$$r_2 = .3r_0 + .5r_1$$
$$r_0 + r_1 + r_2 + r_3 = 1,$$

and the solution is

$$r_0 = \frac{1}{4}, \quad r_1 = \frac{3}{20}, \quad r_2 = \frac{3}{20}, \quad r_3 = \frac{9}{20}.$$

The desired result is thus

$$r_0 + r_1 = \frac{2}{5}.$$

20. If $\sum_{i=0}^{m} P_{ij} = 1$ for all j, then $r_j = 1/(M+1)$ satisfies

$$r_j = \sum_{i=0}^{m} r_i P_{ij}, \quad \sum_0^m r_j = 1.$$

Hence, by uniqueness these are the limiting probabilities.

21. The transition probabilities are

$$P_{i,j} = \begin{cases} 1 - 3\alpha, & \text{if } j = i \\ \alpha, & \text{if } j \neq i \end{cases}$$

By symmetry,

$$P_{ij}^n = \frac{1}{3}(1 - P_{ii}^n), \quad j \neq i$$

So, let us prove by induction that

$$P_{i,j}^n = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^n & \text{if } j \neq i \end{cases}$$

As the preceding is true for $n = 1$, assume it for n. To complete the induction proof, we need to show that

$$P_{i,j}^{n+1} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1} & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n+1} & \text{if } j \neq i \end{cases}$$

Now,

$$\begin{aligned}P_{i,i}^{n+1} &= P_{i,i}^n P_{i,i} + \sum_{j\neq i} P_{i,j}^n P_{j,i}\\ &= \left(\frac{1}{4}+\frac{3}{4}(1-4\alpha)^n\right)(1-3\alpha)\\ &\quad +3\left(\frac{1}{4}-\frac{1}{4}(1-4\alpha)^n\right)\alpha\\ &= \frac{1}{4}+\frac{3}{4}(1-4\alpha)^n(1-3\alpha-\alpha)\\ &= \frac{1}{4}+\frac{3}{4}(1-4\alpha)^{n+1}\end{aligned}$$

By symmetry, for $j \neq i$

$$P_{ij}^{n+1} = \frac{1}{3}\left(1-P_{ii}^{n+1}\right) = \frac{1}{4}-\frac{1}{4}(1-4\alpha)^{n+1}$$

and the induction is complete.

By letting $n \to \infty$ in the preceding, or by using that the transition probability matrix is doubly stochastic, or by just using a symmetry argument, we obtain that $\pi_i = 1/4$.

22. Let X_n denote the value of Y_n modulo 13. That is, X_n is the remainder when Y_n is divided by 13. Now X_n is a Markov chain with states $0, 1, \ldots, 12$. It is easy to verify that $\sum_i P_{ij} = 1$ for all j. For instance, for $j = 3$:

$$\begin{aligned}\sum_i P_{ij} &= P_{2,3}+P_{1,3}+P_{0,3}+P_{12,3}+P_{11,3}+P_{10,3}\\ &= \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = 1.\end{aligned}$$

Hence, from Problem 20, $r_i = \dfrac{1}{13}$.

23. Let the state be 0 if the last two trials were both successes. 1 if the last trial was a success and the one before it a failure. 2 if the last trial was a failure. The transition probability matrix of this Markov chain is

$$P = \begin{bmatrix} .8 & 0 & .2\\ .5 & 0 & .5\\ 0 & .5 & .5\end{bmatrix}$$

This gives $\pi_0 = 5/11, \pi_1 = 2/11, \pi_2 = 4/11$. Consequently, the proportion of trials that are successes is $.8\pi_0 + .5(1-\pi_0) = 7/11$.

24. Let the state be the color of the last ball selected, call it 0 if that color was red, 1 if white, and 2 if blue. The transition probability matrix of this Markov chain is

$$P = \begin{bmatrix} 1/5 & 0 & 4/5\\ 2/7 & 3/7 & 2/7\\ 3/9 & 4/9 & 2/9\end{bmatrix}$$

Solve for the stationary probabilities to obtain the solution.

25. Letting X_n denote the number of pairs of shoes at the door the runner departs from at the beginning of day n, then $\{X_n\}$ is a Markov chain with transition probabilities

$$\begin{aligned}P_{i,i} &= 1/4, & 0<i<k\\ P_{i,i-1} &= 1/4, & 0<i<k\\ P_{i,k-i} &= 1/4, & 0<i<k\\ P_{i,k-i+1} &= 1/4, & 0<i<k\end{aligned}$$

The first equation refers to the situation where the runner returns to the same door she left from and then chooses that door the next day; the second to the situation where the runner returns to the opposite door from which she left from and then chooses the original door the next day; and so on. (When some of the four cases above refer to the same transition probability, they should be added together. For instance, if $i = 4, k = 8$, then the preceding states that $P_{i,i} = 1/4 = P_{i,k-i}$. Thus, in this case, $P_{4,4} = 1/2$.) Also,

$$\begin{aligned}P_{0,0} &= 1/2\\ P_{0,k} &= 1/2\\ P_{k,k} &= 1/4\\ P_{k,0} &= 1/4\\ P_{k,1} &= 1/4\\ P_{k,k-1} &= 1/4\end{aligned}$$

It is now easy to check that this Markov chain is doubly stochastic—that is the column sums of the transition probability matrix are all 1—and so the long run proportions are equal. Hence, the proportion of time the runner runs barefooted is $1/(k+1)$.

26. Let the state be the ordering, so there are $n!$ states. The transition probabilities are

$$P_{(i_1,\ldots,i_n),(i_j,i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_n)} = \frac{1}{n}.$$

It is now easy to check that this Markov chain is doubly stochastic and so, in the limit, all $n!$ possible states are equally likely.

27. The limiting probabilities are obtained from

$$r_0 = \frac{1}{9}r_1$$
$$r_1 = r_0 + \frac{4}{9}r_1 + \frac{4}{9}r_2$$
$$r_2 = \frac{4}{9}r_1 + \frac{4}{9}r_2 + r_3$$
$$r_0 + r_1 + r_2 + r_3 = 1,$$

and the solution is $r_0 = r_3 = \frac{1}{20}$. $r_1 = r_2 = \frac{9}{20}$.

28. There are 4 states: 1 = success on last 2 trials;
2 = success on last, failure on next to last;
3 = failure on last, success on next to last;
4 = failure on last 2 trials.

Transition probabilities are:

$$P_{1,1} = \frac{3}{4}, \quad P_{1,3} = \frac{1}{4}$$
$$P_{2,1} = \frac{2}{3}, \quad P_{2,3} = \frac{1}{3}$$
$$P_{3,2} = \frac{2}{3}, \quad P_{3,4} = \frac{1}{3}$$
$$P_{4,2} = \frac{1}{2}, \quad P_{4,4} = \frac{1}{2}.$$

Limiting probabilities are given by

$$\prod_1 = \frac{3}{4}\prod_1 + \frac{2}{3}\prod_2$$
$$\prod_2 = \frac{2}{3}\prod_3 + \frac{1}{2}\prod_4$$
$$\prod_3 = \frac{1}{4}\prod_1 + \frac{1}{3}\prod_2$$
$$\prod_1 + \prod_2 + \prod_3 + \prod_4 = 1,$$

and the solution is $\prod_1 = 1/2, \prod_2 = 3/16, \prod_3 = 3/16, \prod_4 = 1/8$. Hence, the desired answer is $\prod_1 + \prod_2 = 11/16$.

29. Each employee moves according to a Markov chain whose limiting probabilities are the solution of

$$\prod_1 = .7\prod_1 + .2\prod_2 + .1\prod_3$$
$$\prod_2 = .2\prod_1 + .6\prod_2 + .4\prod_3$$
$$\prod_1 + \prod_2 + \prod_3 = 1.$$

Solving yields $\prod_1 = 6/17, \prod_2 = 7/17, \prod_3 = 4/17$. Hence, if N is large, it follows from the law of large numbers that approximately 6, 7, and 4 of each 17 employees are in categories 1, 2, and 3.

30. Letting X_n be 0 if the n^{th} vehicle is a car and letting it be 1 if the vehicle is a truck gives rise to a two-state Markov chain with transition probabilities

$$P_{00} = 4/5, \quad P_{01} = 1/5$$
$$P_{10} = 3/4, \quad P_{11} = 1/4.$$

The long run proportions are the solutions of

$$r_0 = \frac{4}{5}r_0 + \frac{3}{4}r_1$$
$$r_1 = \frac{1}{5}r_0 + \frac{1}{4}r_1$$
$$r_0 + r_1 = 1.$$

Solving these gives the result

$$r_0 = \frac{15}{19}, \quad r_1 = \frac{4}{19}.$$

That is, 4 out of every 19 cars is a truck.

31. Let the state on day n be 0 if sunny, 1 if cloudy, and 2 if rainy. This gives a three state Markov chain with transition probability matrix.

	0	1	2
0	0	1/2	1/2
P = 1	1/4	1/2	1/4
2	1/4	1/4	1/2

The equations for the long run proportions are

$$r_0 = \frac{1}{4}r_1 + \frac{1}{4}r_2$$
$$r_1 = \frac{1}{2}r_0 + \frac{1}{2}r_1 + \frac{1}{4}r_2$$
$$r_2 = \frac{1}{2}r_0 + \frac{1}{4}r_1 + \frac{1}{2}r_2$$
$$r_0 + r_1 + r_2 = 1.$$

By symmetry it is easy to see that $r_1 = r_2$. This makes it easy to solve and we obtain the result

$$r_0 = \frac{1}{5}, \quad r_1 = \frac{2}{5}, \quad r_2 = \frac{2}{5}.$$

32. With the state being the number of off switches this is a 3 state Markov chain. The equations for the long run proportions are

$$r_0 = \frac{1}{16}\, r_0 + \frac{1}{4}\, r_1 + \frac{9}{16}\, r_2$$
$$r_1 = \frac{3}{8}\, r_0 + \frac{1}{2}\, r_1 + \frac{3}{8}\, r_2$$
$$r_0 + r_1 + r_2 = 1.$$

This gives the solution

$$r_0 = 2/7, \quad r_1 = 3/7, \quad r_2 = 2/7.$$

33. Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following.

$$P_{11} = .3(1/3) + .7(1) = .8$$
$$P_{12} = P_{13} = .3(1/3) = .1$$
$$P_{21} = .6(1/3) + .4(1) = .6$$
$$P_{22} = P_{23} = .6(1/3) = .2$$
$$P_{31} = .9(1/3) + .1(1) = .4$$
$$P_{32} = P_{33} = .9(1/3) = .3$$

Let r_i denote the proportion of exams that are type $i, i = 1, 2, 3$. The r_i are the solutions of the following set of linear equations.

$$r_1 = .8\, r_1 + .6\, r_2 + .4\, r_3$$
$$r_2 = .1\, r_1 + .2\, r_2 + .3\, r_3$$
$$r_1 + r_2 + r_3 = 1$$

Since $P_{i2} = P_{i3}$ for all states i, it follows that $r_2 = r_3$. Solving the equations gives the solution

$$r_1 = 5/7, \quad r_2 = r_3 = 1/7.$$

34. (a) $\pi_i,\ i = 1, 2, 3$, which are the unique solutions of the following equations.

$$\pi_1 = q_2\pi_2 + p_3\pi_3$$
$$\pi_2 = p_1\pi_1 + q_3\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

(b) The proportion of time that there is a counterclockwise move from i which is followed by 5 clockwise moves is $\pi_i q_i p_{i-1} p_i p_{i+1} p_{i+2} p_{i+3}$, and so the answer to (b) is $\sum_{i=1}^{3} \pi_i q_i p_{i-1} p_i p_{i+1} p_{i+2} p_{i+3}$. In the preceding, $p_0 = p_3, p_4 = p_1, p_5 = p_2, p_6 = p_3$.

35. The equations are

$$r_0 = r_1 + \frac{1}{2}\, r_2 + \frac{1}{3}\, r_3 + \frac{1}{4}\, r_4$$
$$r_1 = \frac{1}{2}\, r_2 + \frac{1}{3}\, r_3 + \frac{1}{4}\, r_4$$
$$r_2 = \frac{1}{3}\, r_3 + \frac{1}{4}\, r_4$$
$$r_3 = \frac{1}{4}\, r_4$$
$$r_4 = r_0$$
$$r_0 + r_1 + r_2 + r_3 + r_4 = 1$$

The solution is

$$r_0 = r_4 = 12/37, \quad r_1 = 6/37, \quad r_2 = 4/37,$$
$$r_3 = 3/37.$$

36. (a) $p_0 P_{0,0} + p_1 P_{0,1} = .4p_0 + .6p_1$

(b) $p_0 P^4_{0,0} + p_1 P^4_{0,1} = .2512p_0 + .7488p_1$

(c) $p_0\pi_0 + p_1\pi_1 = p_0/4 + 3p_1/4$

(d) Not a Markov chain.

37. Must show that

$$\pi_j = \sum_i \pi_i P^k_{i,j}$$

The preceding follows because the right hand side is equal to the probability that the Markov chain with transition probabilities $P_{i,j}$ will be in state j at time k when its initial state is chosen according to its stationary probabilities, which is equal to its stationary probability of being in state j.

38. Because j is accessible from i, there is an n such that $P^n_{i,j} > 0$. Because $\pi_i P^n_{i,j}$ is the long run proportion of time the chain is currently in state j and had been in state i exactly n time periods ago, the inequality follows.

39. Because recurrence is a class property it follows that state j, which communicates with the recurrent state i, is recurrent. But if j were positive recurrent, then by the previous exercise so would be i. Because i is not, we can conclude that j is null recurrent.

40. (a) Follows by symmetry.

(b) If $\pi_i = a > 0$ then, for any n, the proportion of time the chain is in any of the states $1, \ldots, n$ is na. But this is impossible when $n > 1/a$.

41. (a) The number of transitions into state i by time n, the number of transitions originating from state i by time n, and the number of time periods the chain is in state i by time n all differ by at most 1. Thus, their long run proportions must be equal.

(b) $r_i P_{ij}$ is the long run proportion of transitions that go from state i to state j.

(c) $\sum_j r_i P_{ij}$ is the long run proportion of transitions that are into state j.

(d) Since r_j is also the long run proportion of transitions that are into state j, it follows that

$$r_j = \sum_j r_i P_{ij}.$$

42. (a) This is the long run proportion of transitions that go from a state in A to one in A^c.

(b) This is the long run proportion of transitions that go from a state in A^c to one in A.

(c) Between any two transitions from A to A^c there must be one from A^c to A. Similarly between any two transitions from A^c to A there must be one from A to A^c. Therefore, the long run proportion of transitions that are from A to A^c must be equal to the long run proportion of transitions that are from A^c to A.

43. Consider a typical state—say, 1 2 3. We must show

$$\prod_{123} = \prod_{123} P_{123,\,123} + \prod_{213} P_{213,\,123} + \prod_{231} P_{231,\,123}.$$

Now $P_{123,\,123} = P_{213,\,123} = P_{231,\,123} = P_1$ and thus,

$$\prod_{123} = P_1 \left[\prod_{123} + \prod_{213} + \prod_{231}\right].$$

We must show that

$$\prod_{123} = \frac{P_1 P_2}{1 - P_1}, \prod_{213} = \frac{P_2 P_1}{1 - P_2}, \prod_{231} = \frac{P_2 P_3}{1 - P_2}$$

satisfies the above which is equivalent to

$$\begin{aligned} P_1 P_2 &= P_1 \left[\frac{P_2 P_1}{1 - P_2} + \frac{P_2 P_3}{1 - P_2}\right] \\ &= \frac{P_1}{1 - P_2} P_2 (P_1 + P_3) \\ &= P_1 P_2 \quad (\text{since } P_1 + P_3 = 1 - P_2). \end{aligned}$$

By symmetry all of the other stationary equations also follow.

44. Given X_n, $X_{n=1}$ is binomial with parameters m and $p = X_n/m$. Hence, $E[X_{n+1}|X_n] = m(X_n/m) = X_n$, and so $E[X_{n+1}] = E[X_n]$. So $E[X_n] = i$ for all n. To solve (b) note that as all states but 0 and m are transient, it follows that X_n will converge to either 0 or m. Hence, for n large

$$\begin{aligned} E[X_n] &= mP\{\text{hits } m\} + 0\, P\{\text{hits } 0\} \\ &= mP\{\text{hits } m\}. \end{aligned}$$

But $E[X_n] = i$ and thus $P\{\text{hits } m\} = i/m$.

45. (a) 1, since all states communicate and thus all are recurrent since state space is finite.

(b) Condition on the first state visited from i.

$$x_i = \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN}, \quad i = 1, \ldots, N-1$$

$$x_0 = 0, \quad x_N = 1.$$

(c) Must show

$$\begin{aligned} \frac{i}{N} &= \sum_{j=1}^{N-1} \frac{j}{N} P_{ij} + P_{iN} \\ &= \sum_{j=0}^{N} \frac{j}{N} P_{ij} \end{aligned}$$

and follows by hypothesis.

46. (i) Let the state be the number of umbrellas he has at his present location. The transition probabilities are

$$P_{0,x} = 1, P_{i,r-i} = 1 - p, P_{i,r-i+1} = p, \\ i = 1, \ldots, r.$$

(ii) We must show that $\prod_j = \sum_1 r_i P_{ij}$ is satisfied by the given solution. These equations reduce to

$$\begin{aligned} r_r &= r_0 + r_1 p \\ r_j &= r_{r-j}(1-p) + r_{r-j+1} p, \quad j = 1, \ldots, r-1 \\ r_0 &= r_r(1-p), \end{aligned}$$

and it is easily verified that they are satisfied.

(iii) $pr_0 = \dfrac{pq}{r+q}$.

(iv) $$\frac{d}{dp}\left[\frac{p(1-p)}{4-p}\right] = \frac{(4-p)(1-2p) + p(1-p)}{(4-p)^2} = \frac{p^2 - 8p + 4}{(4-p)^2}$$

$$p^2 - 8p + 4 = 0 \Rightarrow p = \frac{8 - \sqrt{48}}{2} = .55.$$

47. $\{Y_n, n \geq 1\}$ is a Markov chain with states (i, j).

$$P_{(i,j),(k,\ell)} = \begin{cases} 0, & \text{if } j \neq k \\ P_{j\ell}, & \text{if } j = k, \end{cases}$$

where $P_{j\ell}$ is the transition probability for $\{X_n\}$.

$$\begin{aligned}\lim_{n\to\infty} P\{Y_n = (i,j)\} &= \lim_n P\{X_n = i, X_{n+1} = j\} \\ &= \lim_n [P\{X_n = i\}P_{ij}] \\ &= r_i P_{ij}.\end{aligned}$$

49. (i) No.

$$\lim P\{X_n = i\} = pr^1(i) + (1-p)r^2(i).$$

(ii) Yes.

$$P_{ij} = pP_{ij}^{(1)} + (1-p)P_{ij}^{(2)}.$$

50. Using the Markov chain of Exercise 9, $\mu_{h,t} = 1/.3$, $\mu_{t,h} = 1/.6$. Also, the stationary probabilities of this chain are $\pi_h = 2/3$, $\pi_t = 1/3$. Therefore,

$$E[A(t,t)] = \frac{1}{(1/3)(.4)(.6)(.3)(.6)(.3)(.4)} = 578.7$$

giving

$$\begin{aligned}E[N(tththtt)|X_0 = h] = E[N(t,t)|X_0 = h] \\ +E(A(t,t)]\end{aligned}$$

Also,

$$\begin{aligned}E[N(t,t)|X_0 = h] &= E[N(t)|X_0 = h] + \frac{1}{(1/3)(.4)} \\ &= \frac{13}{1.2} = 10.8\end{aligned}$$

Therefore, $E[N(tththtt)|X_0 = h] = 589.5$

52. Let the state be the successive zonal pickup locations. Then $P_{A,A} = .6$, $P_{B,A} = .3$. The long run proportions of pickups that are from each zone are

$$\pi_A = .6\pi_A + .3\pi_B = .6\pi_A + .3(1 - \pi_A)$$

Therefore, $\pi_A = 3/7$, $\pi_B = 4/7$. Let X denote the profit in a trip. Conditioning on the location of the pickup gives

$$\begin{aligned}E[X] &= \frac{3}{7}E[X|A] + \frac{4}{7}E[X|B] \\ &= \frac{3}{7}[.6(6) + .4(12)] + \frac{4}{7}[.3(12) + .7(8)] \\ &= 62/7\end{aligned}$$

53. With $\pi_i(1/4)$ equal to the proportion of time a policyholder whose yearly number of accidents is Poisson distributed with mean 1/4 is in Bonus-Malus state i, we have that the average premium is

$$\frac{2}{3}(326.375) + \frac{1}{3}[200\pi_1(1/4) + 250\pi_2(1/4) + 400\pi_3(1/4) + 600\pi_4(1/4)]$$

54. $E[X_{n+1}] = E[E[X_{n+1}|X_n]]$.

Now given X_n,

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } \dfrac{M - X_n}{M} \\ X_n - 1, & \text{with probability } \dfrac{X_n}{M}. \end{cases}$$

Hence,

$$\begin{aligned}E[X_{n+1}|X_n] &= X_n + \frac{M - X_n}{M} - \frac{X_n}{M} \\ &= X_n + 1 - \frac{2X_n}{M},\end{aligned}$$

and so $E[X_{n+1}] = \left[1 - \dfrac{2}{M}\right] E[X_n] + 1$.

It is now easy to verify by induction that the formula presented in (ii) is correct.

55. $S_{11} = P\{\text{offspring is aa} \mid \text{both parents dominant}\}$

$$\begin{aligned} &= \frac{P\{\text{aa, both dominant}\}}{P\{\text{both dominant}\}} \\ &= \frac{r^2 \frac{1}{4}}{(1-q)^2} = \frac{r^2}{4(1-q)^2}.\end{aligned}$$

$$\begin{aligned}S_{10} &= \frac{P\{\text{aa, 1 dominant and 1 recessive parent}\}}{P\{\text{1 dominant and 1 recessive parent}\}} \\ &= \frac{P\{\text{aa, 1 parent aA and 1 parent aa}\}}{2q(1-q)} \\ &= \frac{2qr\frac{1}{2}}{2q(1-q)} \\ &= \frac{r}{2(1-q)}.\end{aligned}$$

56. This is just the probability that a gambler starting with m reaches her goal of $n + m$ before going broke, and is thus equal to $\dfrac{1 - (q/p)^m}{1 - (q/p)^{n+m}}$, where $q = 1 - p$.

57. Let A be the event that all states have been visited by time T. Then, conditioning on the direction of the first step gives

$$P(A) = P(A|\text{clockwise})p$$
$$+P(A|\text{counterclockwise})q$$
$$= p\frac{1-q/p}{1-(q/p)^n} + q\frac{1-p/q}{1-(p/q)^n}$$

The conditional probabilities in the preceding follow by noting that they are equal to the probability in the gambler's ruin problem that a gambler that starts with 1 will reach n before going broke when the gambler's win probabilities are p and q.

58. Using the hint, we see that the desired probability is

$$P\{X_{n+1} = i+1|X_n = i\}$$
$$\frac{P\{lim\, X_m = N|X_n = i, X_n + 1 = i+1\}}{P\{lim\, X_m = N|X_n = 1\}}$$
$$= \frac{p^P i + 1}{P_i}$$

and the result follows from Equation (4.74).

59. Condition on the outcome of the initial play.

61. With $P_0 = 0,\ P_N = 1$

$$P_i = \alpha_i P_{i+1} + (1-\alpha_i)P_{i-1}, \quad i = 1, \ldots, N-1$$

These latter equations can be rewritten as

$$P_{i+1} - P_i = \beta_i(P_i - P_{i-1})$$

where $\beta_i = (1-\alpha_i)/\alpha_i$. These equations can now be solved exactly as in the original gambler's ruin problem. They give the solution

$$P_i = \frac{1 + \sum_{j=1}^{i-1} C_j}{1 + \sum_{j=1}^{N-1} C_j}, \quad i = 1, \ldots, N-1$$

where

$$C_j = \prod_{i=1}^{j} \beta_i$$

(c) P_{N-i}, where $\alpha_i = (N-i)/N$

62. (a) Since $r_i = 1/5$ is equal to the inverse of the expected number of transitions to return to state i, it follows that the expected number of steps to return to the original position is 5.

(b) Condition on the first transition. Suppose it is to the right. In this case the probability is just the probability that a gambler who always bets 1 and wins each bet with probability p will, when starting with 1, reach γ before going broke. By the gambler's ruin problem this probability is equal to

$$\frac{1-q/p}{1-(q/p)^\gamma}.$$

Similarly, if the first move is to the left then the problem is again the same gambler's ruin problem but with p and q reversed. The desired probability is thus

$$\frac{p-q}{1-(q/p)^\gamma} = \frac{q-p}{1-(p/q)^\gamma}.$$

64. (a) $E\left[\sum_{k=0}^{\infty} X_k | X_0 = 1\right] = \sum_{k=0}^{\infty} E[X_k|X_0 = 1]$

$$= \sum_{k=0}^{\infty} \mu^k = \frac{1}{1-\mu}.$$

(b) $E\left[\sum_{k=0}^{\infty} X_k | X_0 = n\right] = \frac{n}{1-\mu}.$

65. $r \geq 0 = P\{X_0 = 0\}$. Assume that $r \geq P\{X_{n-1} = 0\}$.

$$P\{X_n = 0 = \sum_j P\{X_n = 0|X_1 = j\}P_j$$
$$= \sum_j [P\{X_{n-1} = \}]^j P_j$$
$$\leq \sum_j r^j P_j$$
$$= r.$$

66. (a) $r_0 = \frac{1}{3}$

(b) $r_0 = 1.$

(c) $r_0 = \left(\sqrt{3}-1\right)/2.$

67. (a) Yes, the next state depends only on the present and not on the past.

(b) One class, period is 1, recurrent.

(c) $P_{i,i+1} = P\frac{N-i}{N}, \quad i = 0, 1, \ldots, N-1$

$$P_{i,i-1} = (1-P)\frac{i}{N}, \quad i = 1, 2, \ldots, N$$

$$P_{i,i} = P\frac{i}{N} + (1-p)\frac{(N-i)}{N}, \quad i = 0, 1, \ldots, N.$$

(d) See (e).

(e) $r_i = \begin{bmatrix} N \\ i \end{bmatrix} p^i(1-p)^{N-i}, \quad i = 0, 1, \ldots, N.$

(f) Direct substitution or use Example 7a.

(g) Time $= \sum_{j=i}^{N-1} T_j$, where T_j is the number of flips to go from j to $j+1$ heads. T_j is geometric with $E[T_j] = N/j$. Thus, $E[\text{time}] = \sum_{j=i}^{N-1} N/j$.

68. (a) $\sum_i r_i Q_{ij} = \sum_i r_j P_{ji} = r_j \sum_i P_{ji} = r_j$.

(b) Whether perusing the sequence of states in the forward direction of time or in the reverse direction the proportion of time the state is i will be the same.

69. $r(n_1, \ldots, n_m) = \dfrac{M!}{n_1, \ldots, n_m!} \left[\dfrac{1}{m}\right]^M$.

We must now show that

$$r(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots) \frac{n_j + 1}{M} \frac{1}{M-1}$$
$$= r(n_1, \ldots, n_i, \ldots, n_j, \ldots) \frac{i}{M} \frac{1}{M-1}$$

or $\dfrac{n_j + 1}{(n_i - 1)!(n_j + 1)!} = \dfrac{n_i}{n_i! n_j!}$, which follows.

70. (a) $P_{i,i+1} = \dfrac{(m-i)^2}{m^2}, \quad P_{i,i-1} = \dfrac{i^2}{m^2},$

$$P_{i,i} = \frac{2i(m-i)}{m^2}$$

(b) Since, in the limit, the set of m balls in urn 1 is equally likely to be any subset of m balls, it is intuitively clear that

$$\pi_i = \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$$

(c) We must verify that, with the π_i given in (b),

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

That is, we must verify that

$$(m-i)\binom{m}{i} = (i+1)\binom{m}{i+1}$$

which is immediate.

71. If $r_j = c\dfrac{P_{ij}}{P_{ji}}$, then

$$r_j P_{jk} = c\frac{P_{ij}P_{jk}}{P_{ji}}$$
$$r_k P_{kj} = c\frac{P_{jk}P_{kj}}{P_{ki}}$$

and are thus equal by hypothesis.

72. Rate at which transitions from i to j to k occur $= r_i P_{ij} P_{jk}$, whereas the rate in the reverse order is $r_k P_{kj} P_{ji}$. So, we must show

$$r_i P_{ij} P_{jk} = r_k P_{kj} P_{ji}.$$

Now, $r_i P_{ij} P_{jk} = r_j P_{ji} P_{jk}$ by reversibility

$= r_j P_{jk} P_{ji}$

$= r_k P_{kj} P_{ji}$ by reversibility.

73. It is straightforward to check that $r_i P_{ij} = r_j P_{ji}$. For instance, consider states 0 and 1. Then

$$r_0 p_{01} = (1/5)(1/2) = 1/10$$

whereas

$$r_1 p_{10} = (2/5)(1/4) = 1/10.$$

74. (a) The state would be the present ordering of the n processors. Thus, there are $n!$ states.

(b) Consider states $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $x^1 = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots, x_n)$. With q_t equal to $1 - p_t$ the time reversible equations are

$$r(x) q_{x_i} p_{x_{i+1}} \prod_{k=1}^{i-1} q_{x_k} = r(x^1) q_{x_{i+1}} p_{x_i} \prod_{k=1}^{i-1} q_{x_k}$$

or

$$r(x) = (q_{x_{i+1}}/p_{x_{i+1}})(q_{x_i}/p_{x_i})^{-1} r(x^1).$$

Suppose now that we successively utilize the above identity until we reach the state $(1, 2, \ldots, n)$. Note that each time j is moved to the left we multiply by q_j/p_j and each time it moves to the right we multiply by $(q_j/p_j)^{-1}$. Since x_j, which is initially in position j, is to have a net move of $j - x_j$ positions to the left (so it will end up in position $j - (j - x_j) = x_j$) it follows from the above that

$$r(x) = C \prod_j (q_{x_j}/p_{x_j})^{j - x_j}.$$

The value of C, which is equal to $r(1, 2, \ldots, n)$, can be obtained by summing over all states x and equating to 1. Since the solution given by the above value of $r(x)$ satisfies the time reversibility equations it follows that the chain is time reversible and these are the limiting probabilities.

75. The number of transitions from i to j in any interval must equal (to within 1) the number from j to i since each time the process goes from i to j in order to get back to i, it must enter from j.

76. We can view this problem as a graph with 64 nodes where there is an arc between 2 nodes if a knight can go from one node to another in a *single move*. The weights on each are equal to 1. It is easy to check that $\sum_i \sum_j w_{ij} = 336$, and for a corner node $i, \sum_j w_{ij} = 2$. Hence, from Example 7b, for one of the 4 corner nodes $i, \prod_i = 2/336$, and thus the mean time to return, which equals $1/r_i$, is $336/2 = 168$.

77. (a) $$\sum_a y_{ja} = \sum_a E_\beta \left[\sum_n a^n I_{\{X_n = j,\, a_n = a\}} \right]$$
$$= E_\beta \left[\sum_n a^n \sum_a I_{\{X_n = j,\, a_n = a\}} \right]$$
$$= E_\beta \left[\sum_n a^n I_{\{X_n = j\}} \right].$$

(b) $$\sum_j \sum_a y_{ja} = E_\beta \left[\sum_n a^n \sum_j I_{\{X_n = j\}} \right]$$
$$= E_\beta \left[\sum a^n \right] = \frac{1}{1-\alpha}.$$

$$\sum_a y_{ja}$$
$$= b_j + E_\beta \left[\sum_{n=1}^{\infty} = a^n I_{\{X_n = j\}} \right]$$
$$= b_j + E_\beta \left[\sum_{n=0}^{\infty} a^{n+1} I_{\{X_{n+1} = j\}} \right]$$
$$= b_j + E_\beta \left[\sum_{n=0}^{\infty} = a^{n+1} \sum_{i,a} I_{\{X_n = i,\, a_n = a\}} \right.$$
$$\left. I_{(X_{n+1} = j\}} \right]$$
$$= b_j + \sum_{n=0}^{\infty} a^{n+1} \sum_{i,a} E_\beta \left[I_{\{X_n = i,\, a_n = a\}} \right] P_{ij}(a)$$
$$= b_j + a \sum_{i,a} \sum_n a^n E_\beta \left[I_{(X_n = i,\, a_n = a\}} \right] P_{ij}(a)$$
$$= b_j + a \sum_{i,a} y_{ia} P_{ij}(a).$$

(c) Let $d_{j,a}$ denote the expected discounted time the process is in j, and a is chosen when policy β is employed. Then by the same argument as in (b):

$$\sum_a d_{ja}$$
$$= b_j + a \sum_{i,a} \sum_n a^n E_\beta [I\{X_n = i,\ a_n = a\}]\, P_{ij}(a)$$
$$= b_j + a \sum_{i,a} \sum_n a^n E_\beta \left[I_{\{X_n = \, i\}} \right] \frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a)$$
$$= b_j + a \sum_{i,a} \sum_a d_{ia}, \frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a),$$

and we see from Equation (9.1) that the above is satisfied upon substitution of $d_{ia} = y_{ia}$. As it is easy to see that $\sum_{i,a} d_{ia} = \dfrac{1}{1-a}$, the result follows since it can be shown that these linear equations have a unique solution.

(d) Follows immediately from previous parts. It is a well-know result in analysis (and easily proven) that if $\lim_{n \to \infty} \sum_i^n a_i/n$ also equals a. The result follows from this since

$$E[R(X_n)] = \sum_j R(j) P\{X_n = j\}$$
$$= \sum_i R(j) r_j.$$

78. Let $\pi_j, j \geq 0$, be the stationary probabilities of the underlying chain.

(a) $\sum_j \pi_j p(s|j)$

(b) $p(j|s) = \dfrac{\pi_j p(s|j)}{\sum_j \pi_j p(s|j)}$

Chapter 5

1. (a) e^{-1}, (b) e^{-1}.

2. Let T be the time you spend in the system; let S_i be the service time of the i person in queue: let R be the remaining service time of the person in service; let S be your service time. Then,

$$\begin{aligned} E[T] &= E[R + S_1 + S_2 + S_3 + S_4 + S] \\ &= E[R] + \sum_{i=1}^{4} E[S_i] + E[S] = 6/\mu \end{aligned}$$

Where we have used the lack of memory property to conclude that R is also exponential with rate μ.

3. The conditional distribution of X, given that $X > 1$, is the same as the unconditional distribution of $1 + X$. Hence, (a) is correct.

4. (a) 0, (b) $\frac{1}{27}$, (c) $\frac{1}{4}$.

5. e^{-1} by lack of memory.

6. Condition on which server initially finishes first. Now,

$$\begin{aligned} &P\{\text{Smith is last}|\text{server 1 finishes first}\} \\ &\quad = P\{\text{server 1 finishes before server 2}\} \\ &\qquad \text{by lack of memory} \\ &\quad = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

Similarly,

$$P\{\text{Smith is last}|\text{server 2 finished first}\} = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

and thus

$$P\{\text{Smith is last}\} = \left[\frac{\lambda_1}{\lambda_1 + \lambda_2}\right]^2 + \left[\frac{\lambda_2}{\lambda_1 + \lambda_2}\right]^2.$$

7. $P\{X_1 < X_2 | \min(X_1, X_2) = t\}$

$$\begin{aligned} &= \frac{P\{X_1 < X_2, \min(X_1, X_2) = t\}}{P\{\min(X_1, X_2) = t\}} \\ &= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + P\{X_2 = t, X_1 > t\}} \\ &= \frac{f_1(t)\bar{F}_2(t)}{f_1(t)\bar{F}_2(t) + f_2(t)\bar{F}_1(t)} \end{aligned}$$

Dividing though by $\bar{F}_1(t)\bar{F}_2(t)$ yields the result. (For a more rigorous argument, replace " $= t$" by " $\in (t, t+\epsilon)$" throughout, and then let $\epsilon \to 0$.)

8. Let X_i have density f_i and tail distribution $\bar{F}_i$.

$$\begin{aligned} r(t) &= \frac{\sum_{i=1}^{n} P\{T = i\} f_i(t)}{\sum_{j=1}^{n} P\{T = j\} \bar{F}_j(t)} \\ &= \frac{\sum_{i=1}^{n} P\{T = i\} r_i(t) \bar{F}_i(t)}{\sum_{j=1}^{n} P\{T = j\} \bar{F}_j(t)} \end{aligned}$$

The result now follows from

$$P\{T = i | X > t\} = \frac{P\{T = i\}\bar{F}_i(t)}{\sum_{j=1}^{n} P\{T = j\}\bar{F}_j(t)}$$

9. Condition on whether machine 1 is still working at time t, to obtain the answer,

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t}\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

11. (a) Using Equation (5.5), the lack of memory property of the exponential, as well as the fact that the minimum of independent exponentials is exponential with a rate equal to the sum of their individual rates, it follows that

$$P(A_1) = \frac{n\mu}{\lambda + n\mu}$$

and, for $j > 1$,

$$P(A_j|A_1 \cdots A_{j-1}) = \frac{(n-j+1)\mu}{\lambda + (n-j+1)\mu}$$

Hence,

$$p = \prod_{j=1}^{n} \frac{(n-j+1)\mu}{\lambda + (n-j+1)\mu}$$

(b) When $n = 2$.
$P\{\max Y_i < X\}$

$$= \int_0^\infty P\{\max Y_i < X | X = x\} \lambda e^{-\lambda x} dx$$
$$= \int_0^\infty P\{\max Y_i < x\} \lambda e^{-\lambda x} dx$$
$$= \int_0^\infty (1 - e^{-\mu x})^2 \lambda e^{-\lambda x} dx$$
$$= \int_0^\infty (1 - 2e^{-\mu x} + e^{-2\mu x})^2 \lambda e^{-\lambda x} dx$$
$$= 1 - \frac{2\lambda}{\lambda + \mu} + \frac{\lambda}{2\mu + \lambda}$$
$$= \frac{2\mu^2}{(\lambda + \mu)(\lambda + 2\mu)}$$

12. (a) $P\{X_1 < X_2 < X_3\}$
$= P\{X_1 = \min(X_1, X_2, X_3)\}$

$P\{X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)\}$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} P\{X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)\}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3}$$

where the final equality follows by the lack of memory property.

(b) $P\{X_2 < X_3 | X_1 = \max(X_1, X_2, X_3)\}$

$$= \frac{P\{X_2 < X_3 < X_1\}}{P\{X_2 < X_3 < X_1\} + P\{X_3 < X_2 < X_1\}}$$

$$= \frac{\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_3}{\lambda_1 + \lambda_3}}{\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_3}{\lambda_1 + \lambda_3} + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_1 + \lambda_2}}$$

$$= \frac{1/(\lambda_1 + \lambda_3)}{1/(\lambda_1 + \lambda_3) + 1/(\lambda_1 + \lambda_2)}$$

(c) $$\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{+\lambda_2 + \lambda_3} + \frac{1}{\lambda_3}$$

(d) $$\sum_{i \neq j \neq k} \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k} \left[\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k} \right]$$

where the sum is over all 6 permutations of 1, 2, 3.

13. Let T_n denote the time until the n^{th} person in line departs the line. Also, let D be the time until the first departure from the line, and let X be the additional time after D until T_n. Then,

$$E[T_n] = E[D] + E[X]$$
$$= \frac{1}{n\theta + \mu} + \frac{(n-1)\theta + \mu}{n\theta + \mu} E[T_{n-1}]$$

where $E[X]$ was computed by conditioning on whether the first departure was the person in line. Hence,

$$E[T_n] = A_n + B_n E[T_{n-1}]$$

where

$$A_n = \frac{1}{n\theta + \mu}, \qquad B_n = \frac{(n-1)\theta + \mu}{n\theta + \mu}$$

Solving gives the solution.

$$E[T_n] = A_n + \sum_{i=1}^{n-1} A_{n-i} \prod_{j=n-i+1}^{n} B_j$$
$$= A_n + \sum_{i=1}^{n-1} 1/(n\theta + \mu)$$
$$= \frac{n}{n\theta + \mu}$$

Another way to solve the preceding is to let I_j equal 1 if customer n is still on line at the time of the $(j-1)^{st}$ departure from the line, and let X_j denote the time between the $(j-1)^{st}$ and j^{th} departure from line. (Of course, these departures only refer to the first n people in line.) Then

$$T_n = \sum_{j=1}^{n} I_j X_j$$

The independence of I_j and X_j gives

$$E[T_n] = \sum_{j=1}^{n} E[I_j] E[X_j]$$

But,

$$E[I_j] = \frac{(n-1)\theta + \mu}{n\theta + \mu} \cdots \frac{(n-j+1)\theta + \mu}{(n-j+2)\theta + \mu}$$
$$= \frac{(n-j+1)\theta + \mu}{n\theta + \mu}$$

and

$$E[X_j] = \frac{1}{(n-j+1)\theta + \mu}$$

which gives the result.

14. (a) The conditional density of X gives that $X < c$ is

$$f(x|X<c) = \frac{f(x)}{P\{x<c\}} = \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda c}},\ 0<x<c$$

Hence,

$$E[X|X<c] = \int_0^c x\lambda e^{-\lambda x} dx/(1-e^{-\lambda c}).$$

Integration by parts yields that

$$\int_0^c x\lambda e^{-\lambda x}\,dx = -xe^{-\lambda x}\Big|_0^c + \int_0^c e^{-\lambda x}dx$$
$$= -ce^{-\lambda c} + (1-e^{-\lambda c})/\lambda.$$

Hence,

$$E[X|X<c] = 1/\lambda - ce^{-\lambda c}/(1-e^{-\lambda c}).$$

(b) $1/\lambda = E[X|X<c](1-e^{-\lambda c}) + (c+1/\lambda)e^{-\lambda c}$
This simplifies to the same answer as given in part (a).

15. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101-i)/200$. Thus,

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101-i}$$

$$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{(200)^2}{(101-i)^2}$$

16. Letting T_i denote the time between departure $i-1$ and departure i, we have

$$E[T] = E[T_1] + E[T_2] + E[T_3]$$

The random variables T_1 and T_2 are both exponential with rate $\lambda_1+\lambda_2$, and so have mean $1/(\lambda_1+\lambda_2)$. To determine $E[T_3]$ consider the time at which the first customer has departed and condition on which server completes the next service. This gives:

$$E[T_3] = E[T_3|\text{server 1}]\ [\lambda_1/(\lambda_1+\lambda_2)]$$
$$= E[T_3|\text{server 2}][\lambda_2/(\lambda_1+\lambda_2)]$$
$$= (1/\lambda_2)[\lambda_1/(\lambda_1+\lambda_2)]$$
$$+ (1/\lambda_1)[\lambda_2/(\lambda_1+\lambda_2)].$$

Therefore,

$$E[\text{Time}] = 2/(\lambda_1+\lambda_2) + (1/\lambda_2)[\lambda_1/(\lambda_1+\lambda_2)]$$
$$+ (1/\lambda_1)[\lambda_2/(\lambda_1+\lambda_2)].$$

17. Let C_i denote the cost of the i^{th} link to be constructed, $i=1,\ldots,n-1$. Note that the first link can be any of the $\binom{n}{2}$ possible links. Given the first one, the second link must connect one of the 2 cities joined by the first link with one of the $n-2$ cities without any links. Thus, given the first constructed link, the next link constructed will be one of $2(n-2)$ possible links. Similarly, given the first two links that are constructed, the next one to be constructed will be one of $3(n-3)$ possible links, and so on. Since the cost of the first link to be built is the minimum of $\binom{n}{2}$ exponentials with rate 1, it follows that

$$E[C_1] = 1\Big/\binom{n}{2}.$$

By the lack of memory property of the exponential it follows that the amounts by which the costs of the other links exceed C_1 are independent exponentials with rate 1. Therefore, C_2 is equal to C_1 plus the minimum of $2(n-2)$ independent exponentials with rate 1, and so

(c) Letting $A = X_{(2)} - X_{(1)}$ we have

$$E[X_{(2)}]$$
$$= E[X_{(1)}] + E[A].$$
$$= \frac{1}{\mu_1+\mu_2} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1+\mu_2}.$$

The formula for $E[A]$ being obtained by conditioning on which X_i is largest.

(d) Let I equal 1 if $X_1 < X_2$ and let it be 2 otherwise. Since the conditional distribution of A (either exponential with rate μ_1 *or* μ_2) is determined by I, which is independent of $X_{(1)}$, it follows that A is independent of $X_{(1)}$.
Therefore

$$Var(X_{(2)}) = Var(X_{(1)}) + Var(A).$$

With $p = \mu_1/(\mu_1+\mu_2)$ we obtain, upon conditioning on I,

$$E[A] = p/\mu_2 + (1-p)/\mu_1,$$
$$E[A^2] = 2p/\mu_2^2 + 2(1-p)/\mu_1^2.$$

Therefore,

$$Var(A) = 2p/\mu_2^2 + 2(1-p)/\mu_1^2 - (p/\mu_2 + (1-p)/\mu_1)^2$$

Thus,

$$Var(X_{(2)}) = 1/(\mu_1+\mu_2)^2 + 2[p/\mu_2^2 + (1-p)/\mu_1^2] - (p/\mu_2 + (1-p)/\mu_1)^2.$$

20. (a) $P_A = \dfrac{\mu_1}{\mu_1+\mu_2}$

(b) $P_B = 1 - \left(\dfrac{\mu_2}{\mu_1+\mu_2}\right)^2$

(c) $E[T] = 1/\mu_1 + 1/\mu_2 + P_A/\mu_2 + P_B/\mu_2$

21. $E[\text{time}] = E[\text{time waiting at 1}] + 1/\mu_1 + E[\text{time waiting at 2}] + 1/\mu_2.$

Now

$E[\text{time waiting at 1}] = 1/\mu_1$,

$E[\text{time waiting at 2}] = (1/\mu_2)\dfrac{\mu_1}{\mu_1+\mu_2}.$

The last equation follows by conditioning on whether or not the customer waits for server 2. Therefore,

$E[\text{time}] = 2/\mu_1 + (1/\mu_2)[1 + \mu_1/(\mu_1+\mu_2)].$

22. $E[\text{time}] = E[\text{time waiting for server 1}] + 1/\mu_1 + E[\text{time waiting for server 2}] + 1/\mu_2.$

Now, the time spent waiting for server 1 is the remaining service time of the customer with server 1 plus any additional time due to that customer blocking your entrance. If server 1 finishes before server 2 this additional time will equal the additional service time of the customer with server 2. Therefore,

$E[\text{time waiting for server 1}]$

$= 1/\mu_1 + E[\text{Additional}]$

$= 1/\mu_1 + (1/\mu_2[\mu_1/(\mu_1+\mu_2)].$

Since when you enter service with server 1 the customer preceding you will be catering service with server 2, it follows that you will have to wait for server 2 if you finish service first. Therefore, conditioning on whether or not you finish first

$E[\text{time waiting for server 2}]$

$= (1/\mu_2)[\mu_1/(\mu_1+\mu_2)].$

Thus,

$E[\text{time}] = 2/\mu_1 + (2/\mu_2)[\mu_1/(\mu_1+\mu_2)] + 1/\mu_2.$

23. (a) 1/2.

(b) $(1/2)^{n-1}$: whenever battery 1 is in use and a failure occurs the probability is 1/2 that it is not battery 1 that has failed.

(c) $(1/2)^{n-i+1}, \quad i > 1.$

(d) T is the sum of $n-1$ independent exponentials with rate 2μ (since each time a failure occurs the time until the next failure is exponential with rate 2μ).

(e) Gamma with parameters $n-1$ and 2μ.

24. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} job completion. Then the T_i are independent, with $T_i, i = 1, \ldots, n-1$ being exponential with rate $\mu_1+\mu_2$. With probability $\dfrac{\mu_1}{\mu_1+\mu_2}$, T_n is exponential with rate μ_2, and with probability $\dfrac{\mu_2}{\mu_1+\mu_2}$ it is exponential with rate μ_1. Therefore,

$$E[T] = \sum_{i=1}^{n-1} E[T_i] + E[T_n] = (n-1)\frac{1}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}\frac{1}{\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{1}{\mu_1}$$

$$Var(T) = \sum_{i=1}^{n-1} Var(T_i) + Var(T_n) = (n-1)\frac{1}{(\mu_1+\mu_2)^2} + Var(T_n)$$

Now use

$$Var(T_n) = E[T_n^2] - (E[T_n])^2 = \frac{\mu_1}{\mu_1+\mu_2}\frac{2}{\mu_2^2} + \frac{\mu_2}{\mu 1+\mu_2}\frac{2}{\mu_1^2} - \left(\frac{\mu_1}{\mu_1+\mu_2}\frac{1}{\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{1}{\mu_1}\right)^2$$

25. Parts (a) and (b) follow upon integration. For part (c), condition on which of X or Y is larger and use the lack of memory property to conclude that the amount by which it is larger is exponential rate λ. For instance, for $x < 0$,

$$fx - y(x)dx = P\{X < Y\}P\{-x < Y - X < -x + dx | Y > X\} = \frac{1}{2}\lambda e^{\lambda x}dx$$

For (d) and (e), condition on I.

26. (a) $$\frac{1}{\mu_1+\mu_2+\mu_3} + \sum_{i=1}^{3}\frac{\mu_i}{\mu_1+\mu_2+\mu_3}\frac{1}{\mu_i} = \frac{4}{\mu_1+mu_2+\mu_3}$$

(b) $\dfrac{1}{\mu_1+\mu_2+\mu_3} + (a) = \dfrac{5}{\mu_1+\mu_2+\mu_3}$

27. (a) $\dfrac{\mu_1}{\mu_1+\mu_3}$

(b) $\dfrac{\mu_1}{\mu_1+\mu_3}\,\dfrac{\mu_2}{\mu_2+\mu_3}$

(c) $\sum_i \dfrac{1}{\mu_i} + \dfrac{\mu_1}{\mu_1+\mu_3}\,\dfrac{\mu_2}{\mu_2+\mu_3}\,\dfrac{1}{\mu_3}$

(d) $\sum_i \dfrac{1}{\mu_i} + \dfrac{\mu_1}{\mu_1+\mu_2}\left[\dfrac{1}{\mu_2} + \dfrac{\mu_2}{\mu_2+\mu_3}\,\dfrac{1}{\mu_3}\right] + \dfrac{\mu_2}{\mu_1+\mu_2}\,\dfrac{\mu_1}{\mu_1+\mu_3}\,\dfrac{\mu_2}{\mu_2+\mu_3}\,\dfrac{1}{\mu_3}$

28. For both parts, condition on which item fails first.

(a) $\displaystyle\sum_{i\neq 1} \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}\,\frac{\lambda_1}{\sum_{j\neq i}\lambda_j}$

(b) $\displaystyle\frac{1}{\sum_{i=1}^n \lambda_j} + \sum_{i=1}^n \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}\,\frac{1}{\sum_{j\neq i}\lambda_j}$

29. (a)
$$
\begin{aligned}
f_{X|X+Y(x|c)} &= C f_{X.\,X+Y(x,c)}\\
&= C_1 f_{X_Y(x,c-x)}\\
&= f_X(x) f_Y(c-x)\\
&= C_2 e^{-\lambda x} e^{-\mu(c-x)}, \quad 0<x<c\\
&= C_3 e^{-(\lambda-\mu)x}, \quad 0<x<c
\end{aligned}
$$

where none of the C_i depend on x. Hence, we can conclude that the conditional distribution is that of an exponential random variable conditioned to be less than c.

(b) $E[X|X+Y=c] = \dfrac{1-e^{-(\lambda-\mu)c}(1+(\lambda-\mu)c)}{\lambda(1-e^{-(\lambda-\mu)c})}$

(c) $c = E[X+Y|X+Y=c] = E[X|X+Y=c] + E[Y|X+Y=c]$,

implying that

$$
E[Y|X+Y=c] = c - \frac{1-e^{-(\lambda-\mu)c}(1+(\lambda-\mu)c)}{\lambda(1-e^{-(\lambda-\mu)c})}
$$

30. Condition on which animal died to obtain

$$
\begin{aligned}
&E[\text{additional life}]\\
&= E[\text{additional life} \mid \text{dog died}]\\
&\frac{\lambda_d}{\lambda_c+\lambda_d} + E[\text{additional life} \mid \text{cat died}]\frac{\lambda_c}{\lambda_c+\lambda_d}\\
&= \frac{1}{\lambda_c}\,\frac{\lambda_d}{\lambda_c+\lambda_d} + \frac{1}{\lambda_d}\,\frac{\lambda_c}{\lambda_c+\lambda_d}
\end{aligned}
$$

31. Condition on whether the 1 PM appointment is still with the doctor at 1:30, and use the fact that if she or he is then the remaining time spent is exponential with mean 30. This gives

$$
\begin{aligned}
&E[\text{time spent in office}]\\
&= 30(1-e^{-30/30}) + (30+30)e^{-30/30}\\
&= 30 + 30e^{-1}
\end{aligned}
$$

32. Let T_i denote the time spent on job i. Then

$$X = T_1 + T_1 + T_2 + T_1 + T_2 + T_3 = 3T_1 + 2T_2 + T_3$$

and thus

$$E[X] = 6, \quad Var(X) = 9 + 4 + 1 = 14$$

33. (a) By the lack of memory property, no matter when Y fails the remaining life of X is exponential with rate λ.

(b)
$$
\begin{aligned}
&E[\min(X,Y)\,|X>Y+c]\\
&= E[\min(X,Y)\,|X>Y, X-Y>c]\\
&= E[\min(X,Y)\,|X>Y]
\end{aligned}
$$

where the final equality follows from (a).

34. (a) $\dfrac{\lambda}{\lambda+\mu_A}$

(b) $\dfrac{\lambda+\mu_A}{\lambda+\mu_A+\mu_B}\cdot\dfrac{\lambda}{\lambda+\mu_B}$

37. $e^{-1/10}$, $e^{-1/4}$, $e^{-1/2}$, e^{-1}

38. Let $k = \min(n, m)$, and condition on $M_2(t)$.

$$
\begin{aligned}
&P\{N_1(t)=n, N_2(t)=m\}\\
&= \sum_{j=0}^k P\{N_1(t)=n, N_2(t)=m|M_2(t)=j\}\\
&\quad\times e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!}\\
&= \sum_{j=0}^k e^{-\lambda_1 t}\frac{(\lambda_1 t)^{n-j}}{(n-j)!}e^{-\lambda_3 t}\frac{(\lambda_3 t)^{m-j}}{(m-j)!}e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!}
\end{aligned}
$$

39. (a) $196/2.5 = 78.4$

(b) $196/(2.5)^2 = 31.36$

We use the central limit theorem to justify approximating the life distribution by a normal distribution with mean 78.4 and standard deviation $\sqrt{31.36} = 5.6$. In the following a standard normal random variable.

(c) $P\{L < 67.2\} \approx P\left\{Z < \frac{67.2 - 78.4}{5.6}\right\}$
$= P\{Z < -2\} = .0227$

(d) $P\{L > 90\} \approx P\left\{Z > \frac{90 - 78.4}{5.6}\right\}$
$= P\{Z > 2.07\} = .0192$

(e) $P\{L > 100\} \approx P\left\{Z > \frac{100 - 78.4}{5.6}\right\}$
$= P\{Z > 3.857\} = .00006$

40. The easiest way is to use definition 5.1. It is easy to see that $\{N(t), t \geq 0\}$ will also possess stationary and independent increments. Since the sum of two independent Poisson random variables is also Poisson, it follows that $N(t)$ is a Poisson random variable with mean $(\lambda_1 + \lambda_2)t$.

41. $\lambda_1/(\lambda_1 + \lambda_2)$.

42. (a) $E[S_4] = 4/\lambda$.

(b) $E[S_4|N(1) = 2]$
$= 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$.

(c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)]$
$= 2\lambda$.

The first equality used the independent increments property.

43. Let S_i denote the service time at server $i, i = 1, 2$ and let X denote the time until the next arrival. Then, with p denoting the proportion of customers that are served by both servers, we have

$$p = P\{X > S_1 + S_2\}$$
$$= P\{X > S_1\}PX > S_1 + S_2|X > S_1\}$$
$$= \frac{\mu_1}{\mu_1 + \lambda}\,\frac{\mu_2}{\mu_2 + \lambda}$$

44. (a) $e^{-\lambda T}$

(b) Let W denote the waiting time and let X denote the time until the first car. Then

$$E[W] = \int_0^\infty E[W|X = x]\lambda e^{-\lambda x}dx$$
$$= \int_0^T E[W|X = x]\lambda e^{-\lambda x}dx$$
$$+ \int_T^\infty E[W|X = x]\lambda e^{-\lambda x}dx$$
$$= \int_0^T (x + E[W|)\lambda e^{-\lambda x}dx + Te^{-\lambda T}$$

Hence,

$$E[W] = T + e^{\lambda T}\int_0^T x\lambda e^{-\lambda x}dx$$

45. $E[N(T)] = E[E[N(T)|T]] = E[\lambda T] = \lambda E[T]$

$E[TN(T)] = E[E[TN(T)|T]] = E[T\lambda T] = \lambda E[T^2]$

$E[N^2(T)] = E\Big[E[N^2(T)|T]\Big] = E[\lambda T + (\lambda T)^2]$
$= \lambda E[T] + \lambda^2 E[T^2]$

Hence,

$$Cov(T, N(T)) = \lambda E[T^2] - E[T]\lambda E[T] = \lambda\sigma^2$$

and

$$Var(N(T)) = \lambda E[T] + \lambda^2 E[T^2] - (\lambda E[T])^2$$
$$= \lambda\mu + \lambda^2\sigma^2$$

46. $E[\sum_{i=1}^{N(t)} X_i] = E\left[E[\sum_{i=1}^{N(t)} X_i|N(t)]\right]$
$= E[\mu N(t)] = \mu\lambda t$

$$E[N(t)\sum_{i=1}^{N(t)} X_i] = E\left[E[N(t)\sum_{i=1}^{N(t)} X_i|N(t)]\right]$$
$$= E[\mu N^2(t)] = \mu(\lambda t + \lambda^2 t^2)$$

Therefore,

$$Cov(N(t), \sum_{i=1}^{N(t)} X_i) = \mu(\lambda t + \lambda^2 t^2) - \lambda t(\mu\lambda t) = \mu\lambda t$$

47. (a) $1/(2\mu) + 1/\lambda$

(b) Let T_i denote the time until both servers are busy when you start with i busy servers $i = 0.1$. Then,

$$E[T_0] = 1/\lambda + E[T_1]$$

Now, starting with 1 server busy, let T be the time until the first event (arrival or departure);

let $X = 1$ if the first event is an arrival and let it be 0 if it is a departure; let Y be the additional time after the first event until both servers are busy.

$$
\begin{aligned}
E[T_1] &= E[T] + E[Y] \\
&= \frac{1}{\lambda+\mu} + E[Y|X=1]\frac{\lambda}{\lambda+\mu} \\
&\quad + E[Y|X=0]\frac{\mu}{\lambda+\mu} \\
&= \frac{1}{\lambda+\mu} + E[T_0]\frac{\mu}{\lambda+\mu}
\end{aligned}
$$

Thus,

$$E[T_0] - \frac{1}{\lambda} = \frac{1}{\lambda+\mu} + E[T_0]\ \frac{\mu}{\lambda+\mu}$$

or

$$E[T_0] = \frac{2\lambda+\mu}{\lambda^2}$$

Also,

$$E[T_1] = \frac{\lambda+\mu}{\lambda^2}$$

(c) Let L_i denote the time until a customer is lost when you start with i busy servers. Then, reasoning as in part (b) gives that

$$
\begin{aligned}
E[L_2] &= \frac{1}{\lambda+\mu} + E[L_1]\frac{\mu}{\lambda+\mu} \\
&= \frac{1}{\lambda+\mu} + (E[T_1] + E[L_2])\frac{\mu}{\lambda+\mu} \\
&= \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda^2} + E[L_2]\frac{\mu}{\lambda+\mu}
\end{aligned}
$$

Thus,

$$E[L_2] = \frac{1}{\lambda} + \frac{\mu(\lambda+\mu)}{\lambda^3}$$

48. Given T, the time until the next arrival. N, the number of busy servers found by the next arrival, is a binomial random variable with parameters n and $p = e^{-\mu T}$.

(a)
$$
\begin{aligned}
E[N] &= \int E[N|T=t]\lambda e^{-\lambda t}dt \\
&= \int ne^{-\mu t}\lambda e^{-\lambda t}dt = \frac{n\lambda}{\lambda+\mu}
\end{aligned}
$$

For (b) and (c), you can either condition on T, or use the approach of part (a) of Exercise 11 to obtain

$$P\{N=0\} = \prod_{j=1}^{n} \frac{(n-j+1)\mu}{\lambda+(n-j+1)\mu}$$

$$
\begin{aligned}
&P\{N = n-i\} \\
&= \frac{\lambda}{\lambda+(n-i)\mu} \prod_{j=1}^{i} \frac{(n-j+1)\mu}{\lambda+(n-j+1)\mu}
\end{aligned}
$$

49. (a) $P\{N(T) - N(s) = 1\} = \lambda(T-s)e^{-\lambda(T-s)}$

(b) Differentiating the expression in part (a) and then setting it equal to 0, gives

$$e^{-\lambda(T-s)} = \lambda(T-s)e^{-\lambda(T-s)}$$

implying that the maximizing value is

$$s = T - 1/\lambda$$

(c) For $s = T - 1/\lambda$, we have that $\lambda(T-s) = 1$ and thus,

$$P\{N(T) - N(s) = 1\} = e^{-1}$$

50. Let T denote the time until the next train arrives; and so T is uniform on $(0, 1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$.

(b) $E[X|T] = 7T$, $Var(X|T) = 7T$. By the conditional variance formula $Var(X) = 7E[T] + 49Var[T] = 7/2 + 49/12 = 91/12$.

51. Condition on X, the time of the first accident to obtain

$$
\begin{aligned}
E[N(t)] &= \int_0^\infty E[N(t)|X=s]\beta e^{-\beta s}ds \\
&= \int_0^t (1+\alpha(t-s))\beta e^{-\beta s}ds
\end{aligned}
$$

52. This is the gambler's ruin probability that, starting with k, the gambler's fortune reaches $2k$ before 0 when her probability of winning each bet is $p = \lambda_1/(\lambda_1+\lambda_2)$. The desired probability is $\dfrac{1-(\lambda_2/\lambda_1)^k}{1-(\lambda_2/\lambda_1)^{2k}}$.

53. (a) e^{-1}

(b) $e^{-1} + e^{-1}(.8)e^{-1}$

54. (a) $P\{L_1 = 0\} = e^{-\lambda m}$

(b) $P\{L_1 < x\} = e^{-\lambda(m-x)}$

(c) $P\{R_1 = 1\} = e^{-\lambda(1-m)}$

(d) $P\{R_1 > x\} = e^{-\lambda(x-m)}$

(e) $E[R] = \int_0^1 P\{R > x\}dx$

$$= m + \int_m^1 P\{R > x\}dx$$

$$= m + \int_m^1 e^{-n\lambda(x-m)}dx$$

$$= m + \frac{1 - e^{-n\lambda(1-m)}}{n\lambda}$$

Now, using that

$$P\{L > x\} = 1 - P\{L \le x\} = 1 - e^{-n\lambda(m-x)}, \quad 0 < x < m$$

gives

$$E\{L\} = \int_0^m (1 - e^{-n\lambda(m-x)})dx = m - \frac{1 - e^{-n\lambda m}}{n\lambda}$$

Hence,

$$E[R - L] = \frac{1 - e^{-n\lambda(1-m)}}{n\lambda} + \frac{1 - e^{-n\lambda m}}{n\lambda}$$

$$\approx \frac{2}{n\lambda} \quad \text{when } n \text{ is large}$$

55. As long as customers are present to be served, every event (arrival or departure) will, independently of other events, be a departure with probability $p = \mu/(\lambda + \mu)$. Thus $P\{X = m\}$ is the probability that there have been a total of m tails at the moment that the n^{th} head occurs, when independent flips of a coin having probability p of coming up heads are made: that is, it is the probability that the n^{th} head occurs on trial number $n + m$. Hence,

$$p\{X = m\} = \binom{n+m-1}{n-1} p^n(1-p)^m$$

56. (a) It is a binomial (n, p) random variable.

(b) It is geometric with parameter p.

(c) It is a negative binomial with parameters r, p.

(d) Let $0 < i_1 < i_2, \cdots < i_r < n$. Then,

$$P\{\text{events at } i_1, \ldots, i_r | N(n) = r\}$$

$$= \frac{P\{\text{events at } i_1, \ldots, i_r, N(n) = r\}}{P\{N(n) = r\}}$$

$$= \frac{P^r(1-p)^{n-r}}{\binom{n}{r} P^r(1-p)^{n-r}}$$

$$= \frac{1}{\binom{n}{r}}$$

57. (a) e^{-2}.

(b) 2 p.m.

58. Let $L_i = P\{i \text{ is the last type collected}\}$.

$$L_i = P\{X_i = \max_{j=1,\ldots,n} X_j\}$$

$$= \int_0^\infty p_i e^{-p_i x} \prod_{j \ne i} (1 - e^{-p_j x})dx$$

$$= \int_0^1 \prod_{j \ne i} (1 - y^{p_j/p_i})dy \quad (y = e^{-p_i x})$$

$$= E\left[\prod_{j \ne i} (1 - U^{p_j/p_i})\right]$$

59. The unconditional probability that the claim is type 1 is 10/11. Therefore,

$$P(1|4000) = \frac{P(4000|1)P(1)}{P(4000|1)P(1) + P(4000|2)P(2)}$$

$$= \frac{e^{-4}10/11}{e^{-4}10/11 + .2e^{-.8}1/11}$$

61. (a) Poisson with mean $cG(t)$.

(b) Poisson with mean $c[1 - G(t)]$.

(c) Independent.

62. Each of a Poisson number of events is classified as either being of type 1 (if found by proofreader 1 but not by 2) or type 2 (if found by 2 but not by 1) or type 3 (if found by both) or type 4 (if found by neither).

(a) The X_i are independent Poisson random variables with means

$$E[X_1] = \lambda p_1(1 - p_2),$$
$$E[X_2] = \lambda(1 - p_1)p_2,$$
$$E[X_3] = \lambda p_1 p_2,$$
$$E[X_4] = \lambda(1 - p_1)(1 - p_2).$$

(b) Follows from the above.

(c) Using that $(1 - p_1)/p_1 = E[X_2]/E[X_3] = X_2/X_3$ we can approximate p_1 by $X_3/(X_2 + X_3)$. Thus p_1 is estimated by the fraction of the errors found by proofreader 2 that are also found by proofreader 1. Similarly, we can estimate p_2 by $X_3/(X_1 + X_3)$.

The total number of errors found, $X_1 + X_2 + X_3$, has mean

$$E[X_1 + X_2 + X_3] = \lambda [1 - (1 - p_1)(1 - p_2)]$$

$$= \lambda \left[1 - \frac{X_2 X_1}{(X_2 + X_3)(X_1 + X_3)}\right]$$

Hence, we can estimate λ by

$$(X_1 + X_2 + X_3) / \left[1 - \frac{X_2 X_1}{(X_2 + X_3)(X_1 + X_3)}\right].$$

For instance, suppose that 1 finds 10 errors, and 2 finds 7 errors, including 4 found by 1. Then $X_1 = 6, X_2 = 3, X_3 = 4$. The estimate of p_1 is 4/7, and that of p_2 is 4/10. The estimate of λ is $13/\,(1 - 18/70) = 17.5$.

(d) Since λ is the expected total number of errors. We can use the estimator of λ to estimate this total. Since 13 errors were discovered we would estimate X_4 to equal 4.5.

63. Let X and Y be respectively the number of customers in the system at time $t + s$ that were present at time s, and the number in the system at $t + s$ that were not in the system at time s. Since there are an infinite number of servers, it follows that X and Y are independent (even if given the number is the system at time s). Since the service distribution is exponential with rate μ, it follows that given that $X(s) = n, X$ will be binomial with parameters n and $p = e^{-\mu t}$. Also Y, which is independent of $X(s)$, will have the same distribution as $X(t)$.

Therefore, Y is Poisson with mean $\lambda \int_0^t e^{-\mu y} dy$

$= \lambda(1 - e^{-\mu t})/\mu$.

(a) $E[X(t + s)|X(s) = n]$

$$= E[X|X(s) = n] + E[Y|X(s) = n].$$

$$= ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu.$$

(b) $Var(X(t + s)|X(s) = n)$

$$= Var(X + Y|X(s) = n)$$

$$= Var(X|X(s) = n) + Var(Y)$$

$$= ne^{-\mu t}(1 - e^{-\mu t}) + \lambda(1 - e^{-\mu t})/\mu.$$

The above equation uses the formulas for the variances of a binomial and a Poisson random variable.

(c) Consider an infinite server queuing system in which customers arrive according to a Poisson process with rate λ, and where the service times are all exponential random variables with rate μ. If there is currently a single customer in the system, find the probability that the system becomes empty when that customer departs.

Condition on R, the remaining service time:

$P\{\text{empty}\}$

$$= \int_0^\infty P\{\text{empty}|R = t\}\mu e^{-\mu t} dt$$

$$= \int_0^\infty \exp\{-\lambda \int_0^t e^{-\mu y} dy\}\mu e^{-\mu t} dt$$

$$= \int_0^\infty \exp\{-\frac{\lambda}{\mu}(1 - e^{-\mu t})\}\mu e^{-\mu t} dt$$

$$= \int_0^1 e^{-\lambda x/\mu} dx$$

$$= \frac{\mu}{\lambda}(1 - e^{-\lambda/\mu})$$

where the preceding used that $P\{\text{empty}|R = t\}$ is equal to the probability that an $M/M/\infty$ queue is empty at time t.

64. (a) Since, given $N(t)$, each arrival is uniformly distributed on $(0, t)$ it follows that

$$E[X|N(t)] = N(t) \int_0^t (t - s) ds/t = N(t)\ t/2.$$

(b) Let $U_1, U_2, \ldots$ be independent uniform $(0,\ t)$ random variables.

Then

$$Var(X|N(t) = n) = Var\left[\sum_{i=1}^n (t - U_i)\right]$$

$$= nVar(U_i) = nt^2/12.$$

(c) By (a), (b), and the conditional variance formula,

$$Var(X) = Var(N(t)t/2) + E[N(t)t^2/12]$$
$$= \lambda t t^2/4 + \lambda t t^2/12 = \lambda t^3/3.$$

65. This is an application of the infinite server Poisson queue model. An arrival corresponds to a new lawyer passing the bar exam, the service time is the time the lawyer practices law. The number in the system at time t is, for large t, approximately a Poisson random variable with mean $\lambda\mu$ where λ is the arrival rate and μ the mean service time. This latter statement follows from

$$\int_0^n [1 - G(y)] dy = \mu$$

where μ is the mean of the distribution G. Thus, we would expect $500 \cdot 30 = 15,000$ lawyers.

66. The number of unreported claims is distributed as the number of customers in the system for the infinite server Poisson queue.

    (a) $e^{-a(t)}(a(t))^n/n!$, where $a(t) = \lambda \int_0^t \bar{G}(y)dy$

    (b) $a(t)\mu_F$, where μ_F is the mean of the distribution F.

67. If we count a satellite if it is launched before time s but remains in operation at time t, then the number of items counted is Poisson with mean $m(t) = \int_0^s \bar{G}(t-y)dy$. The answer is $e^{-m(t)}$.

68. $E[A(t)|N(t)=n]$

$$= E[A]e^{-\alpha t}E\left[\sum_{i=1}^{n} e^{\alpha s_i}|N(t)=n\right]$$

$$= E[A]e^{-\alpha t}E\left[\sum_{i=1}^{n} e^{\alpha U_{(i)}}\right]$$

$$= E[A]e^{-\alpha t}E\left[\sum_{i=1}^{n} e^{\alpha U_i}\right]$$

$$= nE[A]e^{-\alpha t}E\left[e^{\alpha U}\right]$$

$$= nE[A]e^{-\alpha t}\int_0^t e^{\alpha x}\frac{1}{t}dx$$

$$= nE[A]\frac{1-e^{-\alpha t}}{\alpha t}$$

Therefore,

$$E[A(t)] = E\left[N(t)E[A]\frac{1-e^{-\alpha t}}{\alpha t}\right] = \lambda E[A]\frac{1-e^{-\alpha t}}{\alpha t}$$

Going backwards from t to 0, events occur according to a Poisson process and an event occurring a time s (from the starting time t) has value $Ae^{-\alpha s}$ attached to it.

69. (a) Write $D(t+h) = D(t) + D(t+h) - D(t)$, and note that $D(t+h) - D(t)$ has the same distribution as $e^{-\alpha t}D(h)$.

    (b) Write $D(t+h) = D(h) + D(t+h) - D(h)$, and note that $D(t+h) - D(h)$ has the same distribution as $e^{-\alpha h}D(t)$.

    (c) $M(h) = \dfrac{\lambda\mu}{\alpha}(1-e^{-\alpha h}) = \lambda\mu h + o(h)$

    (d) $M'(t) = \lambda\mu e^{-\alpha t}$

    (e) $M'(t) = \lambda\mu - \alpha M(t)$

70. (a) Let A be the event that the first to arrive is the first to depart, let S be the first service time, and let $X(t)$ denote the number of departures by time t.

$$P(A) = \int P(A|S=t)g(t)dt$$
$$= \int P\{X(t)=0\}g(t)dt$$
$$= \int e^{-\lambda\int_0^t G(y)dy}g(t)dt$$

    (b) Given $N(t)$, the number of arrivals by t, the arrival times are iid uniform $(0,t)$. Thus, given $N(t)$, the contribution of each arrival to the total remaining service times are independent with the same distribution, which does not depend on $N(t)$.

    (c) and (d) If, conditional on $N(t)$, X is the contribution of an arrival, then

$$E[X] = \frac{1}{t}\int_0^t\int_{t-s}^{\infty}(s+y-t)g(y)dyds$$

$$E[X^2] = \frac{1}{t}\int_0^t\int_{t-s}^{\infty}(s+y-t)^2g(y)dyds$$

$$E[S(t)] = \lambda t E[X] \quad Var(S(t)) = \lambda t E[X^2]$$

71. Let $U_1, \ldots$ be independent uniform $(0,t)$ random variables that are independent of $N(t)$, and let $U_{(i,n)}$ be the i^{th} smallest of the first n of them.

$$P\left\{\sum_{i=1}^{N(t)} g(S_i) < x\right\}$$

$$= \sum_n P\left\{\sum_{i=1}^{N(t)} g(S_i) < x|N(t)=n\right\}P\{N(t)=n\}$$

$$= \sum_n P\left\{\sum_{i=1}^{n} g(S_i) < x|N(t)=n\right\}P\{N(t)=n\}$$

$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_{(i,n)}) < x\right\}P\{N(t)=n\}$$

(Theorem 5.2)

$$= \sum_n P\left\{\sum_{i=1}^n g(U_i) < x\right\} P\{N(t) = n\}$$

$$\left(\sum_{i=1}^n g(U_{(i,n)}) = \sum_{i=1}^n g(U_i)\right)$$

$$= \sum_n P\left\{\sum_{i=1}^n g(U_i) < x | N(t) = n\right\} P\{N(t) = n\}$$

$$= \sum_n P\left\{\sum_{i=1}^{N(t)} g(U_i) < x | N(t) = n\right\} P\{N(t) = n\}$$

$$= P\left\{\sum_{i=1}^{N(t)} g(U_i) < x\right\}$$

72. (a) Call the random variable S_n. Since it is the sum of n independent exponentials with rate λ, it has a graze distribution with parameters n and λ.

(b) Use the result that given $S_n = t$ the set of times at which the first $n-1$ riders departed are independent uniform $(0, t)$ random variables. Therefore, each of these riders will still be walking at time t with probability

$$p = \int_0^t e^{-\mu(t-s)} ds/t = \frac{1 - e^{-\mu t}}{\mu t}.$$

Hence, the probability that none of the are walking at time t is $(1-p)^{n-1}$.

73. (a) It is the gamma distribution with parameters n and λ.

(b) For $n \geq 1$,
$P\{N = n | T = t\}$

$$= \frac{P\{T = t | N = n\} p(1-p)^{n-1}}{f_T(t)}$$

$$= C\frac{(\lambda t)^{n-1}}{(n-1)!}(1-p)^{n-1}$$

$$= C\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$

where the last equality follows since the probabilities must sum to 1.

(c) The Poisson events are broken into two classes, those that cause failure and those that do not. By Proposition 5.2, this results in 2 independent Poisson processes with respective rates λp and $\lambda(1-p)$. By independence it follows that given that the first event of the first process occurred at time t the number of events of the second process by this time is Poisson with mean $\lambda(1-p)t$.

74. (a) Since each item will, independently, be found with probability $1 - e^{-\mu t}$ it follows that the number found will be Poisson distribution with mean $\lambda(1 - e^{-\mu t})$. Hence, the total expected return is $R\lambda(1 - e^{-\mu t}) - Ct$.

(b) Calculus now yields that the maximizing value of t is given by

$$t = \frac{1}{\mu}\log\left(\frac{R\lambda\mu}{C}\right)$$

provided that $R\lambda\mu > C$; if the inequality is reversed then $t = 0$ is best.

(c) Since the number of items not found by any time t is independent of the number found (since each of the Poisson number of items will independently either be counted with probability $1 - e^{-\mu t}$ or uncounted with probability $e^{-\mu t}$) there is no added gain in letting the decision on whether to stop at time t depend on the number already found.

75. (a) $\{Y_n\}$ is a Markov chain with transition probabilities given by

$$P_{0j} = a_j, \quad P_{i,i-1+j} = a_j, \quad j \geq 0,$$

where

$$a_j = \int \frac{e^{-\lambda t}(\lambda t)^j}{j!} dG(t).$$

(b) $\{X_n\}$ is a Markov chain with transition probabilities

$$P_{i,i+1-j} = \beta_j,\ j = 0, 1, \ldots, i, P_{i,0} = \sum_{k=i+1}^{\infty} \beta_j,$$

where

$$\beta_j = \int \frac{e^{-\mu t}(\mu t)^j}{j!} dF(t).$$

76. Let Y denote the number of customers served in a busy period. Note that given S, the service time of the initial customer in the busy period, it follows by the argument presented in the text that the conditional distribution of $Y - 1$ is that of the

compound Poisson random variable $\sum_{i=1}^{N(S)} Y_i$, where the Y_i have the same distribution as does Y. Hence,

$$E[Y|S] = 1 + \lambda S E[Y]$$

$$Var(Y|S) = \lambda S E[Y^2]$$

Therefore,

$$E[Y] = \frac{1}{1 - \lambda E[S]}$$

Also, by the conditional variance formula

$$\begin{aligned} Var(Y) &= \lambda E[S] E[Y^2] + (\lambda E[Y])^2 Var(S) \\ &= \lambda E[S] Var(Y) + \lambda E[S](E[Y])^2 \\ &\quad + (\lambda E[Y])^2 Var(S) \end{aligned}$$

implying that

$$Var(Y) = \frac{\lambda E[S](E[Y])^2 + (\lambda E[Y])^2 Var(S)}{1 - \lambda E[S]}$$

77. $e^{-11}(11)^n/n!$

78. Poisson with mean 63.

79. Consider a Poisson process with rate λ in which an event at time t is counted with probability $\lambda(t)/\lambda$ independently of the past. Clearly such a process will have independent increments. In addition,

$$\begin{aligned} &P\{2 \text{ or more counted events in} (t, t+h)\} \\ &\quad \le P\{2 \text{ or more events in} (t, t+h)\} \\ &\quad = o(h), \end{aligned}$$

and

$$\begin{aligned} &P\{1 \text{ counted event in } (t, t+h)\} \\ &\quad = P\{1 \text{ counted} \mid 1 \text{ event}\} P(1 \text{ event}) \\ &\qquad + P\{1 \text{ counted} \mid \ge 2 \text{ events}\} P\{\ge 2\} \\ &\quad = \int_t^{t+h} \frac{\lambda(s)}{\lambda} \frac{ds}{h} (\lambda h + o(h)) + o(h) \\ &\quad = \frac{\lambda(t)}{\lambda} \lambda h + o(h) \\ &\quad = \lambda(t) h + o(h). \end{aligned}$$

80. (i) No.

(ii) No.

(iii) $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-m(t)}$ where

$$m(t) = \int_0^t \lambda(s) ds.$$

81. (i) Let S_i denote the time of the ith event, $i \ge 1$. Let $t_i + h_i < t_{i+1}$, $t_n + h_n \le t$. $P\{t_i < S_i < t_i + h_i,\ i = 1, \ldots, n | N(t) = n\}$

$P\{1 \text{ event in } (t_i, t_i + h_i), \quad i = 1, \ldots, n,$

$$= \frac{\text{no events elsewhere in } (0, t)}{P\{N(t) = n\}}$$

$$= \frac{\left[\prod_{i=1}^{n} e^{-(m(t_i+h_i) - m(t_i))} [m(t_i + h_i) - m(t_i)]\right] e^{-[m(t) - \sum_i m(t_i+h_i) - m(t_i)]}}{e^{-m(t)} [m(t)]^n / n!}$$

$$= \frac{n \prod_i^n [m(t_i + h_i) - m(t_i)]}{[m(t)]^n}$$

Dividing both sides by $h_1 \cdots h_n$ and using the fact that $m(t_i + h_i) - m(t_i) = \int_{t_i}^{t_i+h} \lambda(s)\, ds = \lambda(t_i) h + o(h)$ yields upon letting the $h_i \to 0$:

$f_{S_1 \cdots S_2}(t_1, \ldots, t_n | N(t) = n)$

$$= n! \prod_{i=1}^{n} [\lambda(t_i)/m(t)],$$

and the right hand side is seen to be the joint density function of the order statistics from a set of n independent random variables from the distribution with density function $f(x) = m(x)/m(t), x \le t$.

(ii) Let $N(t)$ denote the number of injuries by time t. Now given $N(t) = n$, it follows from part (i) that the n injury instances are independent and identically distributed. The probability (density) that an arbitrary one of those injuries was at s is $\lambda(s)/m(t)$, and so the probability that the injured party will still be out of work at time t is

$$p = \int_0^t P\{\text{out of work at } t | \text{injured at } s\} \frac{\lambda(s)}{m(t)} d\zeta$$

$$= \int_0^t [1 - F(t - s)] \frac{\lambda(s)}{m(t)} d\zeta$$

Hence, as each of the $N(t)$ injured parties have the same probability p of being out of work at t, we see that

$$E[X(t)] | N(t)] = N(t) p$$

and thus,

$$\begin{aligned} E[X(t)] &= pE[N(t)] \\ &= pm(t) \\ &= \int_0^t [1 - F(t-s)]\lambda(s)\, ds. \end{aligned}$$

82. Interpret N as a number of events, and correspond X_i to the i^{th} event. Let $I_1, I_2, \ldots, I_k$ be k nonoverlaping intervals. Say that an event from N is a type j event if its corresponding X lies in I_j, $j = 1, 2, \ldots, k$. Say that an event from N is a type $k+1$ event otherwise. It then follows that the numbers of type j, $j = 1, \ldots, k$, events—call these numbers $N(I_j)$, $j = 1, \ldots, k$,—are independent Poisson random variables with respective means

$$E[N(I_j)] = \lambda P\{X_i \in I_j\} = \lambda \int_{I_j} f(s)ds$$

The independence of th $N(I_j)$ establishes that the process $\{N(t)\}$ has independent increments. Because $N(t+h) - N(t)$ is Poisson distributed with mean

$$\begin{aligned} E[N(t+h) - N(t)] &= \lambda \int_t^{t+h} f(s)ds \\ &= \lambda h f(t) + o(h) \end{aligned}$$

it follows that

$$\begin{aligned} P\{N(t+h) - N(t) = 0\} &= e^{-(\lambda h f(t) + o(h))} \\ &= 1 - \lambda h f(t) + o(h) \end{aligned}$$

$$\begin{aligned} &P\{N(t+h) - N(t) = 1\} \\ &\quad = (\lambda h f(t) + o(h))e^{-(\lambda h f(t) + o(h))} \\ &\quad = (\lambda h f(t) + o(h) \end{aligned}$$

As the preceding also implies that

$$P\{N(t+h) - N(t) \geq 2\} = o(h)$$

the verification is complete.

83. Since $m(t)$ is increasing it follows that nonoverlapping time intervals of the $\{N(t)\}$ process will correspond to nonoverlapping intervals of the $\{N_o(t)\}$ process. As result, the independent increment property will also hold for the $\{N(t)\}$ process. For the remainder we will use the identity

$$m(t+h) = m(t) + \lambda(t)h + o(h).$$

$$\begin{aligned} &P\{N(t+h) - N(t) \geq 2\} \\ &\quad = P\{N_o[m(t+h)] - N_o[m(t)] \geq 2\} \\ &\quad = P\{N_o[m(t) + \lambda(t)h + o(h)] - N_o[m(t)] \geq 2\} \\ &\quad = o[\lambda(t)h + o(h)] = o(h). \end{aligned}$$

$$\begin{aligned} &P\{N(t+h) - N(t) = 1\} \\ &\quad = P\{N_o[m(t) + \lambda(t)h + o(h)] - N_o[m(t)] = 1\} \\ &\quad = P\{1 \text{ event of Poisson process in interval of length } \lambda(t)h + o(h)]\} \\ &\quad = \lambda(t)h + o(h). \end{aligned}$$

84. There is a record whose value is between t and $t + dt$ if the first X larger than t lies between t and $t + dt$. From this we see that, independent of all record values less that t, there will be one between t and $t + dt$ with probability $\lambda(t)dt$ where $\lambda(t)$ is the failure rate function given by

$$\lambda(t) = f(t)/[1 - F(t)].$$

Since the counting process of record values has, by the above, independent increments we can conclude (since there cannot be multiple record values because the X_i are continuous) that it is a nonhomogeneous Poisson process with intensity function $\lambda(t)$. When f is the exponential density, $\lambda(t) = \lambda$ and so the counting process of record values becomes an ordinary Poisson process with rate λ.

85. \$ 40,000 and \$$1.6 \times 10^8$.

86. (a) $P\{N(t) = n\} = .3\, e^{-3t}(3t)^n n! + .7e^{-5t}(5t)^n n!.$
   (b) No!
   (c) Yes! The probability of n events in any interval of length t will, by conditioning on the type of year, be as given in (a).
   (d) No! Knowing how many storms occur in an interval changes the probability that it is a good year and this affects the probability distribution of the number of storms in other intervals.
   (e) $P\{\text{good}|N(1) = 3\}$

$$= \frac{P\{N(1) = 3|\text{good}\}\ P\{\text{good}\}}{P\{N(1) = 3|\text{good}\}P\{\text{good}\} + P\{N(1) = 3|\text{bad}\}P\{\text{bad}\}}$$

$$= \frac{(e^{-3}3^3/3!).3}{(e^{-3}3^3/3!).3 + e^{-5}5^3/3!).7}$$

87. $Cov[X(t), X(t+s)]$

$$\begin{aligned} &= Cov[X(t), X(t) + X(t+s) - X(t)] \\ &= Cov[X(t), X(t)] + Cov[X(t), X(t+s) - X(t)] \\ &= Cov[X(t), X(t)] \text{ by independent increments} \\ &= Var[X(t)] = \lambda t E[Y^2]. \end{aligned}$$

88. Let $X(15)$ denote the daily withdrawal. Its mean and variance are as follows.

$$E[X(15)] = 12 \cdot 15 \cdot 30 = 5400$$

$$Var[X(15)] = 12 \cdot 15 \cdot [30 \cdot 30 + 50 \cdot 50] = 612.000$$

Hence,

$$P\{X(15) \le 6000\}$$

$$= P\left\{\frac{X(15) - 5400}{\sqrt{612,000}} \le \frac{600}{\sqrt{612,000}}\right\}$$

$= P\{Z \le .767\}$ where Z is a standard normal

$= .78$ from Table 7.1 of Chapter 2.

89. Let T_i denote the arrival time of the first type i shock, $i = 1, 2, 3$.

$$P\{X_1 > s, X_2 > t\}$$

$$= P\{T_1 > s, T_3 > s, T_2 > t, T_3 > t\}$$

$$= P\{T_1 > s, T_2 > t, T_3 > \max(s, t)\}$$

$$= e^{-\lambda_1 s} e^{-\lambda_2 t} e^{-\lambda_3 \max(s,t)}$$

90. $P\{X_1 > s\} = P\{X_1 > s, X_2 > 0\}$

$$= e^{-\lambda_1 s} e^{-\lambda_3 s}$$

$$= e^{-(\lambda_1 + \lambda_3)s}.$$

91. To begin, note that

$$P\left[X_1 > \sum_2^n X_i\right]$$

$$= P\{X_1 > X_2\} P\{X_1 - X_2 > X_3 | X_1 > X_2\}$$

$$= P\{X_1 - X_2 - X_3 > X_4 | X_1 > X_2 + X_3\} \ldots$$

$$= P\{X_1 - X_2 \cdots - X_{n-1} > X_n | X_1 > X_2 + \cdots + X_{n-1}\}$$

$$= (1/2)^{n-1}.$$

Hence,

$$P\left\{M > \sum_{i=1}^n X_i - M\right\} = \sum_{i-1}^n P\left\{X_1 > \sum_{j \ne i}^n X_i\right\} = n/2^{n-1}.$$

92. $M_2(t) = \sum_i J_i$

where $J_i = \begin{cases} 1 & \text{if bug } i \text{ contributes 2 errors by } t \\ 0 & \text{otherwise} \end{cases}$

and so

$$E[M_2(t)] = \sum_i P\{N_i(t) = 2\} = \sum_i e^{-\lambda_i t} (\lambda_i t)^2 / 2.$$

93. (i) $\max(X_1, X_2) + \min(X_1, X_2) = X_1 + X_2$.

(ii) This can be done by induction:

$$\begin{aligned}
&\max\{(X_1, \ldots, X_n) \\
&= \max(X_1, \max(X_2, \ldots, X_n)) \\
&= X_1 + \max(X_2, \ldots, X_n) \\
&\quad - \min(X_1, \max(X_2, \ldots, X_n)) \\
&= X_1 + \max(X_2, \ldots, X_n) \\
&\quad - \max(\min(X_1, X_2), \ldots, \min(X_1, X_n)).
\end{aligned}$$

Now use the induction hypothesis.

A second method is as follows:

Suppose $X_1 \le X_2 \le \cdots \le X_n)$. Then the coefficient of X_i on the right-side is

$$1 - \begin{bmatrix} n-i \\ 1 \end{bmatrix} + \begin{bmatrix} n-i \\ 2 \end{bmatrix} - \begin{bmatrix} n-i \\ 3 \end{bmatrix} + \cdots$$

$$= (1-1)^{n-i}$$

$$= \begin{cases} 0, & i \ne n \\ 1, & i = n, \end{cases}$$

and so both sides equal X_n. By symmetry the result follows for all other possible orderings of the $X's$.

(iii) Taking expectations of (ii) where X_i is the time of the first event of the i^{th} process yields

$$\sum_i \lambda_i^{-1} - \sum_i \sum_{<j} (\lambda_i + \lambda_j)^{-1} + \sum_i \sum_{<j} \sum_{<k} (\lambda_i + \lambda_j + \lambda_k)^{-1} - \cdots + (-1)^{n+1} \left[\sum_1^n \lambda_i\right]^{-1}$$

94. (i) $P\{X > t\}$

$$= P\{\text{no events in a circle of area } rt^2\}$$

$$= e^{-\lambda r t^2}.$$

(ii) $E[X] = \int_0^\infty P\{X > t\}dt$

$$= \int_0^\infty e^{-\lambda r t^2} dt$$

$$= \frac{1}{\sqrt{2r\lambda}} \int_0^\infty e^{-x^2/2} dx \quad \text{by } x = t\sqrt{2\lambda r}$$

$$= \frac{1}{2\sqrt{\lambda}}$$

where the last equality follows since

$1/\sqrt{2r} \int_0^\infty e^{-x^2/2} dx = 1/2$ since it represents the probability that a standard normal random variable is greater than its mean.

95. $E[L|N(t) = n] = \dfrac{\int xg(x)e^{-xt}(xt)^n dx}{\int g(x)e^{-xt}(xt)^n dx}$

Conditioning on L yields

$$\begin{aligned} &E[N(s)|N(t) = n] \\ &\quad = E[E[N(s)|N(t) = n, L]|N(t) = n] \\ &\quad = E[n + L(s - t)|N(t) = n] \\ &\quad = n + (s - t)E[L|N(t) = n] \end{aligned}$$

For (c), use that for any value of L, given that there have been n events by time t, the set of n event times are distributed as the set of n independent uniform $(0, t)$ random variables. Thus, for $s < t$

$$E[N(s)|N(t) = n] = ns/t$$

96. $$\begin{aligned} E[N(s)N(t)|L] &= E[E[N(s)N(t)|L, N(s)]|L] \\ &= E[N(s)E[N(t)|L, N(s)]|L] \\ &= E[N(s)[N(s) + L(t - s)]|L] \\ &= E[N^2(s)|L] + L(t - s)E[N(s)|L] \\ &= Ls + (Ls)^2 + (t - s)sL^2 \end{aligned}$$

Thus,

$$Cov(N(s), N(t)) = sm_1 + stm_2 - stm_1^2$$

Chapter 6

1. Let us assume that the state is (n, m). Male i mates at a rate λ with female j, and therefore it mates at a rate λm. Since there are n males, matings occur at a rate λnm. Therefore,

$$v_{(n,m)} = \lambda nm.$$

Since any mating is equally likely to result in a female as in a male, we have

$$P_{(n,m);(n+1,m)} = P_{(n,m)(n,m+1)} = \frac{1}{2}.$$

2. Let $N_A(t)$ be the number of organisms in state A and let $N_B(t)$ be the number of organisms in state B. Then clearly $\{N_A(t); N_B(t)\}$ is a continuous-Markov chain with

$$v_{\{n,m\}} = \alpha n + \beta m$$
$$P_{\{n,m\};\{n-1;m+1\}} = \frac{\alpha n}{\alpha n + \beta m}$$
$$P_{\{n,m\};\{n+2;m-1\}} = \frac{\beta m}{\alpha n + \beta m}.$$

3. This is not a birth and death process since we need more information than just the number working. We also must know which machine is working. We can analyze it by letting the states be

b : both machines are working
1 : 1 is working, 2 is down
2 : 2 is working, 1 is down
0_1: both are down, 1 is being serviced
0_2: both are down, 2 is being serviced.

$$v_b = \mu_1 + \mu_2,\ v_1 = \mu_1 + \mu,\ v_2 = \mu_2 + \mu,$$
$$v_{0_1} = v_{0_2} = \mu$$
$$P_{b,1} = \frac{\mu_2}{\mu_2 + \mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu + \mu_1} = 1 - P_{1,0_2}$$
$$P_{2,b} = \frac{\mu}{\mu + \mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1.$$

4. Let $N(t)$ denote the number of customers in the station at time t. Then $\{N(t)\}$ is a birth and death process with

$$\lambda_n = \lambda\alpha_n, \quad \mu_n = \mu.$$

5. (a) Yes.
   (b) It is a pure birth process.
   (c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability $i(n-i)/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

$$E[\text{time all infected}] = \frac{n(n-1)}{2\lambda} \sum_{i=1}^{n} 1/[i(n-i)].$$

6. Starting with $E[T_0] = \dfrac{1}{\lambda_0} = \dfrac{1}{\lambda}$, employ the identity

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]$$

to successively compute $E[T_i]$ for $i = 1, 2, 3, 4$.

   (a) $E[T_0] + \cdots + E[T_3]$.
   (b) $E[T_2] + E[T_3] + E[T_4]$.

7. (a) Yes!
   (b) For $n = (n_1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$ let

$$S_i(n) = (n_1, \ldots, n_{i-1}, n_{i+1} + 1, \ldots, n_{k-1}),$$
$$i = 1, \ldots, k-2$$
$$S_{k-1}(n) = (n_1, \ldots, n_i, n_{i+1}, \ldots n_{k-1} - 1),$$
$$S_0(n) = (n_1 + 1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1}).$$

   Then

$$q_n, S_1(n) = n_i\mu, \qquad i = 1, \ldots, k-1$$
$$q_n, S_0(n) = \lambda.$$

8. The number of failed machines is a birth and death process with

$$\lambda_0 = 2\lambda \qquad \mu_1 = \mu_2 = \mu$$
$$\lambda_1 = \lambda \qquad \mu_n = 0, n \neq 1, 2$$

$\lambda_n = 0, n > 1.$

Now substitute into the backward equations.

9. Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt. Hence,

$$P_{ij}(t) = e^{-\mu t}(\mu t)^{i-j}/(i-j)!, \quad 0 < j \le i$$
$$P_{i,0}(t) = \sum_{k=i}^{\infty} e^{-\mu t}(\mu t)^k/k!.$$

10. Let $I_j(t) = \begin{cases} 0, & \text{if machine } j \text{ is working at time } t \\ 1, & \text{otherwise.} \end{cases}$

Also, let the state be $(I_1(t), I_2(t))$.

This is clearly a continuous-time Markov chain with

$$v_{(0,0)} = \lambda_1 + \lambda_2 \; \lambda_{(0,0);\,(0,1)} = \lambda_2 \; \lambda_{(0,0);\,(1,0)} = \lambda_1$$
$$v_{(0,1)} = \lambda_1 + \mu_2 \; \lambda_{(0,1);\,(0,0)} = \mu_2 \; \lambda_{(0,1);\,(1,1)} = \lambda_1$$
$$v_{(1,0)} = \mu_1 + \lambda_2 \; \lambda_{(1,0);\,(0,0)} = \mu_1 \; \lambda_{(1,0);\,(1,1)} = \lambda_2$$
$$v_{(1,1)} = \mu_1 + \mu_2 \; \lambda_{(1,1);\,(0,1)} = \mu_1 \; \lambda_{(1,1);\,(1,0)} = \lambda_2$$

By the independence assumption, we have

(a) $P_{(i,j)(k,\ell)}(t) = P_{(i,k)}(t)Q_{(j,\ell)}(t)$

where $P_{i,k}(t) =$ probability that the first machine be in state k at time t given that it was at state i at time 0.

$Q_{j,\ell}(t)$ is defined similarly for the second machine. By example 4(c) we have

$$P_{00}(t) = [\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1]/(\lambda_1 + \mu_1)$$
$$P_{10}(t) = [\mu_1 - \mu_1 e^{-(\mu_1+\lambda_1)t}]/(\lambda_1 + \mu_1).$$

And by the same argument,

$$P_{11}(t) = [\mu_1 e^{-(\mu_1+\lambda_1)t} + \lambda_1]/(\lambda_1 + \mu_1)$$
$$P_{01}(t) = [\lambda_1 - \lambda_1 e^{-(\mu_1+\lambda_1)t}]/(\lambda_1 + \mu_1).$$

Of course, the similar expressions for the second machine are obtained by replacing (λ_1, μ_1) by (λ_2, μ_2). We get $P_{(i,j)(k,\ell)}(t)$ by formula (a). For instance,

$$P_{(0,0)(0,0)}(t) = P_{(0,0)}(t)Q_{(0,0)}(t) = \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)} \times \frac{\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_2+\mu_2)}.$$

Let us check the forward and backward equations for the state $\{(0,\,0);\ (0,\,0)\}$.

Backward equation

We should have

$$P'_{(0,0),(0,0)}(t) = (\lambda_1+\lambda_2)\left[\frac{\lambda_2}{\lambda_1+\lambda_2}P_{(0,1)(0,0)}(t) + \frac{\lambda_1}{\lambda_1+\lambda_2}P_{(1,0)(0,0)}(t) - P_{(0,0)(0,0)}(t)\right],$$

or

$$P'_{(0,0)(0,0)}(t) = \lambda_2 P_{(0,1)(0,0)}(t) + \lambda_1 P_{(1,0)(0,0)}(t) - (\lambda_1+\lambda_2)\,P_{(0,0)(0,0)}(t).$$

Let us compute the right-hand side ($r.h.s.$) of this expression:

$r.h.s.$

$$= \lambda_2 \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]\left[\mu_2 - \mu_2 e^{-(\mu_2+\lambda_2)t}\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} + \frac{\left[\mu_1 - \mu_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} - (\lambda_1+\lambda_2)\frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$= \frac{\lambda_2\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \times \left[\mu_2 - \mu_2 e^{-(\mu_2+\lambda_2)t} - \lambda_2 e^{-(\lambda_2+\mu_2)t} - \mu_2\right] + \frac{\lambda_1\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \times \left[\mu_1 - \mu_1 e^{-(\mu_1+\lambda_1)t} - \mu_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right]$$

$$= \left[-\lambda_2 e^{-(\lambda_2+\mu_2)t}\right]\left[\frac{\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1}{\lambda_1+\mu_1}\right] + \left[-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\frac{\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2}{\lambda_2+\mu_2}\right]$$

$$= Q'_{00}(t)P_{00}(t) + P'_{00}(t)Q_{00}(t) = [P_{00}(t)Q_{00}(t)]'$$

$$= \left[P_{(0,0)(0,0)}(t)\right]'.$$

So, for this state, the backward equation is satisfied.

Forward equation

According to the forward equation, we should now have

$$P'_{(0,\,0)(0,\,0)}(t) = \mu_2 P_{(0,\,0)(0,\,1)}(t) + \mu_1 P_{(0,\,0)(1,\,0)}(t) - (\lambda_1 + \lambda_1) P_{(0,\,0)(0,\,0)}(t).$$

Let us compute the right-side:

r.h.s.

$$= \mu_2 \frac{\left[\lambda_1 e^{-(\mu_1 \quad +\lambda_1)t} + \mu_1\right]\left[\lambda_2 - \lambda_2 e^{-(\lambda_2+\mu_2)t}\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$+\mu_1 \frac{\left[\lambda_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda 2+\mu_2)}$$

$$-(\lambda_1+\lambda_2) \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$= \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)}$$

$$\times \frac{\left[\mu_2\lambda_2 - \lambda_2 e^{-(\lambda_2+\mu_2)t} - \lambda_2\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]\right]}{\lambda_2+\mu_2}$$

$$+ \frac{\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]}{(\lambda_2+\mu_2)}$$

$$\times \frac{\left[\mu_1\left[\lambda_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right] - \lambda_1\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\right]}{(\lambda_1+\mu_1)}$$

$$= P_{00}(t)\left[-\lambda_2 e^{-(\mu_2+\lambda_2)t}\right] + Q_{00}(t)\left[-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right]$$

$$= P_{00}(t)Q'_{00}(t) + Q_{00}(t)P'00(t) = \left[P_{(0,0)(0,0)}(t)\right].$$

In the same way, we can verify the Kolmogorov's equations for all the other states.

11. (b) Follows from the hint upon using the lack of memory property and the fact that ϵ_i, the minimum of $j-(i-1)$ independent exponentials with rate λ, is exponential with rate $(j-i+1)\lambda$.

(c) From (a) and (b)

$$P\{T_1 + \cdots + T_j \le t\} = P\left\{\max_{1\le i\le j} X_i \le t\right\} = (1-e^{-\lambda t})^j$$

(d) With all probabilities conditional on $X(0) = 1$

$$\begin{aligned} P_{1j}(t) &= P\{X(t) = j\} \\ &= P\{X(t) \ge j\} - P\{X(t) \ge j+1\} \\ &= P\{T_1 + \cdots + T_j \le t\} \\ &\quad - P\{T_1 + \cdots + T_{j+1} \le t\} \end{aligned}$$

(e) The sum of independent geometrics, each having parameter $p = e^{-\lambda t}$ is negative binomial with parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual.

12. (a) If the state is the number of individuals at time t, we get a birth and death process with

$$\begin{aligned} \lambda_n &= n\lambda + \theta \qquad n < N \\ \lambda_n &= n\lambda \qquad n \ge N \\ \mu_n &= n\mu. \end{aligned}$$

(b) Let P_i be the long-run probability that the system is in state i. Since this is also the proportion of time the system is in state i, we are looking for $\sum_{i=3}^{\infty} P_i$.

We have $\lambda_k P_k = \mu_{k+1} P_{k+1}$.

This yields

$$P_1 = \frac{\theta}{\mu} P_0$$

$$P_2 = \frac{\lambda+\theta}{2\mu} P_1 = \frac{\theta(\lambda+\theta)}{2\mu^2} P_0$$

$$P_3 = \frac{2\lambda+\theta}{2\mu} P_2 = \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3} P_0.$$

For $k \ge 4$, we get

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1},$$

which implies

$$P_k = \frac{(k-1)(k-2)\cdots(3)}{(k)(k-1)\cdots(4)} \left[\frac{\lambda}{\mu}\right]^{k-3}$$

$$P_3 = \frac{3}{k}\left[\frac{\lambda}{\mu}\right]^{k-3} P_3;$$

therefore $\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k,$

$$\text{but} \quad \sum_{k=1}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k = \log\left[\frac{1}{1-\dfrac{\lambda}{\mu}}\right]$$

$$= \log\left[\frac{\mu}{\mu-\lambda}\right] \text{ if } \frac{\lambda}{\mu} < 1.$$

$$\text{So} \quad \sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$$

$$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$$

$$\frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3}P_0.$$

Now $\sum_0^{\infty} P_i = 1$ implies

$$P_0 = \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{1}{2\lambda^3}\theta(\lambda+\theta)(2\lambda+\theta)\right.$$

$$\left.\times\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]^{-1}.$$

And finally,

$$\sum_{k=3}^{\infty} P_k = \left[\left[\frac{1}{2\lambda^3}\right]\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right.$$

$$\left.\theta(\lambda+\theta)(2\lambda+\theta)\right] \Big/ \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2}\right.$$

$$+\frac{\theta(\lambda+\theta)(2\lambda+\theta)}{2\lambda^3}$$

$$\left.\times\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right].$$

13. With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3 \quad \mu_1 = \mu_2 = 4.$$

Therefore

$$P_1 = \frac{3}{4}P_0 \quad P_2 = \frac{3}{4} \quad P_1 = \left[\frac{3}{4}\right]^2 P_0.$$

And since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{16}{37}.$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left[\frac{3}{4}\right]^2\right] P_0$$

$$= \frac{30}{16}\left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{30}{37}.$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16}\cdot\frac{16}{37} = \frac{28}{37}.$$

(c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8}\right]^2\right]^{-1} = \frac{64}{97}.$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8}\right]^2\frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}.$$

The rate of added customers is therefore

$$\lambda\left[\frac{88}{97}\right] - \lambda\left[\frac{28}{37}\right] = 3\left[\frac{88}{97} - \frac{28}{37}\right] = 0.45.$$

The business he does would improve by 0.45 customers per hour.

14. Letting the number of cars in the station be the state, we have a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 20, \quad \lambda_i = 0,\ i > 2,$$

$$\mu_1 = \mu_2 = 12.$$

Hence,

$$P_1 = \frac{5}{3}P_0,\ P_2 = \frac{5}{3}P_1 = \left[\frac{5}{3}\right]^2 P_0,$$

$$P_3 = \frac{5}{3}P_2 = \left[\frac{5}{3}\right]^3 P_0$$

and as $\sum_0^3 P_i = 1$, we have

$$P_0 = \left[1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^2\right]^{-1} = \frac{27}{272}.$$

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore

$$P_3 = \left[\frac{5}{3}\right]^3 P_0 = 125/272.$$

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3 \ \lambda_i = 0, \quad i \geq 4,$$
$$\mu_1 = 2 \ \mu_2 = \mu_3 = 4.$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0 \ P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0 \ P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0.$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}.$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}.$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0 \ P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0 \ P_3 = \left[\frac{3}{4}\right]^3 P_0,$$

and $P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}.$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}.$$

16. Let the state be

0: an acceptable molecule is attached

1: no molecule attached

2: an unacceptable molecule is attached.

Then this is a birth and death process with balance equations

$$P_{12} = \frac{\mu}{\lambda} P_0$$
$$P_2 = \frac{\lambda(1-\alpha)}{\mu_1} P_1 = \frac{(1-\alpha)}{\alpha}\frac{\mu_2}{\mu_1} P_0.$$

Since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{\mu_2}{\lambda\alpha} + \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}\right]^{-1}$$
$$= \frac{\lambda\alpha\mu_1}{\lambda\alpha\mu_1 + \mu_1\mu_2 + \lambda(1-\alpha)\mu_2}.$$

P_0 is the percentage of time the site is occupied by an acceptable molecule.

The percentage of time the site is occupied by an unacceptable molecule is

$$P_2 = \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1} P_0 = \frac{\lambda(1-\alpha)\mu_2}{\lambda\alpha\mu_1 + \mu_1 + \lambda(1-\alpha)\mu_2}.$$

17. Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$\mu_1 P_1 = \lambda p P_0$$
$$\mu_2 P_2 = \lambda(1-p) P_0$$
$$P_0 + P_1 + P_2 = 1.$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$$
$$P_1 = \lambda p P_0/\mu_1, \qquad P_2 = \lambda(1-p)P_0/\mu_2.$$

18. There are $k+1$ states; state 0 means the machine is working, state i means that it is in repair phase i, $i = 1, \ldots, k$. The balance equations for the limiting probabilities are

$$\lambda P_0 = \mu_k P_k$$
$$\mu_1 P_1 = \lambda P_0$$
$$\mu_i P_i = \mu_{i-1} P_{i-1}, \quad i = 2, \ldots, k$$
$$P_0 + \cdots + P_k = 1.$$

To solve, note that

$$\mu_i P_i = \mu_{i-1} P_{i-1} = \mu_{i-2} P_{i-2} = \cdots = \lambda P_0.$$

Hence,

$$P_i = (\lambda/\mu_i) P_0,$$

and, upon summing,

$$1 = P_0 \left[1 + \sum_{i=1}^{k} (\lambda/\mu_i)\right].$$

Therefore,

$$P_0 = \left[1 + \sum_{i=1}^{k} (\lambda/\mu_i)\right]^{-1}, \quad P_i = (\lambda/\mu_i)P_0,$$

$i = 1, \ldots, k.$

The answer to part (a) is P_i and to part (b) is P_0.

19. There are 4 states. Let state 0 mean that no machines are down, state 1 that machine one is down and two is up, state 2 that machine one is up and two is down, and 3 that both machines are down. The balance equations are as follows.

$$\begin{aligned}(\lambda_1 + \lambda_2)P_0 &= \mu_1 P_1 + \mu_2 P_2\\ (\mu_1 + \lambda_2)P_1 &= \lambda_1 P_0 + \mu_1 P_3\\ (\lambda_1 + \mu_2)P_2 &= \lambda_2 P_0\\ \mu_1 P_3 &= \mu_2 P_1 + \mu_1 P_2\end{aligned}$$

$$P_0 + P_1 + P_2 + P_3 = 1.$$

These equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.

20. Letting the state be the number of down machines, this is a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 0, 1$$
$$\mu_i = \mu, \quad i = 1, 2.$$

By the results of Example 3g, we have that

$$E[\text{time to go from 0 to 2}] = 2/\lambda + \mu/\lambda^2.$$

Using the formula at the end of Section 3, we have that

Var(time to go from 0 to 2)

$$\begin{aligned}&= Var(T_0) + Var(T_1)\\ &= \frac{1}{\lambda^2} + \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\mu+\lambda}(2/\lambda + \mu/\lambda^2)^2.\end{aligned}$$

Using Equation (5.3) for the limiting probabilities of a birth and death process, we have that

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2}.$$

21. How we have a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 1, 2$$
$$\mu_i = i\mu, \quad i = 1, 2.$$

Therefore,

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2/2},$$

and so the probability that at least one machine is up is higher in this case.

22. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n+1), \quad n \geq 0$$
$$\mu_n = \mu, \quad n \geq 1.$$

From Equation (5.3),

$$1/P_0 = 1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_n = P_0(\lambda/\mu)^n/n! = e^{-\lambda/\mu}(\lambda/\mu)^n/n!, \quad n \geq 0.$$

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \ \lambda_1 = \frac{2}{10}, \ \lambda_2 = \frac{1}{10}, \ \lambda_i = 0, \quad i \geq 3$$
$$\mu_1 = \frac{1}{8}, \ \mu_2 = \frac{2}{8}, \ \mu_3 = \frac{2}{8}.$$

The balance equations reduce to

$$\begin{aligned}P_1 &= \frac{3/10}{1/8}P_0 = \frac{12}{5}P_0\\ P_2 &= \frac{2/10}{2/8}P_1 = \frac{4}{5}P_1 = \frac{48}{25}P_0\\ P_3 &= \frac{1/10}{2/8}P_2 = \frac{4}{10}P_3 = \frac{192}{250}P_0.\end{aligned}$$

Hence, using $\sum_0^3 P_i = 1$, yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}.$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}.$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}.$$

24. We will let the state be the number of taxis waiting. Then, we get a birth and death process with $\lambda_n = 1\mu_n = 2$. This is a $M/M/1$, and therefore,

(a) Average number of taxis waiting $= \dfrac{1}{\mu - \lambda}$

$$= \frac{1}{2-1} = 1.$$

(b) The proportion of arriving customers that get taxis is the proportion of arriving customers that find at least one taxi waiting. The rate of arrival of such customers is $2(1 - P_0)$. The proportion of such arrivals is therefore

$$\frac{2(1-P_0)}{2} = 1 - P_0 = 1 - \left[1 - \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu} = \frac{1}{2}.$$

25. If $N_i(t)$ is the number of customers in the ith system $(i = 1, 2)$, then let us take $\{N_1(t), N_2(t)\}$ as the state. The balance equation are with $n \geq 1, m \geq 1$.

(a) $\lambda P_{0,\,0} = \mu_2 P_{0,\,1}$

(b) $P_{n,\,0}(\lambda + \mu_1) = \lambda P_{n-1,\,0} + \mu_2 P_{n,\,1}$

(c) $P_{0,\,m}(\lambda + \mu_2) = \mu_1 P_{1,\,m-1} + \mu_2 P_{0,\,m+1}$

(d) $P_{n,\,m}(\lambda + \mu_1 + \mu_2) = \lambda P_{n-1,\,m} + \mu_1 P_{n+1,\,m-1} + \mu_2 P_{n,\,m+1}.$

We will try a solution of the form $C\alpha^n\beta^m = P_{n,\,m}$. From (a), we get

$$\lambda C = \mu_2 C\beta = \beta = \frac{\lambda}{\mu_2}.$$

From (b),

$$(\lambda + \mu_1)\, C\alpha^n = \lambda C\alpha^{n-1} + \mu_2 C\alpha^n\beta,$$

or

$$(\lambda + \mu_1)\,\alpha = \lambda + \mu_2\alpha\beta = \lambda + \mu_2\alpha\frac{\lambda}{\mu} = \lambda + \lambda\alpha,$$

and $\mu_1\alpha = \lambda \Rightarrow \alpha = \dfrac{\lambda}{\mu_1}.$

To get C, we observe that $\displaystyle\sum_{n,\,m} P_{n,\,m} = 1,$

but

$$\sum_{n,\,m} P_{n,\,m} = C\sum_n \alpha^n \sum_m \beta^m = C\left[\frac{1}{1-\alpha}\right]\left[\frac{1}{1-\beta}\right],$$

and $C = \left[1 - \dfrac{\lambda}{\mu_1}\right]\left[1 - \dfrac{\lambda}{\mu_2}\right].$

Therefore a solution of the form $C\alpha^n\beta^n$ must be given by

$$P_{n,\,m} = \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_1}\right]^n\left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_2}\right]^m.$$

It is easy to verify that this also satisfies (c) and (d) and is therefore the solution of the balance equations.

26. Since the arrival process is Poisson, it follows that the sequence of future arrivals is independent of the number presently in the system. Hence, by time reversibility the number presently in the system must also be independent of the sequence of past departures (since looking backwards in time departures are seen as arrivals).

27. It is a Poisson process by time reversibility. If $\lambda > \delta\mu$, the departure process will (in the limit) be a Poisson process with rate $\delta\mu$ since the servers will always be busy and thus the time between departures will be independent random variables each with rate $\delta\mu$.

28. Let P_{ij}^x, V_i^x denote the parameters of the $X(t)$ and P_{ij}^y, V_i^y of the $Y(t)$ process; and let the limiting probabilities by P_i^x, P_i^y, respectively. By independence we have that for the Markov chain $\{X(t), Y(t)\}$ its parameters are

$$V_{(i,\,\ell)} = V_i^x + V_\ell^y$$

$$P_{(i,\,\ell),(j,\,\ell)} = \frac{V_i^x}{V_i^x + V_\ell^y} P_{ij}^x$$

$$P_{(i,\,\ell),\,(i,\,k)} = \frac{V_\ell^y}{V_i^x + V_\ell^y} P_{\ell k}^y$$

and

$$\lim_{t\to\infty} P\{(X(t), Y(t)) = (i, j)\} = P_i^x P_j^y.$$

Hence, we need show that

$$P_i^x P_\ell^y V_i^x P_{ij}^x = P_j^x P_\ell^y V_j^x P_{ji}^x.$$

[That is, rate from $(i,\,\ell)$ to $(j,\,\ell)$ equals the rate from $(j,\,\ell)$ to $(i,\,\ell)$]. But this follows from the fact that the rate from i to j in $X(t)$ equals the rate from j to i; that is,

$$P_i^x V_i^x P_{ij}^x = P_j^x V_j^x P_{ji}^x.$$

The analysis is similar in looking at pairs $(i,\,\ell)$ and (i, k).

29. (a) Let the state be S, the set of failed machines.

(b) For $i \in S$, $j \in S^c$,

$$q_{S,\, S-i} = \mu_i/|S|, q_{S,\, S+j} = \lambda_j,$$

where $S - i$ is the set S with i deleted and $S + j$ is similarly S with j added. In addition, $|S|$ denotes the number of elements in S.

(c) $P_S q_{S,\, S-i} = P_{S-i} q_{S\,-\,i,\; S}$.

(d) The equation (c) are equivalent to

$$P_S \mu_i/|S| = P_{S\,-\,i}\lambda_i$$

or

$$P_S = P_{S-i}|S|\lambda_i/\mu_i.$$

Iterating this recursion gives

$$P_S = P_0(|S|)! \prod_{i\in S} (\lambda_i/\mu_i),$$

where 0 is the empty set. Summing over all S gives

$$1 = P_0 \sum_S (|S|)! \prod_{i\in S} (\lambda_i/\mu_i),$$

and so

$$P_S = \frac{(|S|)! \prod_{i\in S} (\lambda_i/\mu_i)}{\sum_S (|S|)! \prod_{i\in S} (\lambda_i/\mu_i)}.$$

As this solution satisfies the time reversibility equations, it follows that, in the steady state, the chain is time reversible with these limiting probabilities.

30. Since λ_{ij} is the rate it enters j when in state i, all we need do to prove both time reversibility and that P_j is as given is to verify that

$$\lambda_{kj} P_k = \lambda_{jk} P_j \sum_1^n P_j = 1.$$

Since $\lambda_{kj} = \lambda_{jk}$, we see that $P_j \equiv 1/n$ satisfies the above.

31. (a) This follows because of the fact that all of the service times are exponentially distributed and thus memoryless.

(b) Let $n = (n_1, \ldots, n_i, \ldots, n_j, \ldots, n_r)$, where $n_i > 0$ and let $n' = (n_1, \ldots, n_i - 1, \ldots, n_j - 1, \ldots, n_r)$. Then $q_{n,\, n'} = \mu_i/(r-1)$.

(c) The process is time reversible if we can find probabilities $P(n)$ that satisfy the equations

$$P(n)\mu_i/(r-1) = P(n')\mu_j/(r-1)$$

where n and n' are as given in part (b). The above equations are equivalent to

$$\mu_i P(n) = \mu_j/P(n').$$

Since $n_i = n'_i + 1$ and $n'_j = n_j + 1$ (where n_k refers to the k^{th} component of the vector n), the above equation suggests the solution

$$P(n) = C \prod_{k=1}^{r} (1/\mu_k)^n k$$

where C is chosen to make the probabilities sum to 1. As $P(n)$ satisfies all the time reversibility equations it follows that the chain is time reversible and $P(n)$ given above are the limiting probabilities.

32. The states are 0, 1, $1'$, $n, n \geq 2$. State 0 means the system is empty, state 1 ($1'$) means that there is one in the system and that one is with server 1 (2); state $n, n \geq 2$, means that there are n customers in the system. The time reversibility equations are as follows.

$$(\lambda/2)P_0 = \mu_1 P_1$$
$$(\lambda/2)P_0 = \mu_2 P_{1'}$$
$$\lambda P_1 = \mu_2 P_2$$
$$\lambda P_{1'} = \mu_1 P_2$$
$$\lambda P_n = \mu P_{n+1}, n \geq 2$$

where $\mu = \mu_1 + \mu_2$. Solving the last set of equations (with $n \geq 2$) in terms of P_2 gives

$$P_{n+1} = (\lambda/\mu)P_n$$
$$= (\lambda/\mu)^2 P_{n-1} = \cdots = (\lambda/\mu)^{n-1} P_2.$$

That is,

$$P_{n+2} = (\lambda/\mu)^n P_2, \quad n \geq 0.$$

Equations three and four yield that

$$P_1 = (\mu_2/\lambda)P_2$$
$$P_{1'} = (\mu_1/\lambda)P_2$$

The second equation yields that

$$P_0 = (2\mu_2/\lambda)P_{1'} = (2\mu_1\mu_2/\lambda^2)P_2.$$

Thus all the other probabilities are determined in terms of P_0. However, we must now verify that the top equation holds for this solution. This is shown as follows.

$$P_0 = (2\mu_1/\lambda)P_1 = (2\mu_1\mu_2/\lambda^2)P_2.$$

Thus all the time reversible equations hold when the probabilities are given (in terms of P_2) as shown above. The value of P_2 is now obtained by requiring all the probabilities to sum to 1. The fact that this sum will be finite follows from the assumption that $\lambda/\mu < 1$.

33. Suppose first that the waiting room is of infinite size. Let $X_i(t)$ denote the number of customers at server $i, i = 1, 2$. Then since each of the $M/M/1$ processes $\{X_i(t)\}$ is time reversible, it follows by Problem 28 that the vector process $\{(X_1(t), X_2(t)), t \geq 0\}$ is a time reversible Markov chain. Now the process of interest is just the truncation of this vector process to the set of states A where

$$A = \{(0, m) : m \leq 4\} \cup \{(n, 0) : n \leq 4\}$$
$$\cup \{(n, m) : nm > 0, n + m \leq 5\}.$$

Hence, the probability that there are n with server 1 and n with server 2 is

$$P_{n,\,m} = k(\lambda_1/\mu_1)^n(1 - \lambda_1/\mu_1)(\lambda_2/\mu_2)^m(1 - \lambda_2/\mu_2),$$
$$= C(\lambda_1/\mu_1)^n(\lambda_2/\mu_2)^m, \quad (n, m) \in A.$$

The constant C is determined from

$$\sum P_{n,\,n} = 1,$$

where the sum is over all $(n,\ m)$ in A.

34. The process $\{X_i(t)\}$ is a two state continuous time Markov chain and its limiting probability is

$$\lim_{t\to\infty} P\{X_i(t) = 1\} = \mu_i/(\mu_i + \lambda_i), \quad i = 1, \ldots, 4$$

(a) By independence,
proportion of time all working
$$= \prod_{i=1}^{4} \mu_i/(\mu_i + \lambda_i).$$

(b) It is a continuous time Markov chain since the processes $\{X_i(t)\}$ are independent with each being a continuous time Markov chain.

(c) Yes, by Problem 28 since each of the processes $\{X_i(t)\}$ is time reversible.

(d) The model which supposes that one of the phones is down is just a truncation of the process $\{X(t)\}$ to the set of states A, where A includes all 16 states except (0, 0, 0, 0). Hence, for the truncated model

$$P\{\text{all working/truncated}\}$$
$$= P\{\text{all working}\}/(1 - P(0, 0, 0, 0)$$
$$= \frac{\prod_{i=1}^{4}(\mu_i/(\mu_i + \lambda_i)}{1 - \prod_{i=1}^{4}(\lambda_i/(\lambda_i + \mu_i)}.$$

35. We must find probabilities P_i^n such that

$$P_i^n q_{ij}^n = P_j^n q_{ji}^n$$

or

$$\begin{aligned} cP_i^n q_{ij} &= P_j^n q_{ji}, && \text{if } i \in A, j \notin A \\ P_i q_{ij} &= cP_j^n q_{ji}, && \text{if } i \notin A, j \in A \\ P_i q_{ij} &= P_j q_{ji}, && \text{otherwise.} \end{aligned}$$

Now, $P_i q_{ij} = P_j q_{ji}$ and so if we let

$$P_i^n = \begin{matrix} kP_i/c & \text{if } i \in A \\ kP_i & \text{if } i \notin A \end{matrix}$$

then we have a solution to the above equations. By choosing k to make the sum of the P_j^n equal to 1, we have the desired result. That is,

$$k = \left(\sum_{i\in A} P_i/c - \sum_{i\notin A} P_i\right)^{-1}.$$

36. In Problem 3, with the state being the number of machines down, we have

$$v_0 = 2\lambda \ P_{0,\,1} = 1$$
$$v_1 = \lambda + \mu \ P_{1,\,0} = \frac{\mu}{(\lambda+\mu)} P_{1,\,2} = \frac{1}{(\lambda+\mu)}$$
$$v_2 = \mu \ P_{2,\,1} = 1.$$

We will choose $v = 2\lambda = 2\mu$, then the uniformized version is given by

$$v_i^n = 2(\lambda + \mu) \quad \text{for } i = 0, 1, 2$$
$$P_{00}^n = 1 - \frac{2\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(\lambda+\mu)}$$
$$P_{01}^n = \frac{2\lambda}{2(\lambda+\mu)} \cdot 1 = \frac{\lambda}{(\lambda+\mu)}$$
$$P_{10}^n = \frac{\lambda+\mu}{2(\lambda+\mu)} \cdot \frac{\mu}{(\lambda+\mu)} = \frac{\mu}{2(\lambda+\mu)}$$

$$P_{11}^n = 1 - \frac{\lambda+\mu}{2(\lambda+\mu)} = \frac{1}{2}$$

$$P_{12}^n = \frac{\lambda+\mu}{2(\lambda+\mu)}\frac{\lambda}{(\lambda+\mu)} = \frac{\lambda}{2(\lambda+\mu)}$$

$$P_{21}^n = \frac{\mu}{2(\lambda+\mu)}$$

$$P_{22}^n = 1 - \frac{\mu}{2(\lambda+\mu)} = \frac{2\lambda+\mu}{2(\lambda+\mu)}.$$

37. The state of any time is the set of down components at that time. For $S \subset \{1, 2, \ldots, n\}$, $i \notin S$, $j \in S$

$$q(S, S+i) = \lambda_i$$

$$q(S, S-j) = \mu_j \alpha^{|S|}$$

where $S+i = S \cup \{i\}$, $S-j = S \cap \{j\}^c$, $|S|$ = number of elements in S.

The time reversible equations are

$$P(S)\mu_i\alpha^{|S|} = P(S-i)\lambda_i, \quad i \in S$$

The above is satisfied when, for $S = \{i_1, i_2, \ldots, i_k\}$

$$P(S) = \frac{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}}{\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}\alpha^{k(k+1)/2}}P(\phi)$$

where $P(\phi)$ is determined so that

$$\sum P(S) = 1$$

where the sum is over all the 2^n subsets of $\{1, 2, \ldots, n\}$.

38. Say that the process is "on" when in state 0.

(a) $E[0(t+h)] = E[0(t) + \text{on time in } (t, t+h)]$

$= n(t) + E[\text{on time in } (t, t+h)]$

Now

$E[\text{on time in } (t, t+h) | X(t) = 0] = h + o(h)$

$E[\text{on time in } (t, t+h) | X(t) = 1] = o(h).$

So, by the above

$$n(t+h) = n(t) + P_{00}(t)h + o(h).$$

(b) From (a) we see that

$$\frac{n(t+h) - n(t)}{h} = P_{00}(t) + o(h)/h.$$

Let $h - 0$ to obtain

$$n'(t) = P_{00}(t)$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}.$$

Integrating gives

$$n(t) = \frac{\mu t}{\lambda+\mu} - \frac{\lambda}{(\lambda+\mu)^2}e^{-(\lambda+\mu)t} + C$$

Since $m(0) = 0$ it follows that $C = \lambda/(\lambda+\mu)^2$.

39. $E[0(t)|x(0) = 1] = t - E[\text{time in } 1 | X(0) = 1]$

$$= t - \frac{\lambda t}{\lambda+\mu} - \frac{\mu}{(\lambda+\mu)^2}[1 - e^{-(\lambda+\mu)t}].$$

The final equality is obtained from Example 7b (or Problem 38) by interchanging λ and μ.

40. $Cov[X(s), X(t)] = E[X(s)X(t)] - E[X(s)]EX(t)]$

Now,

$$X(s)X(t) = \begin{cases} 1 & \text{if } X(s) = X(t) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, for $s \le t$

$E[X(s)X(t)]$

$= P\{X(s) = X(t) = 1 | X(0) = 0\}$

$= P_{00}(s)P_{00}(t-s)$ by the Markovian property

$$= \frac{1}{(\lambda+\mu)^2}[\mu + \lambda e^{-(\lambda+\mu)s}][\mu + \lambda e^{-(\lambda+\mu)(t-s)}].$$

Also,

$E[X(s)]E[X(t)]$

$$= \frac{1}{(\lambda+\mu)^2}[\mu + \lambda e^{-(\lambda+\mu)s}][\mu + \lambda e^{-(\lambda+\mu)t}].$$

Hence,

$Cov[X(s), X(t)]$

$$= \frac{1}{(\lambda+\mu)^2}[\mu + \lambda e^{-(\lambda+\mu)s}]\lambda e^{-(\lambda+\mu)t}[e^{(\lambda+\mu)s} - 1].$$

41. (a) Letting T_i denote the time until a transition out of i occurs, we have

$$P_{ij} = P\{X(Y) = j\} = P\{X(Y) = j \mid T_i < Y\}$$

$$\times \frac{v_i}{v_i+\lambda} + P\{X(Y) = j | Y \le T_i\}\frac{\lambda}{\lambda+v_i}$$

$$= \sum_k P_{ik}P_{kj}\frac{v_i}{v_i+\lambda} + \frac{\delta_{ij}\lambda}{\lambda+v_i}.$$

The first term on the right follows upon conditioning on the state visited from i (which is k with probability P_{ik}) and then using the lack of memory property of the exponential to assert that given a transition into k occurs before time Y then the state at Y is probabilistically the same as if the process had started in state k and we were interested in the state after an exponential time with rate λ. As $q_{ik} = v_i P_{ik}$, the result follows.

(b) From (a)

$$(\lambda + v_i)\bar{P}_{ij} = \sum_k q_{ik}\bar{P}_{kj} + \lambda\delta_{ij}$$

or

$$-\lambda\delta_{ij} = \sum_k r_{ik}\bar{P}_{kj} - \lambda\bar{P}_{ij}$$

or, in matrix terminology,

$$\begin{aligned}-\lambda I &= R\bar{P} - \lambda I\bar{P}\\ &= (R - \lambda I)\bar{P},\end{aligned}$$

implying that

$$\begin{aligned}\bar{P} &= -\lambda I(R - \lambda I)^{-1} = -(R/\lambda - I)^{-1}\\ &= (I - R/\lambda)^{-1}.\end{aligned}$$

(c) Consider, for instance,

$$\begin{aligned}&P\{X(Y_1 + Y_2) = j|X(0) = i\}\\ &= \sum_k P\{X(Y_1 + Y_2) = j|X(Y_1) = k, X(0) = i)\\ &\quad P\{X(Y_1) = k|X(0) = i\}\\ &= \sum_k P\{X(Y_1 + Y_2) = j|X(Y_1) = k\}\bar{P}_{ik}\\ &= \sum_k P\{X(Y_2) = j|X(0) = k\}\bar{P}_{ik}\\ &= \sum_k \bar{P}_{kj}\bar{P}_{ik},\end{aligned}$$

and thus the state at time $Y_1 + Y_2$ is just the 2-stage transition probabilities of $\bar{P}_{ij}$. The general case can be established by induction.

(d) The above results in exactly the same approximation as Approximation 2 in Section 8.

42. (a) The matrix P^* can be written as

$$P^* = I + R/v,$$

and so P_{ij}^{*n} can be obtained by taking the i, j element of $(I + R/v)^n$, which gives the result when $v = n/t$.

(b) Uniformization shows that $P_{ij}(t) = E\left[P_{ij}^{*N}\right]$, where N is independent of the Markov chain with transition probabilities P_{ij}^* and is Poisson distributed with mean vt. Since a Poisson random variable with mean vt has standard deviation $(vt)^{1/2}$, it follows that for large values of vt it should be near vt. (For instance, a Poisson random variable with mean 10^6 has standard deviation 10^3 and thus will, with high probability, be within 3000 of 10^6.) Hence, since for fixed i and j, P_{ij}^{*m} should not vary much for values of m about vt when vt is large, it follows that, for large vt

$$E\left[P_{ij}^{*N}\right] \approx P_{ij}^{*n}, \quad \text{where } n = vt.$$

Chapter 7

1. (a) Yes, (b) no, (c) no.

2. (a) S_n is Poisson with mean $n\mu$.

   (b) $P\{N(t) = n\}$

   $$= P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$
   $$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$
   $$= \sum_{k=0}^{[t]} e^{-n\mu}(n\mu)^k/k!$$
   $$- \sum_{k=0}^{[t]} e^{-(n+1)\mu}[(n+1)\mu]^k/k!,$$

   where $[t]$ is the largest integer not exceeding t.

3. By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence,

   $$P\{N(5) = 0\} = e^{-5/2}.$$

4. (a) No! Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4.

   (b) No! Use the same processes as in (a) for a counter example. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$, where λ is the rate of the Poisson process. The probability will be different for the next interarrival.

   (c) No, because of (a) or (b).

5. The random variable N is equal to $N(I) + 1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on (0, 1). By the results of Example 2c,

   $$E[N] = a(1) + 1 = e.$$

6. (a) Consider a Poisson process having rate λ and say that an event of the renewal process occurs when ever one of the events numbered r, $2r$, $3r$, ... of the Poisson process occur. Then

   $P\{N(t) \geq n\}$

   $$= P\{nr \text{ or more Poisson events by } t\}$$
   $$= \sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i/i!.$$

   (b) $E[N(t)]$

   $$= \sum_{n=1}^{\infty} P\{N(t) \geq n\} = \sum_{n=1}^{\infty}\sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i/i!$$
   $$= \sum_{i=r}^{\infty}\sum_{n=1}^{[i/r]} e^{-\lambda t}(\lambda t)^i/i! = \sum_{i=r}^{\infty} [i/r]e^{-\lambda t}(\lambda t)^i/i!.$$

7. Once every five months.

8. (a) The number of replaced machines by time t constitutes a renewal process. The time between replacements equals

   T, if lifetime of new machine is $\geq T$

   x, if lifetime of new machine is x, $x < T$.

   Hence,

   $E[\text{time between replacements}]$

   $$= \int_0^T xf(x)dx + T[1 - F(T)],$$

   and the result follows by Proposition 3.1.

   (b) The number of machines that have failed in use by time t constitutes a renewal process. The mean time between in-use failures, $E[F]$, can be calculated by conditioning on the lifetime of the initial machine as

   $$E[F] = E[E[F|\text{lifetime of initial machine}]].$$

   Now

   $E[F|\text{lifetime of machine is } x]$

   $$= \begin{cases} x, & \text{if } x \leq T \\ T + E[F], & \text{if } x > T. \end{cases}$$

Hence,

$$E[F] = \int_0^T xf(x)dx + (T + E[F])[1 - F(T)]$$

or

$$E[F] = \frac{\int_0^T xf(x)dx + T[1 - F(T)]}{F(T)},$$

and the result follows from Proposition 3.1.

9. A job completion constitutes a reneval. Let T denote the time between renewals. To compute $E[T]$ start by conditioning on W, the time it takes to finish the next job.

$$E[T] = E[E[T|W]].$$

Now, to determine $E[T|W = w]$ condition on S, the time of the next shock. This gives

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x]\lambda e^{-\lambda x}dx$$

Now, if the time to finish is less than the time of the shock then the job is completed at the finish time; otherwise everything starts over when the shock occurs. This gives

$$E[T|W = w, S = x] = \begin{cases} x + E[T] & if x < w \\ w & if x \geq w \end{cases}.$$

Hence,

$$E[T|W = w]$$

$$= \int_0^w (x + E[T])\lambda e^{-\lambda x}dx + w\int_w^\infty \lambda e^{-\lambda x}dx$$

$$= E[T][1 - e^{-\lambda w}] + 1/\lambda - we^{-\lambda w} - \frac{1}{\lambda}e^{-\lambda w} - we^{-\lambda w}$$

Thus,

$$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W}).$$

Taking expectations gives

$$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}]),$$

and so

$$E[T] = \frac{1 - E[e^{-\lambda W}]}{\lambda E[e^{-\lambda W}]}.$$

In the above, W is a random variable having distribution F and so

$$E[e^{-\lambda W}] = \int_0^\infty e^{-\lambda w}f(w)dw.$$

10. Yes, ρ/μ

11. $$\frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t)}{t}$$

Since $X_1 < \infty$, Proposition 3.1 implies that

$$\frac{\text{number of renewals in } (X_1, t)}{t} - \frac{1}{\mu} \text{ as } t - \infty.$$

12. Let X be the time between successive d-events. Conditioning on T, the time until the next event following a d-event, gives

$$E[X] = \int_0^d x\lambda e^{-\lambda x}dx + \int_d^\infty (x + E[X]\lambda e^{-\lambda x}dx$$
$$= 1/\lambda + E[X]e^{-\lambda d}$$

Therefore, $E[X] = \dfrac{1}{\lambda(1 - e^{-\lambda d})}$

(a) $\dfrac{1}{E[X]} = \lambda(1 - e^{-\lambda d})$

(b) $1 - e^{-\lambda d}$

13. (a) N_1 and N_2 are stopping times. N_3 is not.

(b) Follows immediately from the definition of I_i.

(c) The value of I_i is completely determined from $X_1, \ldots, X_{i-1}$ (e.g., $I_i = 0$ or 1 depending upon whether or not we have stopped after observing $X_1, \ldots, X_{i-1}$). Hence, I_i is independent of X_i.

(d) $\sum_{i=1}^{\infty} E[I_i] = \sum_{i=1}^{\infty} P\{N \geq i\} = E[N]$.

(e) $E\left[X_1 + \cdots + X_{N_1}\right] = E[N_1]E[X]$

But $X_1 + \cdots + X_{N_1} = 5$, $E[X] = p$ and so

$E[N_1] = 5/p$.

$E\left[X_1 + \cdots + X_{N_2}\right] = E[N_2]E[X]$

$E[X] = p, E[N_2] = 5p + 3(1 - p) = 3 + 2p,$

$E\left[X_1 + \cdots + X_{N_2}\right] = (3 + 2p)p.$

14. (a) It follows from the hint that $N(t)$ is not a stopping time since $N(t) = n$ depends on X_{n+1}.

Now $N(t) + 1 = n(\Leftrightarrow)N(t) = n - 1$

$(\Leftrightarrow)X_1 + \cdots + X_{n-1} \leq t,$

$X_1 + \cdots + X_n > t,$

and so $N(t) + 1 = n$ depends only on $X_1, \ldots, X_n$. Thus $N(t) + 1$ is a stopping time.

(b) Follows upon application of Wald's equation—using $N(t)+1$ as the stopping time.

(c) $\sum_{i=1}^{N(t)+1} X_i$ is the time of the first renewal after t. The inequality follows directly from this interpretation since there must be at least one renewal in the interval between t and $t+m$.

(e) $t < \sum_{i=1}^{N(t)+1} X_i < t+M$.

Taking expectations and using (b) yields

$t < \mu(m(t)+1) < t+M$,

or

$t-\mu < \mu m(t) < t+M-\mu$,

or

$\frac{1}{\mu}-\frac{1}{t} < \frac{m(t)}{t} < \frac{1}{\mu}+\frac{M-\mu}{\mu t}$.

Let $t \to \infty$ to see that $\frac{m(t)}{t} - \frac{1}{\mu}$.

15. (a) X_i = amount of time he has to travel after his *ith* choice (we will assume that he keeps on making choices even after becoming free). N is the number of choices he makes until becoming free.

(b) $E[T] = E\left[\sum_1^N X_i\right] = E[N]E[X]$.

N is a geometric random variable with $P=1/3$, so

$E[N]=3, E[X]=\frac{1}{3}(2+4+6)=4$.

Hence, $E[T]=12$.

(c) $E\left[\sum_1^N X_i|N=n\right] = (n-1)\frac{1}{2}(4+6)+2 = 5n-3$, since given $N=n, X_1,\ldots,X_{n-1}$ are equally likely to be either 4 or 6, $X_n=2$, $E\left(\sum_1^n X_i\right)=4n$.

(d) From (c),

$E\left[\sum_1^N X_i\right] = E[5N-3] = 15-3=12$.

16. No, since $\sum_{1=i}^{N} X_i = 4$ and $E[X_i] = 1/13$ which would imply that $E[N]=52$, which is clearly incorrect. Wald's equation is not applicable since the X_i are not independent.

17. (i) Yes. (ii) No—Yes, if F exponential.

18. We can imagine that a renewal corresponds to a machine failure, and each time a new machine is put in use its life distribution will be exponential with rate μ_1 with probability p, and exponential with rate μ_2 otherwise. Hence, if our state is the index of the exponential life distribution of the machine presently in use, then this is a 2-state continuous time Markov chain with intensity rates

$q_{1,2}=\mu_1(1-p), q_{2,1}=\mu_2 p$.

Hence,

$$P_{11}(t) = \frac{\mu_1(1-p)}{\mu_1(1-p)+\mu_2 p}\exp\{-[\mu_1(1-p)+\mu_2 p]\,t\} + \frac{\mu_2 p}{\mu_1(1-p)+\mu_2 p}$$

with similar expressions for the other transition probabilities [$P_{12}(t)=1-P_{11}(t)$, and $P_{22}(t)$ is the same with $\mu_2 p$ and $\mu_1(1-p)$ switching places]. Conditioning on the initial machine now gives

$$E[Y(t)] = pE[Y(t)|X(0)=1]+(1-p)E[Y(t)|X(0)=2] = p\left[\frac{P_{11}(t)}{\mu_1}+\frac{P_{12}(t)}{\mu_2}\right]+(1-p)\left[\frac{P_{21}(t)}{\mu_1}+\frac{P_{22}(t)}{\mu_2}\right].$$

Finally, we can obtain $m(t)$ from

$\mu[m(t)+1] = t+E[Y(t)]$,

where

$\mu = p/\mu_1+(1-p)/\mu_2$

is the mean interarrival time.

19. Since, from Example 2c, $m(t)=e^t-1, 0<t\le 1$, we obtain upon using the identity $t+E[Y(t)] = \mu[m(t)+1]$ that $E[Y(1)]=e/2-1$.

20. $W_n = \frac{(R_1+\cdots+R_n)}{(X_1+\cdots+X_n)/n} - \frac{ER}{EX}$

by the Strong law of large numbers.

21. $\frac{\mu_G}{\mu+1/\lambda}$, where μ_G is the mean of G.

22. Cost of a cycle $= C_1 + C_2 I - R(T)(1-I)$.

$$I = \begin{cases} 1, & if\ X < T \\ 0, & if\ X \geq T \end{cases} \text{ where X } = \text{ life of car.}$$

Hence,

E[cost of a cycle]

$$= C_1 + C_2 H(T) - R(T)[1 - H(T)].$$

Also,

$$E[\text{time of cycle} = \int E[time|X = x]h(x)dx$$
$$= \int_0^t xh(x)dx + T[1 - H(T)].$$

Thus the average cost per unit time is given by

$$\frac{C_1 + C_2 H(T) - R(T)[1 - H(T)]}{\int_0^t xh(x)dx + T[1 - H(T)]}$$

23. From Problem 22 we want to minimize

$$\frac{4 + \frac{T-2}{6} - \left[4 - \frac{T}{2}\right]\left[\frac{8-T}{6}\right]}{\int_2^T x\frac{dx}{6} + T\left[\frac{8-T}{6}\right]}, 2 \leq T \leq 8$$

$$= \frac{18T - 20 - T^2}{16T - 4 - T^2}.$$

Using calculus the equation can be shown to be increasing in T for $2 \leq T \leq 8$, and so the optimal value is $T = Z$.

24. Let $N_1 = N$ denote the stopping time. Because X_i, $i \geq 1$, are independent and identically distributed, it follows by the definition of a stopping time that the event $\{N_1 = n\}$ is independent of the values X_{n+i}, $i \geq 1$. But this implies that the sequence of random variables X_{N_1+1}, $X_{N_1+2}, \ldots$ is independent of $X_1, \ldots, X_N$ and has the same distribution as the original sequence X_i, $i \geq 1$. Thus if we let N_2 be a stopping time on X_{N_1+1}, $X_{N_1+2}, \ldots$ that is defined exactly as is N_1 is on the original sequence, then $X_{N_1+1}, X_{N_1+2}, \ldots, X_{N_1+N_2}$ is independent of and has the same distribution as does $X_1, \ldots, X_{N_1}$. Similarly, we can define a stopping time N_3 on the sequence $X_{N_1+N_2+1}$, $X_{N_1+N_2+2}, \ldots$ that is identically defined on this sequence as is N_1 on the original sequence, and so on. If we now consider a reward process for which X_i is the reward earned during period i, then this reward process is a renewal reward process whose cycle lengths are $N_1, N_2, \ldots$. By the renewal reward theorem,

$$\text{average reward per unit time} = \frac{E[X_1 + \cdots + X_N]}{E[N]}$$

But the average reward per unit time is $\lim_{n\to\infty} \sum_{i=1}^{n} X_i/n$, which, by the strong law of large numbers, is equal to $E[X]$. Thus,

$$E[X] = \frac{E[X_1 + \ldots X_N]}{E[N]}$$

25. Say that a new cycle begins each time a train is dispatched. Then, with C being the cost of a cycle, we obtain, upon conditioning on $N(t)$, the number of arrivals during a cycle, that

$$E[C] = E[E[C|N(t)]] = E[K + N(t)ct/2]$$
$$= k + \lambda ct^2/2$$

Hence,

$$\text{average cost per unit time} = \frac{E[C]}{t} = \frac{K}{t} + \lambda ct/2$$

Calculus shows that the preceding is minimized when $t = \sqrt{2K/(\lambda c)}$, with the average cost equal to $\sqrt{2\lambda Kc}$.

On the other hand, the average cost for the N policy of Example 7.12 is $c(N-1)/2 + \lambda K/N$. Treating N as a continuous variable yields that its minimum occurs $N = \sqrt{2\lambda K/c}$, with a resulting minimal average cost of $\sqrt{2\lambda Kc} - c/2$.

26. $$\frac{[c + 2c + \cdots + (N-1)c]/\lambda + KNc + \lambda K^2 c/2}{N/\lambda + K}$$
$$= \frac{c(N-1)N/2\lambda + KNc + \lambda K^2 c/2}{N/\lambda + K}.$$

27. Say that a new cycle begins when a machine fails; let C be the cost per cycle; let T be the time of a cycle.

$$E[C] = K + \frac{c_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_1}$$

$$E[T] = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1}$$

T the long run average cost per unit time is $E[C]/E[T]$.

28. For N large, out of the first N items produced there will be roughly Nq defective items. Also, there will be roughly NP_I inspected items, and as each inspected item will still be, independently, defective with probability q, it follows that there will be roughly $NP_I q$ defective items discovered. Hence, the proportion of defective items that are discovered is, in the limit,

$$NP_1 q/Nq = P_I = \frac{(1/p)^k}{(1/p)^k - 1 + 1/\alpha}.$$

29. (a) Imagine that you are paid a reward equal to W_i on day i. Since everything starts over when a busy period ends, it follows that the reward process constitutes a renewal reward process with cycle time equal to N and with the reward during a cycle equal to $W_1 + \cdots + W_N$. Thus $E[W]$, the average reward per unit time is $E[W_1 \ + \cdots + \ W_N]/E[N]$.
   (b) The sum of the times in the system of all customers and the total amount of work that has been processed both start equal to 0 and both increase at the same rate. Hence, they are always equal.
   (c) This follows from (b) by looking at the value of the two totals at the end of the first busy period.
   (d) It is easy to see that N is a stopping time for the $L_i, i \geq 1$, and so, by Wald's Equation, $E[\sum_{i=1}^{N} L_i] = E[L]E[N]$. Thus, from (a) and (c), we obtain that $E[W] = E[L]$.

30. $$\frac{A(t)}{t} = \frac{t - S_{N(t)}}{t}$$
$$= 1 - \frac{S_{N(t)}}{t}$$
$$= 1 - \frac{S_{N(t)}}{N(t)} \frac{N(t)}{t}.$$

The result follows since $S_{N(t)}/N(t) — \mu$ (by the Strong law of large numbers) and $N(t)/t — 1/\mu$.

31. $$P\{E(t) > x|A(t) = s\}$$
$$= P\{0 \text{ renewals in } (t, t+x]|A(t) = s\}$$
$$= P\{\text{interarrival} > x + s|A(t) = s\}$$
$$= P\{\text{interarrival} > x + s|\text{interarrival} > s\}$$
$$= \frac{1 - F(x+s)}{1 - F(s)}.$$

32. Say that the system is off at t if the excess at t is less than c. Hence, the system is off the last c time units of a renewal interval. Hence,

Proportion of time excess is less than c

$$= E[\text{off time in a renewal cycle}]/[X]$$
$$= E[\min(X, c)]/E[X]$$
$$= \int_0^c (1 - F(x))dx/E[X].$$

33. Let B be the amount of time the server is busy in a cycle; let X be the remaining service time of the person in service at the beginning of a cycle.

$$E[B] = E[B|X < t](1 - e^{-\lambda t}) + E[B|X > t]e^{-\lambda t}$$
$$= E[X|X < t](1 - e^{-\lambda t}) + \left(t + \frac{1}{\lambda + \mu}\right) e^{-\lambda t}$$
$$= E[X] - E[X|X > t]e^{-\lambda t} + \left(t + \frac{1}{\lambda + \mu}\right) e^{-\lambda t}$$
$$= \tfrac{1}{\mu} - \left(t + \frac{1}{\mu}\right) e^{-\lambda t} + \left(t + \frac{1}{\lambda + \mu}\right) e^{-\lambda t}$$
$$= \frac{1}{\mu}\left[1 - \frac{\lambda}{\lambda + \mu} e^{-\lambda t}\right]$$

More intuitively, writing $X = B + (X - B)$, and noting that $X - B$ is the additional amount of service time remaining when the cycle ends, gives

$$E[B] = E[X] - E[X - B]$$
$$= \frac{1}{\mu} - \frac{1}{\mu} P(X > B)$$
$$= \frac{1}{\mu} - \frac{1}{\mu} e^{-\lambda t} \frac{\lambda}{\lambda + \mu}$$

The long run proportion of time that the server is busy is $\dfrac{E[B]}{t + 1/\lambda}$.

34. A cycle begins immediately after a cleaning starts. Let C be the cost of a cycle.

$$E[C] = \lambda C_2 T/4 + C_1 \lambda \int_0^{3T/4} \bar{G}(y)dy$$

where the preceding uses that the number of customers in an $M/G/\infty$ system at time t is Poisson distributed with mean $\lambda \int_0^t \bar{G}(y)dy$. The long run average cost is $E[C]/T$. The long run proportion of time of the system is being cleaned is $\frac{T/4}{T} = 1/4$.

35. (i) We can view this as an $M/G/\infty$ system where a satellite launching corresponds to an arrival and F is the service distribution. Hence,

$$P\{X(t) = k\} = e^{-\lambda(t)}[\lambda(t)]^k/k!,$$

where $\lambda(t) = \lambda \int_0^t (1 - F(s))ds$.

(ii) By viewing the system as an alternating renewal process that is on when there is at least one satellite orbiting, we obtain

$$\lim P\{X(t) = 0\} = \frac{1/\lambda}{1/\lambda + E[T]},$$

where T, the on time in a cycle, is the quantity of interest. From part (i)

$$\lim P\{X(t) = 0\} = e^{-\lambda\mu},$$

where $\mu = \int_0^\infty (1 - F(s))ds$ is the mean time that a satellite orbits. Hence,

$$e^{-\lambda\mu} = \frac{1/\lambda}{1/\lambda + E[T]},$$

and so

$$E[T] = \frac{1 - e^{-\lambda\mu}}{\lambda e^{-\lambda\mu}}.$$

36. (i) If we let $N_i(t)$ denote the number of times person i has skied down by time t, then $\{N_i(t)\}$ is a (delayed) renewal process. As $N(t) = \sum N_i(t)$, we have

$$\lim \frac{N(t)}{t} = \sum_i \lim \frac{N_i(t)}{t} = \sum_i \frac{1}{\mu_i + \theta_i},$$

where μ_i and θ_i are respectively the mean of the distributions F_i and G_i.

(ii) For each skier, whether they are climbing up or skiing down constitutes an alternating renewal process, and so the limiting probability that skier i is climbing up is $p_i = \mu_i/(\mu_i + \theta_i)$. From this we obtain

$$\lim P\{U(t) = k\} = \sum_S \{\prod_{i \in S} p_i \prod_{i \in S^c} (1 - p_i)\},$$

where the above sum is over all of the $\begin{bmatrix} n \\ k \end{bmatrix}$ subsets S of size k.

(iii) In this case the location of skier i, whether going up or down, is a 2-state continuous time Markov chain. Letting state 0 correspond to going up, then since each skier acts independently according to the same probability, we have

$$P\{U(t) = k\} = \begin{bmatrix} n \\ k \end{bmatrix} [P_{00}(t)]^k [1 - P_{00}(t)]^{n-k}$$

where $P_{00}(t) = (\lambda e^{-(\lambda+\mu)t} + \mu)/(\lambda + \mu)$.

37. (a) This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$\begin{aligned} E[\text{off}] &= \sum_{i=1}^{3} E[\text{off}|i \text{ fails}] P\{i \text{ fails}\} \\ &= (1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (2)\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \\ &\quad + (3/2)\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \end{aligned}$$

As the on time in a cycle is exponential with rate equal to $\lambda_1 + \lambda_2 + \lambda_3$, we obtain that p, the proportion of time that the system is working is

$$p = \frac{1/(\lambda_1 + \lambda_2 + \lambda_3)}{E[C]}$$

where

$$\begin{aligned} E[C] &= E[\text{cycle time}] \\ &= 1/(\lambda_1 + \lambda_2 + \lambda_3) + E[\text{off}] \end{aligned}$$

(b) Think of the system as a renewal reward process by supposing that we earn 1 per unit time that machine 1 is being repaired. Then, r_1, the proportion of time that machine 1 is being repaired is

$$r_1 = \frac{(1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}$$

(c) By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with s_2 being the proportion of time that 2 is in a state of suspended animation,

$$s_2 = \frac{(1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (3/2)\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}$$

38. Let $T_{e,f}$ denote the time it takes to go from e to f, and let d be the distance between A to B. Then, with S being the driver's speed

$$E[T_{A,B}] = \frac{1}{20}\int_{40}^{60} E[T_{A,B}|S=s]ds$$
$$= \frac{1}{20}\int_{40}^{60}\frac{d}{s}ds$$
$$= \frac{d}{20}\log(3/2)$$

Also,

$$E[T_{B,A}] = E[T_{B,A}|S=40](1/2) + E[T_{B,A}|S$$
$$= 60](1/2) = \frac{1}{2}(d/40 + d/60)$$
$$= d/48$$

(a) $$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]} = \frac{\frac{1}{20}\log(3/2)}{\frac{1}{20}\log(3/2) + 1/48}$$

(b) By assuming that a reward is earned at a rate of 1 per unit time whenever he is driving at a speed of 40 miles per hour, we see that p, the proportion of time this is the case, is

$$p = \frac{(1/2)d/40}{E\,[T_{A,B}] + E\,[T_{B,A}]} = \frac{\frac{1}{80}}{\frac{1}{20}\log(3/2) + 1/48}$$

39. Let B be the length of a busy period. With S equal to the service time of the machine whose failure initiated the busy period, and T equal to the remaining life of the other machine at that moment, we obtain

$$E[B] = \int E[B|S=s]g(s)ds$$

Now,

$$E[B|S=s] = E[B|S=s,\ T \le s](1-e^{-\lambda s}) + E[B|S$$
$$= s, T > s]e^{-\lambda s}$$
$$= (s + E[B])(1-e^{-\lambda s}) + se^{-\lambda s}$$
$$= s + E[B](1-e^{-\lambda s})$$

Substituting back gives that

$$E[B] = E[S] + E[B]E[1-e^{-\lambda s}]$$

or

$$E[B] = \frac{E[S]}{E[e^{-\lambda s}]}$$

Hence,

$$E[\text{idle}] = \frac{1/(2\lambda)}{1/(2\lambda) + E[B]}$$

40. Proportion of time 1 shoots = $\dfrac{1/(1-P_1)}{\sum_{j=1}^{3} 1/(1-P_j)}$ by alternating renewal process (or by semi-Markov process) since $1/(1-P_j)$ is the mean time marksman j shoots. Similarly, proportion of time i shoots $= \dfrac{1/(1-P_i)}{\sum 1/(1-P_j)}$.

41. $$\int_0^1 \frac{(1-F(x)dx}{\mu}$$
$$= \begin{cases} \int_0^1 \frac{2-x}{2}\,dx = \frac{3}{4} \text{ in part (i)} \\ \int_0^1 e^{-x} \quad dx = 1 - e^{-1} \text{ in part (ii).} \end{cases}$$

42. (a) $F_e(x) = \dfrac{1}{\mu}\displaystyle\int_0^x e^{-y/\mu}dy = 1 - e^{-x/\mu}$.

(b) $F_e(x) = \dfrac{1}{c}\displaystyle\int_0^x dy = x/c, \quad 0 \le x \le c.$

(c) You will receive a ticket if, starting when you park, an official appears within 1 hour. From Example 5.1c the time until the official appears has the distribution F_e, which, by part a, is the uniform distribution on (0, 2). Thus, the probability is equal to 1/2.

43. Since half the interarrival times will be exponential with mean 1 and half will be exponential with mean 2, it would seem that because the exponentials with mean 2 will last, on average, twice as long, that

$$\bar{F}_e(x) = \frac{2}{3}e^{-x/2} + \frac{1}{3}e^{-x}$$

With $\mu = (1)1/2 + (2)1/2 = 3/2$ equal to the mean interarrival time

$$\bar{F}_e(x) = \int_x^\infty \frac{\bar{F}(y)}{\mu}dy$$

and the earlier formula is seen to be valid.

44. Let T be the time it takes the shuttle to return. Now, given T, X is Poisson with mean λT. Thus,

$$E[X|T] = \lambda T, \quad Var(X|T) = \lambda T$$

Consequently,

(a) $E[X] = E[E[X|T]] = \lambda E[T]$

(b) $Var(X) = E[Var(X|T)] + Var(E[X|T])$

$$= \lambda E[T] + \lambda^2 Var(T)$$

(c) Assume that a reward of 1 is earned each time the shuttle returns empty. Then, from renewal reward theory, r, the rate at which the shuttle returns empty, is

$$\begin{aligned} r &= \frac{P\{\text{empty}\}}{E[T]} \\ &= \frac{\int P\{\text{empty}|T = t\} f(t)dt}{E[T]} \\ &= \frac{\int e^{-\lambda t} f(t)dt}{E[T]} \\ &= \frac{E[e^{-\lambda T}]}{E[T]} \end{aligned}$$

(d) Assume that a reward of 1 is earned each time that a customer writes an angry letter. Then, with N_a equal to the number of angry letters written in a cycle, it follows that r_a, the rate at which angry letters are written, is

$$\begin{aligned} r_a &= E[N_a]/E[T] \\ &= \int E[N_a|T = t] f(t)dt/E[T] \\ &= \int_c^\infty \lambda(t - c) f(t)dt/E[T] \\ &= \lambda E[(T - c)^+]/E[T] \end{aligned}$$

Since passengers arrive at rate λ, this implies that the proportion of passengers that write angry letters is r_a/λ.

(e) Because passengers arrive at a constant rate, the proportion of them that have to wait more than c will equal the proportion of time that the age of the renewal process (whose event times are the return times of the shuttle) is greater than c. It is thus equal to $\bar{F}_e(c)$.

45. The limiting probabilities for the Markov chain are given as the solution of

$$r_1 = r_2 \frac{1}{2} + r_3$$

$$r_2 = r_1$$

$$r_1 + r_2 + r_3 = 1.$$

or

$$r_1 = r_2 = \frac{2}{5}, \quad r_3 = \frac{1}{5}.$$

(a) $r_1 = \dfrac{2}{5}$.

(b) $P_i = \dfrac{r_i \mu_i}{\sum_i r_i \mu_i}$ and so,

$$P_1 = \frac{2}{9}, \ P_2 = \frac{4}{9}, \ P_3 = \frac{3}{9}.$$

46. Continuous-time Markov chain.

47. (a) By conditioning on the next state, we obtain the following:

$$\begin{aligned} \mu_j &= E[\text{time in } i] \\ &= \sum E[\text{time in } i|\text{next state is } j] P_{ij} \\ &= \sum_i t_{ij} P_{ij}. \end{aligned}$$

(b) Use the hint. Then,

$$\begin{aligned} &E[\text{reward per cycle}] \\ &\quad = E[\text{reward per cycle}|\text{next state is } j] P_{ij} \\ &\quad = t_{ij} P_{ij}. \end{aligned}$$

Also,

$E[\text{time of cycle} = E[\text{time between visits to } i]$.

Now, if we had supposed a reward of 1 per unit time whenever the process was in state i and 0 otherwise then using the same cycle times as above we have that

$$P_i = \frac{E[\text{reward is cycle}]}{E[\text{time of cycle}]} = \frac{\mu_i}{E[\text{time of cycle}]}.$$

Hence,

$$E[\text{time of cycle}] = \mu_i/P_i,$$

and so

$$\text{average reward per unit time} = t_{ij} P_{ij} P_i/\mu_i.$$

The above establishes the result since the average reward per unit time is equal to the proportion of time the process is in i and will next enter j.

48. Let the state be the present location if the taxi is waiting or let it be the most recent location if it is on the road. The limiting probabilities of the embedded Markov chain satisfy

$$r_1 = \frac{2}{3} r_3$$

$$r_2 = r_1 + \frac{1}{3} r_3$$

$$r_1 + r_2 + r_3 = 1.$$

Solving, yields

$$r_1 = \frac{1}{4}, \quad r_2 = r_3 = \frac{3}{8}.$$

The mean time spent in state i before entering another state is

$$\mu_1 = 1 + 10 = 11, \quad \mu_2 = 2 + 20 = 22,$$

$$\mu_3 = 4 + \left[\frac{2}{3}\right]15 + \left[\frac{1}{3}\right]25 = \frac{67}{3},$$

and so the limiting probabilities are

$$P_1 = \frac{66}{465}, P_2 = \frac{198}{465}, P_3 = \frac{201}{465}.$$

The time the state is i is broken into 2 parts—the time t_i waiting at i, and the time traveling. Hence, the proportion of time the taxi is waiting at state i is $P_i t_i/(t_i/\mu_i)$. The proportion of time it is traveling from i to j is $P_i m_{ij}/(t_i + \mu_i)$.

49. Think of each interarrival time as consisting of n independent phases—each of which is exponentially distributed with rate λ—and consider the semi–Markov process whose state at any time is the phase of the present interarrival time. Hence, this semi-Markov process goes from state 1 to 2 to 3 ... to n to 1, and so on. Also the time spent in each state has the same distribution. Thus, clearly the limiting probabilities of this semi-Markov chain is $P_i = 1/n, i = 1, \ldots, n$. To compute $\lim P\{Y(t) < x\}$, we condition on the phase at time t and note that if it is $n - i + 1$, which will be the case with probability $1/n$, then the time until a renewal occurs will be the sum of i exponential phases, which will thus have a gamma distribution with parameters i and λ.

50. (a) $\sum_{j=1}^{N_i(m)} X_i^j$.

(b) $\dfrac{\sum_{j=1}^{N_i(m)} X_i^j}{\sum_i \sum_{j=1}^{N_i(m)} X_i^j}$.

(c) Follows from the Strong law of large numbers since the X_i^j are independent and identically distributed and have mean μ_i.

(d) This is most easily proven by first considering the model under the assumption that each transition takes one unit of time. Then $N_i(m)/m$ is the rate at which visits to i occur, which, as such visits can be thought of as being renewals, converges to

$$(E[\text{number of transitions between visits}])^{-1}$$

by Proposition 3.1. But, by Markov-chain theory, this must equal x_i. As the quantity in (d) is clearly unaffected by the actual times between transition, the result follows.

Equation (6.2) now follows by dividing numerator and denominator of (b) by m; by writing

$$\frac{X_i^j}{m} = \frac{X_i^j}{N_i(m)} \frac{N_i(m)}{(m)}$$

and by using (c) and (d).

51. It is an example of the inspection paradox. Because every tourist spends the same time in departing the country, those questioned at departure constitute a random sample of all visiting tourists. On the other hand, if the questioning is of randomly chosen hotel guests then, because longer staying guests are more likely to be selected, it follows that the average time of the ones selected will be larger than the average of all tourists. The data that the average of those selected from hotels was approximately twice as large as from those selected at departure are consistent with the possibility that the time spent in the country by a tourist is exponential with a mean approximately equal to 9.

52. (a) $P\{X_1 + \cdots + X_n < Y\}$

$$= P\{X_1 + \cdots + X_n < Y | X_n < Y\} P\{X_n < Y\}$$

$$= P\{X_1 + \cdots + X_{n-1} < Y\} P\{X < Y\}$$

where the above follows because given that $Y > X_n$ the amount by which it is greater is, by the lack of memory property, also exponential with rate λ. Repeating this argument yields the result.

(b)
$$E[N(Y)] = \sum_{n=1}^{\infty} P\{N(Y) \geq n\}$$
$$= \sum_{n=1}^{\infty} P\{X_1 + \cdots + X_n \leq Y\}$$
$$= \sum_{n=1}^{\infty} P\{X < Y\}^n = \frac{P}{1 - P},$$

where

$$P = P\{X < Y\} = \int P\{X < Y | X = x\} f(x) dx$$
$$= \int e^{-\lambda x} f(x) dx = E[e^{-\lambda x}].$$

54. Let T denote the number of variables that need be observed until the pattern first appears. Also, let T^{∞} denote the number that need be observed once the pattern appears until it next appears. Let $p = p_1^2 p_2^2 p_3$

$$p^{-1} = E[T^{\infty}] \\ = E[T] - E[T_{1,2}] \\ = E[T] - (p_1 p_2)^{-1}$$

Hence, $E[T] = 8383.333$. Now, since $E[I(5)I(8)] = (.1)^3(.2)^3(.3)^2$, we obtain from Equation (7.45) that

$$Var(T^{\infty}) = (1/p)^2 - 9/p + 2(1/p)^3(.1)^3(.2)^3(.3)^2 \\ = 6.961943 \times 10^7$$

Also,

$$Var(T_{1,2}) = (.02)^{-2} - 3(.02)^{-1} = 2350$$

and so

$$Var(T) = Var(T_{1,2}) + Var(T^{\infty}) \approx 6.96 \times 10^7$$

55. $E[T(1)] = (.24)^{-2} + (.4)^{-1} = 19.8611,$

$E[T(2)] = 24.375$, $E[T_{12}] = 21.875$,

$E[T_{2,1}] = 17.3611$. The solution of the equations

$$19.861 = E[M] + 17.361P(2)$$

$$24.375 = E[M] + 21.875P(1)$$

$$1 = P(1) + P(2)$$

give the results

$P(2) \approx .4425$. $E[M] \approx 12.18$

56. (a) $\dfrac{(10)^{10}}{10!} \sum_{i=0}^{9} i!/(10)^i$

(b) Define a renewal process by saying that a renewal occurs the first time that a run of 5 consecutive distinct values occur. Also, let a reward of 1 be earned whenever the previous 5 data values are distinct. Then, letting R denote the reward earned between renewal epochs, we have that

$$E[R] = 1 + \sum_{i=1}^{4} E\big[\text{reward earned a time } i \text{ after a renewal}\big]$$

$$= 1 + \sum_{i=1}^{4} \binom{5+i}{i} \Big/ \binom{10}{i} \\ = 1 + 6/10 + 7/15 + 7/15 + 6/10 \\ = 47/15$$

If R_i is the reward earned at time i then for $i \geq 5$

$$E[R_i] = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6/(10)^{10} = 189/625$$

Hence,

$$E[T] = (47/15)(625/189) \approx 10.362$$

57. $$P\{\sum_{i=1}^{T} X_i > x\} = P\{\sum_{i=1}^{T} X_i > x | T = 0\}(1-\rho) \\ + P\{\sum_{i=1}^{T} X_i > x | T > 0\}\rho$$

$$= P\{\sum_{i=1}^{T} X_i > x | T > 0\}\rho$$

$$= \rho \int_0^{\infty} P\{\sum_{i=1}^{T} X_i > x | T > 0, X_1 = y\} \frac{\bar{F}(y)}{\mu} dy$$

$$= \frac{\rho}{\mu} \int_0^{x} P\{\sum_{i=1}^{T} X_i > x | T > 0, X_1 = y\} \bar{F}(y) dy \\ + \frac{\rho}{\mu} \int_x^{\infty} \bar{F}(y) dy$$

$$= \frac{\rho}{\mu} \int_0^{x} h(x-y) \bar{F}(y) dy + \frac{\rho}{\mu} \int_x^{\infty} \bar{F}(y) dy$$

$$= h(0) + \frac{\rho}{\mu} \int_0^{x} h(x-y) \bar{F}(y) dy - \frac{\rho}{\mu} \int_0^{x} \bar{F}(y) dy$$

where the final equality used that

$$h(0) = \rho = \frac{\rho}{\mu} \int_0^{\infty} \bar{F}(y) dy$$

Chapter 8

1. (a) $E[\text{number of arrivals}]$

$$= E[E\{\text{number of arrivals}|\text{service period is } S\}]$$
$$= E[\lambda S]$$
$$= \lambda/\mu.$$

(b) $P\{0 \text{ arrivals}\}$

$$= E[P\{0 \text{ arrivals}|\text{service period is } S\}]$$
$$= E[P\{N(S) = 0\}]$$
$$= E[e^{-\lambda S}]$$
$$= \int_0^x e^{-\lambda s}\mu e^{-\mu s} ds$$
$$= \frac{\mu}{\lambda + \mu}.$$

2. This problem can be modeled by an $M/M/1$ queue in which $\lambda = 6$, $\mu = 8$. The average cost rate will be

$10 per hour per machine $\times$ average number of broken machines.

The average number of broken machines is just L, which can be computed from Equation (3.2):

$$L = \lambda/(\mu - \lambda)$$
$$= \frac{6}{2} = 3.$$

Hence, the average cost rate = $30/hour.

3. Let $C_M =$ Mary's average cost/hour and $C_A =$ Alice's average cost/hour.

Then, $C_M = \$3 + \$1\times$ (Average number of customers in queue when Mary works),

and $C_A = \$C + \$1\times$ (Average number of customers in queue when Alice works).

The arrival stream has parameter $\lambda = 10$, and there are two service parameters—one for Mary and one for Alice:

$$\mu_M = 20 \quad \mu_A = 30.$$

Set $L_M =$ average number of customers in queue when Mary works and
$L_A =$ average number of customers in queue when Alice works.

Then using Equation (3.2), $L_M = \dfrac{10}{(20-10)} = 1$

$$L_A = \frac{10}{(20-10)} = \frac{1}{2}.$$

So $C_M = \$3 + \$1/\text{customer} \times L_M \text{ customers}$
$= \$3 + \$1$
$= \$4/\text{hour}.$

Also, $C_A = \$C + \$1/\text{customer} \times L_A \text{ customers}$
$= \$C + \$1 \times \frac{1}{2}$
$= \$C + \frac{1}{2}$ / hour.

(b) We can restate the problem this way: If $C_A = C_M$, solve for C.

$$4 = C + \frac{1}{2} \Rightarrow C = \$3.50/\text{hour},$$

i.e., $3.50/hour is the most the employer should be willing to pay Alice to work. At a higher wage his average cost is lower with Mary working.

4. Let N be the number of other customers that were in the system when the customer arrived, and let $C = 1/f_{W_Q^*}(x)$. Then

$$f_{N|W_Q^*}(n|x) = C f_{W_Q^*|N}(x|n)P\{N = n\}$$
$$= C\mu e^{-\mu x}\frac{(\mu x)^{n-1}}{(n-1)!}(\lambda/\mu)^n(1 - \lambda/\mu)$$
$$= K\frac{(\lambda x)^{n-1}}{(n-1)!}$$

where

$$K = \frac{1}{f_{W_Q^*}(x)} \mu e^{-\mu x} (\lambda/\mu)(1-\lambda/\mu)$$

Using

$$1 = \sum_{n=1}^{\infty} f_{N|W_Q^*}(n|x) = K \sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!} = K e^{\lambda x}$$

shows that

$$f_{N|W_Q^*}(n|x) = e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, \qquad n > 0$$

Thus, $N-1$ is Poisson with mean λx.

The preceding also yields that for $x > 0$

$$f_{W_Q^*}(x) = e^{\lambda x} \mu e^{-\mu x} (\lambda/\mu)(1-\lambda/\mu)$$
$$= \frac{\lambda}{\mu}(\mu - \lambda) e^{-(\mu-\lambda)x}$$

Hence, for $x > 0$

$$P\{W_Q^* \le x\} = P\{W_Q^* = 0\} + \int_0^x f_{W_Q^*}(y)dy$$
$$= 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}(1 - e^{-(\mu-\lambda)x})$$

5. Let I equal 0 if $W_Q^* = 0$ and let it equal 1 otherwise. Then,

$$E[W_Q^*|I=0] = 0$$
$$E[W_Q^*|I=1] = (\mu-\lambda)^{-1}$$
$$Var(W_Q^*|I=0) = 0$$
$$Var(W_Q^*|I=1) = (\mu-\lambda)^{-2}$$

Hence,

$$E[Var(W_Q^*|I] = (\mu-\lambda)^{-2}\lambda/\mu$$
$$Var(E[W_Q^*|I]) = (\mu-\lambda)^{-2}\lambda/\mu(1-\lambda/\mu)$$

Consequently, by the conditional variance formula,

$$Var(W_Q^*) = \frac{\lambda}{\mu(\mu-\lambda)^2} + \frac{\lambda}{\mu^2(\mu-\lambda)}$$

6. Let the state be the idle server. The balance equations are

Rate Leave = Rate Enter,

$$(\mu_2+\mu_3)P_1 = \frac{\mu_1}{\mu_1+\mu_2}P_3 + \frac{\mu_1}{\mu_1+\mu_3}P_2,$$
$$(\mu_1+\mu_3)P_2 = \frac{\mu_2}{\mu_2+\mu_3}P_1 + \frac{\mu_2}{\mu_2+\mu_1}P_3,$$
$$\mu_1+\mu_2+\mu_3 = 1.$$

These are to be solved and the quantity P_i represents the proportion of time that server i is idle.

7. To compute W for the $M/M/2$, set up balance equations as

$$\lambda p_0 = \mu p_1 \qquad \text{(each server has rate } \mu)$$
$$(\lambda+\mu)p_1 = \lambda p_0 + 2\mu p_2$$
$$(\lambda+2\mu)p_n = \lambda p_{n-1} + 2\mu p_{n+1} \qquad n \ge 2.$$

These have solutions $P_n = \rho^n/2^{n-1}p_0$ where $\rho = \lambda/\mu$.

The boundary condition $\sum_{n=0}^{\infty} P_n = 1$ implies

$$P_0 = \frac{1-\rho/2}{1+\rho/2} = \frac{(2-\rho)}{(2+\rho)}.$$

Now we have P_n, so we can compute L, and hence W from $L = \lambda W$:

$$L = \sum_{n=0}^{\infty} np_n = \rho p_0 \sum_{n=0}^{\infty} n \left[\frac{\rho}{2}\right]^{n-1}$$
$$= 2p_0 \sum_{n=0}^{\infty} n \left[\frac{\rho}{2}\right]^n$$
$$= 2\frac{(2-\rho)}{(2+\rho)} \frac{(\rho/2)}{(1-\rho/2)^2}$$
$$= \frac{4\rho}{(2+\rho)(2-\rho)}$$
$$= \frac{4\mu\lambda}{(2\mu+\lambda)(2\mu-\lambda)}.$$

From $L = \lambda W$ we have

$$W = W_{m/m/2} = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)}.$$

The $M/M/1$ queue with service rate 2μ has

$$Wm/m/1 = \frac{1}{2\mu-\lambda}$$

from Equation (3.3). We assume that in the $M/M/1$ queue, $2\mu > \lambda$ so that the queue is stable. But then $4\mu > 2\mu+\lambda$, or $\frac{4\mu}{2\mu+\lambda} > 1$, which implies $Wm/m/2 > Wm/m/1$.

The intuitive explanation is that if one finds the queue empty in the $M/M/2$ case, it would do no good to have two servers. One would be better off with one faster server.

Now let $W_Q^1 = W_Q(M/M/1)$

$$W_Q^2 = W_Q(M/M/2).$$

Then,

$W_Q^1 = Wm/m/1 - 1/2\mu$

$W_Q^2 = Wm/m/2 - 1/\mu.$

So,

$$W_Q^1 = \frac{\lambda}{2\mu(2\mu - \lambda)} \qquad (3.3)$$

and

$$W_Q^2 = \frac{\lambda^2}{\mu(2\mu - \lambda)(2\mu + \lambda)}.$$

Then,

$$W_Q^1 > W_Q^2 \Leftrightarrow \frac{1}{2} > \frac{\lambda}{(2\mu + \lambda)}$$
$$\lambda < 2\mu.$$

Since we assume $\lambda < 2\mu$ for stability in the $M/M/1$, $W_Q^2 < W_Q^1$ whenever this comparison is possible, i.e., whenever $\lambda < 2\mu$.

8. Take the state to be the number of customers at server 1. The balance equations are

$$\mu P_0 = \mu P_1$$
$$2\mu P_j = \mu P_{j+1} + \mu P_{j-1}, \quad 1 \le j < n$$
$$\mu P_n = \mu P_{n-1}$$
$$1 = \sum_{j=0}^{n} P_j$$

It is easy to check that the solution to these equations is that all the $P_j^{'}s$ are equal, so $P_j = 1/(n+1)$, $j = 0, \ldots, n$.

9. This model is mathematically equivalent to the $M/M/1$ queue with finite capacity k. The produced items constitute the arrivals to the queue, and the arriving customers constitute the services. That is, if we take the state of the system to be the number of items presently available then we just have the model of Section 8.3.2.

(a) The proportion of customers that go away empty-handed is equal to P_0, the proportion of time there are no items on the shelves. From Section 8.3.2,

$$P_0 = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{k+1}}.$$

(b) $W = \dfrac{L}{\lambda(1 - P_k)}$ where L is given by equation (8.12).

(c) The average number of items in stock is L.

(d) The state space is $j, -n \le j \le k$. The state is $j, j \ge 0$, when there are j items in stock and no waiting customers; it is $j, j < 0$, when there are no items in stock and no waiting customers. The balance equations are

$$\mu P_k = \lambda P_{k-1}$$
$$(\lambda + \mu)P_j = \lambda P_{j-1} + \mu P_{j+1,} - n < j < k$$
$$\lambda P_{-n} = \mu P_{-(n-1)}$$
$$1 = \sum_{j=-n}^{k} P_j$$

(e) $-\sum_{jm-n}^{-1} jP_j$

10. The state is the number of customers in the system, and the balance equations are

(a)
$$m\theta P_0 = \mu P_1$$
$$((m-j)\theta + \mu)P_j = (m - j + 1)\theta P_{j-1} + \mu P_{j+1}, \quad 0 < j < m$$
$$\mu P_m = \theta P_{m-1}$$
$$1 = \sum_{j=0}^{m} P_j$$

(b) $\lambda_\alpha = \sum_{j=0}^{m} (m-j)\theta P_j$

(c) $L/\lambda_\alpha = \sum_{j=0}^{m} jP_j / \sum_{j=0}^{m} (m-j)\theta P_j$

11. (a) $\lambda P_0 = \alpha\mu P_1$

$$(\lambda + \alpha\mu)P_n = \lambda P_{n-1} + \alpha\mu P_{n+1}, \quad n \ge 1.$$

These are exactly the same equations as in the $M/M/1$ with $\alpha\mu$ replacing μ. Hence,

$$P_n = \left[\frac{\lambda}{\alpha\mu}\right]^n \left[1 - \frac{\lambda}{\alpha\mu}\right], \quad n \ge 0,$$

and we need the condition $\lambda < \alpha\mu$.

(b) If T is the waiting time until the customer first enters service, then by conditioning on the number present when he arrives yields

$$E[T] = \sum_n E[T|n \text{ present}]P_n$$
$$= \sum_n \frac{n}{\mu} P_n$$
$$= \frac{L}{\mu}.$$

Since $L = \sum nP_n$, and the P_n are the same as in the $M/M/1$ with λ and $\alpha\mu$, we have that $L = \lambda/(\alpha\mu - \lambda)$ and so

$$E[T] = \frac{\lambda}{\mu(\alpha\mu - \lambda)}.$$

(c) $P\{\text{enters service exactly } n \text{ times}\}$

$$= (1-\alpha)^{n-1}\alpha.$$

(d) This is expected number of services $\times$ mean services time $= 1/\alpha\mu$.

(e) The distribution is easily seen to be memoryless.
Hence, it is exponential with rate $\alpha\mu$.

12. (a) $\lambda_n = \begin{cases} \lambda, & \text{if } n<N \\ 0, & \text{if } n \geq N \end{cases}$

$\mu_n = \mu, \quad 1 \leq n \leq N$

(b) $$\lambda_0 P_0 = \mu_1 P_1$$

$$(\lambda_n + \mu_n)P_n = \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}, \quad n > 0$$

(d) The rate at which an arrival finds n is equal to the rate at which a departure leaves behind n.

(e) $$P_n = \frac{\lambda_{n-1}}{\mu_n}P_{n-1}$$

$$= \frac{\lambda_{n-1}\lambda_{n-2}}{\mu_n \mu_{n-1}}P_{n-2}$$

$$= \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n}P_0$$

Using that $\sum_{n=0}^{\infty} P_n$ gives

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n}}$$

The necessary condition for a solution is that

$$\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n} < \infty$$

(f) $\lambda_a = \sum_{n=0}^{\infty} \lambda_n P_n$

(g) $W = \dfrac{L}{\lambda_a} = \dfrac{\sum_{n=0}^{\infty} nP_n}{\sum_{n=0}^{\infty} \lambda_n P_n}$

13. (a) $$\lambda p_0 = \mu p_1$$

$$(\lambda + \mu)p_1 = \lambda p_0 + 2\mu p_2$$

$$(\lambda + 2\mu)p_n = \lambda p_{n-1} + 2\mu p_{n+1} \quad n \geq 2.$$

These are the same balance equations as for the $M/M/2$ queue and have solution

$$p_0 = \left[\frac{2\mu - \lambda}{2\mu + \lambda}\right], \quad p_n = \frac{\lambda^n}{2^{n-1}\mu^n}p_0.$$

(b) The system goes from 0 to 1 at rate $\lambda p_0 = \dfrac{\lambda(2\mu - \lambda)}{(2\mu + \lambda)}$. The system goes from 2 to 1 at rate

$$2\mu p_2 = \frac{\lambda^2(2\mu - \lambda)}{\mu(2\mu + \lambda)}.$$

(c) Introduce a new state cl to indicate that the stock clerk is checking by himself. The balance equation for P_{cl} is

$$(\lambda + \mu)p_{cl} = \mu p_2.$$

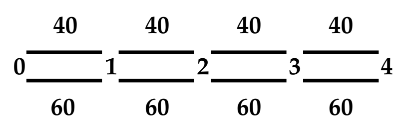
The reason for p_2 that it is only if the checker completes service first in p_2 that the system moves to state cl. Then

$$p_{cl} = \frac{\mu}{\lambda + \mu}p_2 = \frac{\lambda^2}{2\mu(\lambda + \mu)}\frac{(2\mu - \lambda)}{(2\mu + \lambda)}.$$

Finally, the proportion of time the stock clerk is checking is

$$p_{cl} + \sum_{n=2}^{\infty} p_n = p_{cl} + \frac{2\lambda^2}{\mu(2\mu + \lambda)}.$$

14. The system has given states whose transition diagram is



Hence, the balance equations are

$$\begin{aligned} 40p_0 &= 30p_1 \\ 70p_1 &= 40p_0 + 30p_2 \\ 70p_2 &= 40p_1 + 60p_3 \\ 100p_3 &= 40p_2 + 60p_4 \\ 60p_4 &= 40p_3. \end{aligned}$$

Solution of these gives $p_1 = 4/3p_0$, $p_2 = 16/9p_0$, $p_3 = 32/27p_0$, $p_4 = 64/81p_0$. The condition $\sum p_i = 1$ implies p_0 is 81/493, which is about 1/6.

Clearly, p_0 = Proportion of time both servers are free.

(b) The original attendant works a proportion $(1 - p_0)$ of the time, and the second attendant works a proportion $p_3 + p_4$ of the time. So if the first attendant receives Sz, the second should receive only $S(p_3 + p_4)(1 - p_0)z$. Computation shows this factor to be about 0.414, so $0.414x + x = 100$, or $x = 70.72$, and the second attendant receives \$29.28.

15. There are four states $= 0, 1_A, 1_B, 2$. Balance equations are

$$2P_0 = 2P_{1_B}$$
$$4P_{1_A} = 2P_0 + 2P_2$$
$$4P_{1_B} = 4P_{1_A} + 4P_2$$
$$6P_2 = 2P_{1_B}$$

$$P_0 + P_{1_A} + P_{1_B} + P_2 = 1 \Rightarrow P_0 = \frac{3}{9},$$
$$P_{1_A} = \frac{2}{9}, P_{1_B} = \frac{3}{9}, P_2 = \frac{1}{9}.$$

(a) $P_0 + P_{1_B} = \frac{2}{3}$.

(b) By conditioning upon whether the state was 0 or 1_B when he entered we get that the desired probability is given by

$$\frac{1}{2} + \frac{1}{2}\frac{2}{6} = \frac{4}{6}.$$

(c) $P_{1_A} + P_{1_B} + 2P_2 = \frac{7}{9}$.

(d) Again, condition on the state when he enters to obtain

$$\frac{1}{2}\left[\frac{1}{4} + \frac{1}{2}\right] + \frac{1}{2}\left[\frac{1}{4} + \frac{2}{6}\frac{1}{2}\right] = \frac{7}{12}.$$

This could also have been obtained from (a) and (c) by the formula $W = \frac{L}{\lambda a}$.

That is, $W = \dfrac{\frac{7}{9}}{2\left[\frac{2}{3}\right]} = \frac{7}{12}$.

16. Let the states be $(0, 0), (1, 0), (0, 1)$, and $(1, 1)$, where state (i, j) means that there is i customers with server 1 and j with server 2. The balance equations are as follows.

$$\lambda P_{00} = \mu_1 P_{10} + \mu_2 P_{01}$$
$$(\lambda + \mu_1)P_{10} = \lambda P_{00} + \mu_2 P_{11}$$
$$(\lambda + \mu_2)P_{01} = \mu_1 P_{11}$$
$$(\mu_1 + \mu_2)P_{11} = \lambda P_{01} + \lambda P_{10}$$

$$P_{00} + P_{01} + P_{10} + P_{11} = 1$$

Substituting the values $\lambda = 5, \mu_1 = 4, \mu_2 = 2$ and solving yields the solution

$P_{00} = 128/513$, $P_{10} = 110/513$, $P_{01} = 100/513$, $P_{11} = 175/513$

(a) $W = L/\lambda_a = [1(P_{01} + P_{10}) + 2P_{11}]/[\lambda(1 - P_{11})] = 56/119$.

Another way is to condition on the state as seen by the arrival. Letting T denote the time spent, this gives

$$W = E[T|00]128/338 + E[T|01]100/338$$
$$+ E[T|10]110/338$$
$$= (1/4)(228/338) + (1/2)(110/338)$$
$$= 56/119.$$

(b) $P_{01} + P_{11} = 275/513$.

17. The state space can be taken to consist of states $(0, 0), (0, 1), (1, 0), (1, 1)$, where the i^{th} component of the state refers to the number of customers at server i, $i = 1, 2$. The balance equations are

$$2P_{0,0} = 6P_{0,1}$$
$$8P_{0,1} = 4P_{1,0} + 4P_{1,1}$$
$$6P_{1,0} = 2P_{0,0} + 6P_{1,1}$$
$$10P_{1,1} = 2P_{0,1} + 2P_{1,0}$$
$$1 = P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1}$$

Solving these equations gives $P_{0,0} = 1/2$, $P_{0,1} = 1/6, P_{1,0} = 1/4, P_{1,1} = 1/12$.

(a) $P_{1,1} = 1/12$.

(b) $W = \dfrac{L}{\lambda_a} = \dfrac{P_{0,1} + P_{1,0} + 2P_{1,1}}{2(1 - P_{1,1})} = \dfrac{7}{22}$

(c) $\dfrac{P_{0,0} + P_{0,1}}{1 - P_{1,1}} = \dfrac{8}{11}$

18. (a) The states are 0, 1, 2, 3 where the state is i when there are i in the system.

(b) The balance equations are

$$\lambda P_0 = \mu P_1$$
$$(\lambda + \mu)P_1 = \lambda P_0 + 2\mu P_2$$
$$(\lambda + 2\mu)P_2 = \lambda P_1 + 2\mu P_3$$
$$2\mu P_3 = \lambda P_2$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

The solution of these equations is

$$P_1 = (\lambda/\mu)P_0, P_2 = (\lambda^2/2\mu^2)P_0, P_3 = (\lambda^3/4\mu^3)P_0$$
$$P_0 = [1 + \lambda/\mu + \lambda^2/(2\mu^2) + \lambda^3/(4\mu^3)]^{-1}$$

(c) $E[\text{Time}] = E[\text{Time in queue}]$
$+E[\text{time in service}]$
$= 1/(2\mu) + 1/\mu.$

(d) $1 - P_3$.

(e) Conditioning on the state as seen by the arrival
$W = [(1/\mu)(P_0 + P_1) + (2/\mu)P_2]/(1 - P_3)$.
Could also use $W = L/\lambda_a$.

19. (a) Say that the state is $(n, 1)$ whenever it is a good period and there are n in the system, and say that it is $(n, 2)$ whenever it is a bad period and there are n in the system, $n = 0, 1$

(b) $(\lambda_1 + \alpha_1)P_{0,\,1} = \mu P_{1,\,1} + \alpha_2 P_{0,\,2}$

$(\lambda_2 + \alpha_2)P_{0,\,2} = \mu P_{1,\,2} + \alpha_1 P_{0,\,1}$

$(\mu + \alpha_1)P_{1,\,1} = \lambda_1 P_{0,\,1} + \alpha_2 P_{1,\,2}$

$(\mu + \alpha_2)P_{1,\,2} = \lambda_2 P_{0,\,2} + \alpha_1 P_{1,\,1}$

$P_{0,\,1} + P_{0,\,2} + P_{1,\,1} + P_{1,\,2} = 1$

(c) $P_{0,\,1} + P_{0,\,2}$

(d) $\lambda_1 P_{0,\,1} + \lambda_2 P_{0,\,2}$

20. (a) The states are $0, (1,\ 0), (0,\ 1)$ and $(1,\ 1)$, where 0 means that the system is empty, $(1,\ 0)$ that there is one customer with server 1 and none with server 2, and so on.

(b) $(\lambda_1 + \lambda_2)P_0 = \mu_1 P_{10} + \mu_2 P_{01}$

$(\lambda_1 + \lambda_2 + \mu_1)P_{10} = \lambda_1 P_0 + \mu_2 P_{11}$

$(\lambda_1 + \mu_2)P_{01} = \lambda_2 P_0 + \mu_1 P_{11}$

$(\mu_1 + \mu_2)P_{11} = \lambda_1 P_{01} + (\lambda_1 + \lambda_2)P_{10}$

$P_0 + P_{10} + P_{01} + P_{11} = 1$

(c) $L = P_{01} + P_{10} + 2P_{11}$

(d) $W = L/\lambda_a = L/[\lambda_1(1 - P_{11}) + \lambda_2(P_0 + P_{10})]$

21. (a) $\lambda_1 P_{10}$

(b) $\lambda_2(P_0 + P_{10})$

(c) $\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$

(d) This is equal to the fraction of server 2's customers that are type 1 multiplied by the proportion of time server 2 is busy. (This is true since the amount of time server 2 spends with a customer does not depend on which type of customer it is.) By (c) the answer is thus
$(P_{01} + P_{11})\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$.

22. The state is the pair $(i, j), i = 0, 1, 0 \le j \le n$ where i signifies the number of customers in service and j the number in orbit. The balance equations are

$(\lambda + j\theta)P_{0,j} = \mu P_{1,j}, \quad j = 0, \ldots, N$

$(\lambda + \mu)P_{1,j} = \lambda P_{0,j} + (j+1)\theta P_{0,j+1},$
$j = 0, \ldots, N-1$

$\mu P_{1,N} = \lambda P_{0,N}$

(c) $1 - P_{1,\,N}$

(d) The average number of customers in the system is

$$L = \sum_{i,\,j} (i + j)P_{i,\,j}$$

Hence, the average time that an entering customer spends in the system is $W = L/\lambda(1 - P_{1,N})$, and the average time that an entering customer spends in orbit is $W - 1/\mu$.

23. (a) The states are $n, n \ge 0$, and b. State n means there are n in the system and state b means that a breakdown is in progress.

(b) $\beta P_b = a(1 - P_0)$

$\lambda P_0 = \mu P_1 + \beta P_b$

$(\lambda + \mu + a)P_n = \lambda P_{n-1} + \mu P_{n+1}, \quad n \ge 1$

(c) $W = L/\lambda_n = \sum_{n=1}^{\infty} nP_a/[\lambda(1 - P_b)]$.

(d) Since rate at which services are completed $= \mu(1 - P_0 - P_b)$ it follows that the proportion of customers that complete service is
$\mu(1 - P_0 - P_b)/\lambda_a$
$= \mu(1 - P_0 - P_b)/[\lambda(1 - P_b)]$.
An equivalent answer is obtained by conditioning on the state as seen by an arrival. This gives the solution

$$\sum_{n=0}^{\infty} P_n[\mu/(\mu + a)]^{n+1},$$

where the above uses that the probability that $n + 1$ services of present customers occur before a breakdown is $[\mu/(\mu + a)]^{n+1}$.

(e) P_b.

24. The states are now $n, n \ge 0$, and $n', n \ge 1$ where the state is n when there are n in the system and no

breakdown, and it is n' when there are n in the system and a breakdown is in progress. The balance equations are

$$\lambda P_0 = \mu P_1$$

$$(\lambda+\mu+\alpha)P_n = \lambda P_{n-1} + \mu P_{n+1} + \beta P_{n'}, \quad n \geq 1$$

$$(\beta+\lambda)P_{1'} = \alpha P_1$$

$$(\beta+\lambda)P_{n'} = \alpha P_n + \lambda P_{(n-1)'}, \quad n \geq 2$$

$$\sum_{n=0}^{\infty} P_n + \sum_{n=1}^{\infty} P_{n'} = 1.$$

In terms of the solution to the above,

$$L = \sum_{n=1}^{\infty} n(P_n + P_{n'})$$

and so

$$W = L/\lambda_\alpha = L/\lambda.$$

25. (a)
$$\lambda P_0 = \mu_A P_A + \mu_B P_B$$
$$(\lambda+\mu_A)P_A = a\lambda P_0 + \mu_B P_2$$
$$(\lambda+\mu_B)P_B = (1-a)\lambda P_0 + \mu_A P_2$$
$$(\lambda+\mu_A+\mu_B)P_n = \lambda P_{n-1} + (\mu_A+\mu_B)P_{n+1'}$$
$$n \geq 2 \quad \text{where} \quad P_1 = P_A + P_B.$$

(b) $L = P_A + P_B + \sum_{n=2}^{\infty} nP_n$

Average number of idle servers $= 2P_0 + P_A + P_B$.

(c) $P_0 + P_B + \dfrac{\mu_A}{\mu_A+\mu_B} \sum_{n=2}^{\infty} P_n.$

26. States are $0, 1, 1', \ldots, k-1(k-1)', k, k+1, \ldots$ with the following interpretation

$0 =$ system is empty

$n = n$ in system and server is working

$n' = n$ in system and server is idle,

$n = 1, 2, \ldots, k-1$

(a)
$$\lambda P_0 = \mu P_1, (\lambda+\mu)P_1 = \mu P_2$$
$$\lambda P_n' = \lambda P_{(n-1)'}\ n = 1, \ldots, k-1$$
$$(\lambda+\mu)P_k = \lambda P_{(k-1)'} + \mu P_{k+1} + \lambda P_{k-1}$$
$$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1'}\ n > k$$

(b) $\dfrac{k-1}{\lambda}P_0 + \sum_{n=1}^{k-1}\left[\dfrac{k-1-n}{\lambda} + \dfrac{n}{\mu}\right]P_{n'} + \sum_{n-1}^{\infty} P_n \dfrac{n}{\mu}.$

(c) $\lambda < \mu$.

27. (a) The special customer's arrival rate is act θ because we must take into account his service time. In fact, the mean time between his arrivals will be $1/\theta + 1/\mu_1$. Hence, the arrival rate is $(1/\theta + 1/\mu_1)^{-1}$.

(b) Clearly we need to keep track whether the special customer is in service. For $n \geq 1$, set

$P_n = Pr\{n$ customers in system regular customer in service$\}$,

$P_n^S = Pr\{n$ customers in system, special customer in service$\}$, and

$P_0 = Pr\{0$ customers in system$\}$.

$$(\lambda+\theta)P_0 = \mu P_1 + \mu_1 P_1^S$$

$$(\lambda+\theta+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1} + \mu_1 P_{n+1}^S$$

$$(\lambda+\mu)P_n^S = \theta P_{n-1} + \lambda P_{n-1}^S,$$

$$n \geq 1 \left[P_0^S = P_0\right].$$

(c) Since service is memoryless, once a customer resumes service it is as if his service has started anew. Once he begins a particular service, he will complete it if and only if the next arrival of the special customer is after his service. The probability of this is Pr {Service $<$ Arrival of special customer} $= \mu/(\mu+\theta)$, since service and special arrivals are independent exponential random variables. So,

Pr {bumped exactly n times}

$$= (1 - \mu/(\mu+\theta))^n(\mu/(\mu+\theta))$$
$$= (\theta/(\mu+\theta))^n(\mu/(\mu+\theta)).$$

In essence, the number of times a customer is bumped in service is a geometric random variable with parameter $\mu/(\mu+\theta)$.

28. If a customer leaves the system busy, the time until the next departure is the time of a service. If a customer leaves the system empty, the time until the next departure is the time until an arrival plus the time of a service.

Using moment-generating functions we get

$$E\{e^{\delta D}\} = \frac{\lambda}{\mu}E\{e^{\delta D}|\text{system left busy}\}$$
$$+ \left[1 - \frac{\lambda}{\mu}\right] E\{e^{\delta D}|\text{system left empty}\}$$
$$= \left[\frac{\lambda}{\mu}\right]\left[\frac{\mu}{\mu-\delta}\right] + \left[1 - \frac{\lambda}{\mu}\right]\left[E\{e^{\delta(X+Y)}\}\right],$$

where X has the distribution of interarrival times, Y has the distribution of service times, and X and Y are independent.

Then

$$E\{e^{\delta(X+Y)}\} = E\{e^{\delta X}e^{\delta Y)}\}$$

$$= E[e^{\delta X}]E[e^{\delta Y)}] \text{ by independence}$$

$$= \left[\frac{\lambda}{\lambda-\delta}\right]\left[\frac{\mu}{\mu-\delta}\right].$$

So,

$$E\{e^{\delta D}\} = \left[\frac{\lambda}{\mu}\right]\left[\frac{\mu}{\mu-\delta}\right] + \left[1-\frac{\lambda}{\mu}\right]\left[\frac{\lambda}{\lambda-\delta}\right]\left[\frac{\mu}{\mu-\delta}\right]$$

$$= \frac{\lambda}{(\lambda-\delta)}.$$

By the uniqueness of generating functions, it follows that D has an exponential distribution with parameter λ.

29. (a) Let state 0 mean that the server is free; let state 1 mean that a type 1 customer is having a wash; let state 2 mean that the server is cutting hair; and let state 3 mean that a type 3 is getting a wash.

(b) $\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$

$\mu_1 P_1 = \lambda p_1 P_0$

$\mu_2 P_2 = \lambda p_2 P_0 + \mu_1 P_3$

$\mu_1 P_3 = \lambda p_3 P_0$

$P_0 + P_1 + P_2 + P_3 = 1$

(c) P_2

(d) λP_0

Direct substitution now verifies the equation.

31. Solving for the total arrival rates we have

$\lambda_1 = 5$

$\lambda_2 = 10 + \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_3$

$\lambda_3 = 15 + \frac{1}{2}\lambda_1 + \lambda_2,$

implying that $\lambda_1 = 5, \lambda_2 = 85/2$, and $\lambda_3 = 60$.

(i) $L = \sum \frac{\lambda_i}{\mu_i - \lambda_i} = \frac{5}{5} + \frac{85/2}{50-85/2} + \frac{60}{100-60} = \frac{49}{6}.$

(ii) $W = \frac{L}{\sum T_j} = \frac{49}{180}.$

32. Letting the state be the number of customers at server 1, the balance equations are

$$(\mu_2/2)P_0 = (\mu_1/2)P_1$$

$$(\mu_1/2 + \mu_2/2)P_1 = (\mu_2/2)P_0 + (\mu_1/2)P_2$$

$$(\mu_1/2)P_2 = (\mu_2/2)P_1$$

$$P_0 + P_1 + P_2 = 1$$

Solving yields that

$P_1 = (1 + \mu_1/\mu_2 + \mu_2/\mu_1)^{-1}, \quad P_0 = \mu_1/\mu_2 P_1,$

$P_2 = \mu_2/\mu_1 P_1$

Hence, letting L_i be the average number of customers at server i, then

$L_1 = P_1 + 2P_2, \qquad L_2 = 2 - L_1$

The service completion rate for server 1 is $\mu_1(1-P_0)$, and for server 2 it is $\mu_2(1-P_2)$.

33. (a) Use the Gibbs sampler to simulate a Markov chain whose stationary distribution is that of the queuing network system with $m-1$ customers. Use this simulated chain to estimate $P_{i,\,m-1}$, the steady state probability that there are i customers at server j for this system. Since, by the arrival theorem, the distribution function of the time spent at server j in the m customer system is $\sum_{i=0}^{m-1} P_{i,\,m-1}G_{i+1}(x)$, where $G_k(x)$ is the probability that a gamma (k, μ) random variable is less than or equal to x, this enables us to estimate the distribution function.

(b) This quantity is equal to the average number of customers at server j divided by m.

34. $W_Q = L_Q/\lambda_\alpha = \dfrac{\sum_j \dfrac{\lambda_j^2}{\mu_j(\mu_j-\lambda_j)}}{\sum_j r_j}$

35. Let S and U denote, respectively, the service time and value of a customer. Then U is uniform on $(0, 1)$ and

$E[S|U] = 3 + 4U, \quad Var(S|U) = 5.$

Hence,

$$E[S] = E\{E[S|U]\} = 3 + 4E[U] = 5$$

$$Var(S) = E[Var(S|U)] + Var(E[S|U])$$
$$= 5 + 16Var(U) = 19/3.$$

Therefore,

$$E[S^2] = 19/3 + 25 = 94/3.$$

(a) $W = W_Q + E[S] = \dfrac{94\lambda/3}{1-\delta\lambda} + 5.$

(b) $W_Q + E[S|U = x] = \dfrac{94\lambda/3}{1-\delta\lambda} + 3 + 4x.$

36. The distributions of the queue size and busy period are the same for all three disciplines; that of the waiting time is different. However, the means are identical. This can be seen by using $W = L/\lambda$, since L is the same for all. The smallest variance in the waiting time occurs under first-come, first-served and the largest under last-come, first-served.

37. (a) The proportion of departures leaving behind 0 work
$=$ proportion of departures leaving an empty system
$=$ proportion of arrivals finding an empty system
$=$ proportion of time the system is empty (by Poisson arrivals)
$= P_0.$

(b) The average amount of work as seen by a departure is equal to the average number it sees multiplied by the mean service time (since no customers seen by a departure has yet started service). Hence,

Average work as seen by a departure
$=$ average number it sees $\times\ E[S]$
$=$ average number an arrival sees $\times\ E[S]$
$= LE[S]$ by Poisson arrivals

$$= \lambda(W_Q + E[S])E[S]$$

$$= \frac{\lambda^2 E[S]E[S^2]}{\lambda - \lambda E[S]} + \lambda(E[S])^2.$$

38. (a) $Y_n =$ number of arrivals during the $(n+1)$st service.

(b) Taking expectations we get

$$EX_{n+1} = EX_n - 1 + EY_n + E\delta_n.$$

Letting $n \to \infty$, EX_{n+1} and EX_n cancel, and $EY_\infty = EY_1$. Therefore,

$$E\delta_\infty = 1 - EY_1.$$

To compute EY_1, condition on the length of service S; $E[Y_1|S = t] = \lambda t$ by Poisson arrivals. But $E[\lambda S]$ is just λES. Hence,

$$E\delta_\infty = 1 - \lambda ES.$$

(c) Squaring Equation (8.1) we get

$$(^*)X_{n+1}^2 = X_n^2 + 1 + Y_n^2 + 2(X_nY_n - X_n) - 2Y_n + \delta_n(2Y_n + 2X_n - 1).$$

But taking expectations, there are a few facts to notice:

$$E\delta_n S_n = 0 \quad \text{since} \quad \delta_n S_n \equiv 0.$$

Y_n and X_n are independent random variables because $Y_n =$ number of arrivals during the $(n+1)^{st}$ service. Hence,

$$EX_nY_n = EX_nEY_n.$$

For the same reason, Y_n and δ_n are independent random variables, so $E\delta_nY_n = E\delta_nEY_n$. $EY_n^2 = \lambda ES + \lambda^2 ES^2$ by the same conditioning argument of part (b).
Finally also note $\delta_n^2 \equiv \delta_n$.
Taking expectations of (*) gives

$$EX_{n+1}^2 = EX_n^2 + 1 + \lambda E(S) + \lambda^2 E(S^2)$$
$$+ 2EX_n(\lambda E(S) - 1)$$
$$- 2\lambda E(S) + 2\lambda E(S)E\delta_n - E\delta_n.$$

Letting $n \to \infty$ cancels EX_n^2 and EX_{n+1}^2, and $E\delta_n \to E\delta_\infty = 1 - \lambda E(S)$. This leaves

$$0 = \lambda^2 E(S^2) + 2EX_\infty(\lambda E(S) - 1) + 2\lambda E(S)[1 - \lambda E(S)],$$

which gives the result upon solving for EX_∞.

(d) If customer n spends time W_n in system, then by Poisson arrivals $E[X_n|W_n] = \lambda W_n$. Hence, $EX_n = \lambda EW_n$ and letting $n \to \infty$ yields $EX_\infty = \lambda W = L$. It also follows since the average number as seen by a departure is always equal to the average number as seen by an arrival, which in this case equals L by Poisson arrivals.

39. (a) $a_0 = P_0$ due to Poisson arrivals. Assuming that each customer pays 1 per unit time while in service the cost identity (2.1) states that Average number in service $= \lambda E[S]$,
or
$$1 - P_0 = \lambda E[S].$$

(b) Since a_0 is the proportion of arrivals that have service distribution G_1 and $1 - a_0$ the proportion having service distribution G_2, the result follows.

(c) We have
$$P_0 = \frac{E[I]}{E[I] + E[B]}$$
and $E[I] = 1/\lambda$ and thus,
$$E[B] = \frac{1 - P_0}{\lambda P_0} = \frac{E[S]}{1 - \lambda E[S]}.$$
Now from (a) and (b) we have
$$E[S] = (1 - \lambda E[S])E[S_1] + \lambda E[S]E[S_2],$$
or
$$E[S] = \frac{E[S_1]}{1 + \lambda E[S_1] + \lambda E[S_2]}.$$
Substitution into $E[B] = E[S]/(1 - \lambda E[S])$ now yields the result.

40. (a) (i) A little thought reveals that time to go from n to $n-1$ is independent of n.

(ii) $nE[B] = \dfrac{nE[S]}{1 - \lambda E[S]}.$

(b) (i) $E[T|N] = A + NE[B]$.

(ii) $E[T] = A + E[N]E[B]$
$$= A + \frac{\lambda A E[S]}{1 - \lambda E[S]} = \frac{A}{1 - \lambda E[S]}.$$

41. $E[N] = 2, E[N^2] = 9/2, E[S^2] = 2E^2[S] = 1/200$
$$W = \frac{\frac{1}{20}\frac{5}{2}/4 + 4 \cdot 2/400}{1 - 8/20} = \frac{41}{480}$$
$$W_Q = \frac{41}{480} - \frac{1}{20} = \frac{17}{480}.$$

42. For notational ease, set $\alpha = \lambda_1/(\lambda_1 + \lambda_2)$ = proportion of customers that are type I.

$\rho_1 = \lambda_1 E(S_1)$, $\rho_2 E(S_2)$.

Since the priority rule does not affect the amount of work in system compared to $FIFO$ and $W^Q_{FIFO} = V$, we can use Equation (6.5) for W^Q_{FIFO}. Now $W_Q = \alpha W^1_Q + (1-\alpha)W^2_Q$ by averaging over both classes of customers. It is easy to check that W_Q then becomes
$$W_Q = \frac{\left[\lambda_1 ES_1^2 + \lambda_2 ES_2^2\right][\alpha(1-\rho_1-\rho_2) + (1-\alpha)]}{2(1-\rho_1-\rho_2)(1-\rho_1)},$$
which we wish to compare to
$$W^Q_{FIFO} = \frac{\left[\lambda_1 ES_1^2 + \lambda_2 ES_2^2\right]}{2(1-\rho_1-\rho_2)} \cdot \frac{(1-\rho_1)}{(1-\rho_1)}.$$
Then $W_Q < W^Q_{FIFO} \Leftrightarrow \alpha(-\rho_1 - \rho_2) \le -\rho_1$
$$\Leftrightarrow \alpha\rho_2 > (1-\alpha)\rho_1$$
$$\Leftrightarrow \frac{\lambda_1}{\lambda_1+\lambda_2} \cdot \lambda_2 E(S_2) > \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \lambda_1 ES_1$$
$$\Leftrightarrow E(S_2) > E(S_1).$$

43. Problem 42 shows that if $\mu_1 > \mu_2$, then serving 1's first minimizes average wait. But the same argument works if $c_1\mu_1 > c_2\mu_2$, i.e.,
$$\frac{E(S_1)}{c_1} < \frac{E(S_2)}{\mu_1}.$$

44. (a) As long as the server is busy, work decreases by 1 per unit time and jumps by the service of an arrival even though the arrival may go directly into service. Since the bumped customer's remaining service does not change by being bumped, the total work in system remains the same as for nonpreemptive, which is the same as $FIFO$.

(b) As far as type I customers are concerned, the type II customers do not exist. A type I customer's delay only depends on other type I customers in system when he arrives. Therefore, $W^1_Q = V^1$ = amount of type I work in system.

By part (a), this is the same V^1 as for the nonpreemptive case (6.6). Therefore,
$$W^1_Q = \lambda_1 E(S_1)W^1_Q + \frac{\lambda_1 E\left[S_1^2\right]}{2},$$

or

$$W_Q^1 = \frac{\lambda_1 E\left[S_1^2\right]}{2(1-\lambda_1 E(S_1))}.$$

Note that this is the same as for an $M/G/1$ queue which has only type I customers.

(c) This does not account for the fact that some type II work in queue may result from customers that have been bumped from service, and so their average work would not be $E[S]$.

(d) If a type II arrival finds a bumped type II in queue, then a type I is in service. But in the nonpreemptive case, the only difference is that the type II bumped customer is served ahead of the type I, both of whom still go before the arrival. So the total amount of work found facing the arrival is the same in both cases. Hence,

$$\mathbf{W_Q^2} = \underbrace{\mathbf{V_Q^2}\,\textbf{(nonpreemptive)}}_{\substack{\textbf{total work found}\\\textbf{by type II}}} + \underbrace{\textbf{E (extra time)}}_{\substack{\textbf{extra time due}\\\textbf{to being bumped}}}$$

(e) As soon as a type II is bumped, he will not return to service until all type I's arriving during the first type I's service have departed, all further type I's who arrived during the additional type I services have departed, and so on. That is, each time a type II customer is bumped, he waits back in queue for one type I busy period. Because the type I customers do not see the type II's at all, their busy period is just an $M/G_1/1$ busy period with mean

$$\frac{E(S_1)}{1-\lambda_1 E(S_1)}.$$

So given that a customer is bumped N times, we have

$$E\{\text{extra time}|N\} = \frac{NE(S_1)}{1-\lambda_1 E(S_1)}.$$

(f) Since arrivals are Poisson, $E[N|S_2] = \lambda_1 S_2$, and so $EN = \lambda_1 ES_2$.

(g) From (e) and (f), $E(\text{extra time}) = \dfrac{\lambda_1 E(S_2)E(S_1)}{1-\lambda_1 E(S_1)}$. Combining this with (e) gives the result.

45. By regarding any breakdowns that occur during a service as being part of that service, we see that this is an $M/G/1$ model. We need to calculate the first two moments of a service time. Now the time of a service is the time T until something happens (either a service completion or a breakdown) plus any additional time A. Thus,

$$\begin{aligned} E[S] &= E[T+A] \\ &= E[T] + E[A] \end{aligned}$$

To compute $E[A]$ we condition upon whether the happening is a service or a breakdown. This gives

$$\begin{aligned} E[A] &= E[A|\text{service}]\frac{\mu}{\mu+\alpha} \\ &\quad + E[A|\text{breakdown}]\frac{\alpha}{\mu+\alpha} \\ &= E[A|\text{breakdown}]\frac{\alpha}{\mu+\alpha} \\ &= (1/\beta + E[S])\frac{\alpha}{\mu+\alpha}. \end{aligned}$$

Since, $E[T] = 1/(\alpha+\mu)$ we obtain that

$$E[S] = \frac{1}{\alpha+\mu} + (1/\beta + E[S])\frac{\alpha}{\mu+\alpha}$$

or

$$E[S] = 1/\mu + \alpha/(\mu\beta)$$

We also need $E[S^2]$, which is obtained as follows.

$$\begin{aligned} E[S^2] &= E[(T+A)^2] \\ &= E[T^2] + 2E[AT] + E[A^2] \\ &= E[T^2] + 2E[A]E[T] + E[A^2] \end{aligned}$$

The independence of A and T following because the time of the first happening is independent of whether the happening was a service or a breakdown. Now,

$$\begin{aligned} E[A^2] &= E[A^2|\text{breakdown}]\frac{\alpha}{\mu+\alpha} \\ &= \frac{\alpha}{\mu+\alpha}E[(\text{down time} + S^\alpha)^2] \\ &= \frac{\alpha}{\mu+\alpha}\left\{E[\text{down}^2] + 2E[\text{down}]E[S] + E[S^2]\right\} \\ &= \frac{\alpha}{\mu+\alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}. \end{aligned}$$

Hence,

$$E[S^2] = \frac{2}{(\mu+\beta)^2} + 2\left[\frac{\alpha}{\beta(\mu+\alpha)} + \frac{\alpha}{\mu+\alpha}\left(\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right)\right] + \frac{\alpha}{\mu+\alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}.$$

Now solve for $E[S^2]$. The desired answer is

$$W_Q = \frac{\lambda E[S^2]}{2(1-\lambda E[S])}$$

In the above, S^α is the additional service needed after the breakdown is over. S^α has the same distribution as S. The above also uses the fact that the expected square of an exponential is twice the square of its mean.

Another way of calculating the moments of S is to use the representation

$$S = \sum_{i=1}^{N} (T_i + B_i) + T_{N+1}$$

where N is the number of breakdowns while a customer is in service, T_i is the time starting when service commences for the i^{th} time until a happening occurs, and B_i is the length of the i^{th} breakdown. We now use the fact that, given N, all of the random variables in the representation are independent exponentials with the T_i having rate $\mu+\alpha$ and the B_i having rate β. This yields

$$E[S|N] = (N+1)/(\mu+\alpha) + N/\beta$$

$$Var(S|N) = (N+1)/(\mu+\alpha)^2 + N/\beta^2.$$

Therefore, since $1+N$ is geometric with mean $(\mu+\alpha)/\mu$ [and variance $\alpha(\alpha+\mu)/\mu^2$] we obtain

$E[S] = 1/\mu + \alpha/(\mu\beta)$;

and, using the conditional variance formula,

$$Var(S) = [1/(\mu+\alpha) + 1/\beta]^2\alpha(\alpha+\mu)/\mu^2 + 1/[\mu(\mu+\alpha)] + \alpha/\mu\beta^2).$$

46. β is to be the solution of Equation (7.3):

$$\beta = \int_0^\infty e^{-\mu t(1-\beta)}dG(t)$$

$$If G(t) = 1 - e^{-\lambda t}(\lambda < \mu) \quad \text{and} \quad \beta = \lambda/\mu$$

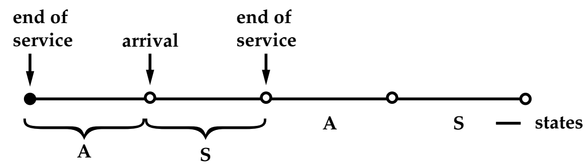
$$\int_0^\infty e^{-\mu t(1-\lambda/\mu)}dG(t) = \int_0^\infty e^{-\mu t(1-\lambda/\mu)}\lambda e^{-\lambda t}dt = \int_0^\infty e^{-\mu t}dt = \frac{\lambda}{\mu} = \beta.$$

The equation checks out.

47. For $k = 1$, Equation (8.1) gives

$$P_0 = \frac{1}{1+\lambda E(S)} = \frac{(\lambda)}{(\lambda)+E(S)} \quad P_1 = \frac{\lambda(ES)}{1+\lambda E(S)} = \frac{E(S)}{\lambda + E(S)}.$$

One can think of the process as an *alteracting renewal process*. Since arrivals are Poisson, the time until the next arrival is still exponential with parameter λ.



The basic result of alternating renewal processes is that the limiting probabilities are given by

$$Pr\{\text{being in “state } S\text{”}\} = \frac{E(S)}{E(A)+E(S)} \text{ and}$$

$$PR\{\text{being in “state } A\text{”}\} = \frac{E(A)}{E(A)+E(S)}$$

These are exactly the Erlang probabilities given above since $EA = 1/\lambda$. Note this uses Poisson arrivals in an essential way, viz., to know the distribution of time until the next arrival after a service is still exponential with parameter λ.

48. The easiest way to check that the P_i are correct is simply to check that they satisfy the balance equations:

$$\lambda p_0 = \mu p_1$$
$$(\lambda+\mu)p_1 = \lambda p_0 + 2\mu p_2$$
$$(\lambda+2\mu)p_2 = \lambda p_1 + 3\mu p_3$$
$$(\lambda+i\mu)p_i = \lambda p_{i-1} + (i+1)\mu p_{i+1} \quad 0 < i \le k$$
$$(\lambda+k\mu)p_n = \lambda p_{n-1} + k\mu p_{n+1} \quad n \ge k,$$

or

$$p_1 = \frac{1}{\mu} P_0$$

$$p_2 = \frac{\lambda^2}{2\mu^2} P_0$$

$$p_i = \frac{\lambda^i}{\mu^1 i!} P_0 \quad 0 < i \le k$$

$$p_{k+n} = \frac{\lambda^{k+n}}{\mu^{k+n} k! k^n} P_0 \quad n \ge 1.$$

In this form it is easy to check that the p_i of Equation (8.2) solves the balance equations.

49. $$P_3 = \frac{\dfrac{(\lambda E[S])^3}{3!}}{\displaystyle\sum_{j=0}^{3} \frac{(\lambda E[S])^j}{j!}}, \quad \lambda = 2, E[S] = 1$$
$$= \frac{8}{38}.$$

50. (i) $P\{\text{arrival finds all servers busy}\}$

$$= \sum_{i=k}^{\infty} P_i = \frac{\left[\dfrac{\lambda}{\mu}\right]^k \dfrac{k\mu}{k\mu - \lambda}}{k! \displaystyle\sum_{i=0}^{k-1} \frac{\left[\dfrac{\lambda}{\mu}\right]^i}{1!} + \left[\dfrac{\lambda}{\mu}\right]^k \dfrac{k\mu}{k\mu - \lambda}}.$$

(ii) $W = W_Q + 1/\mu$ where W_Q is as given by Equation (7.3) $L = \lambda W$.

51. Note that when all servers are busy, the departures are exponential with rate $k\mu$. Now see Problem 26.

52. S_n is the service time of the n^{th} customer. T_n is the time between the arrival of the n^{th} and $(n+1)^{\text{st}}$ customer.

53. $1/\mu_F < k/\mu_G$, where μ_F and μ_G are the respective means of F and G.

Chapter 9

1. If $x_i = 0$, $\phi(x) = \phi(0_i, x)$
   If $x_i = 1$, $\phi(x) = \phi(1_i, x)$.

2. (a) If $\min_i x_i = 1$, then $\underline{x} = (1, 1, \ldots, 1)$ and so $\phi(\underline{x}) = 1$.
   If $\max_i x_i = 0$, then $\underline{x} = (0, 0, \ldots, 0)$ and so $\phi(\underline{x}) = 0$.
   (b) $\max(x, y) \geq x \Rightarrow \phi(\max(x, y)) \geq \phi(x)$
   $\max(x, y) \geq y \Rightarrow \phi(\max(x, y)) \geq \phi(y)$
   $\therefore\ \phi(\max(x, y)) \geq \max(\phi(x), \phi(y))$.
   (c) Similar to (b).

3. (a) If ϕ is series, then $\phi(x) = \min_i x_i$ and so $\phi^D(\underline{x}) = 1 - \min_i (1 - x_i) = \max x_i$, and vice versa.
   (b) $\phi^{D,D}(x) = 1 - \phi^D(1 - x)$
   $= 1 - [1 - \phi(1 - (1 - x))]$
   $= \phi(x)$.
   (c) An $n - k + 1$ of n.
   (d) Say $\{1, 2, \ldots, r\}$ is a minimal path set. Then $\phi(\underbrace{1, 1, \ldots,}_{r} 1, 0, 0, \ldots 0) = 1$, and so
   $\phi^D(\underbrace{0,\ 0, \ldots,}_{r} 0, 1, 1, \ldots, 1) = 1 - \phi(1, 1, \ldots,$
   $1, 0, 0, \ldots, 0) = 0$, implying that $\{1, 2, \ldots, r\}$ is a cut set. We can easily show it to be minimal. For instance,
   $$\phi^D(\underbrace{0, 0, \ldots,}_{r-1} 0, 1, 1, \ldots, 1)$$
   $$= 1 - \phi(\underbrace{1, 1, \ldots,}_{r-1} 1, 0, 0, \ldots, 0) = 1,$$
   since $\phi(\underbrace{1, 1, \ldots,}_{r-1} 1, 0, 0, \ldots, 0) = 0$ since $\{1, 2, \ldots, r - 1\}$ is not a path set.

4. (a) $\phi(x) = x_1 \max(x_2, x_3, x_4) x_5$.
   (b) $\phi(x) = x_1 \max(x_2 x_4, x_3 x_5) x_6$.
   (c) $\phi(x) = \max(x_1, x_2 x_3) x_4$.

5. (a) Minimal path sets are
   $\{1, 8\}$, $\{1, 7, 9\}$, $\{1, 3, 4, 7, 8\}$, $\{1, 3, 4, 9\}$, $\{1, 3, 5, 6, 9\}$, $\{1, 3, 5, 6, 7, 8\}$, $\{2, 5, 6, 9\}$, $\{2, 5, 6, 7, 8\}$, $\{2, 4, 9\}$, $\{2, 4, 7, 8\}$, $\{2, 3, 7, 9\}$, $\{2, 3, 8\}$.
   Minimal cut sets are
   $\{1, 2\}$, $\{2, 3, 7, 8\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 3, 7, 9\}$, $\{4, 5, 7, 8\}$, $\{4, 6, 7, 8\}$, $\{8, 9\}$.

6. A minimal cut set has to contain at least one component of each minimal path set. There are 6 minimal cut sets:
   $\{1, 5\}$, $\{1, 6\}$, $\{2, 5\}$, $\{2, 3, 6\}$, $\{3, 4, 6\}$, $\{4, 5\}$.

7. $\{1, 4, 5\}$, $\{3\}$, $\{2, 5\}$.

8. The minimal path sets are $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$. The minimal cut sets are $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{1, 4\}$, $\{2, 3\}$.

9. (a) A component is irrelevant if its functioning or not functioning can never make a difference as to whether or not the system functions.
   (b) Use the representation (2.1.1).
   (c) Use the representation (2.1.2).

10. The system fails the first time at least one component of each minimal path set is down—thus the left side of the identity. The right side follows by noting that the system fails the first time all of the components of at least one minimal cut set are failed.

11. $r(p) = P\{\text{either } x_1 x_3 = 1 \text{ or } x_2 x_4 = 1\}$
    $P\{\text{either of 5 or 6 work}\}$
    $= (p_1 p_3 + p_2 p_4 - p_1 p_3 p_2 p_4)$
    $(p_5 + p_6 - p_5 p_5)$.

12. The minimal path sets are
$\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$.
With $q_i = 1 - P_i$, the structure function is
$$r(p) = P\{\text{either of } 1, 2, \text{ or } 3 \text{ works}\}$$
$$P\{\text{either of 4 or 5 works}\}$$
$$= (1 - q_1q_2q_3)(1 - q_4q_5).$$

13. Taking expectations of the identity
$$\phi(X) = X_i\phi(1_i, X) + (1 - X_i)\phi(0_i, X),$$
noting the independence of X_i and $\phi(1_i, X)$ and of $\phi(0_i, X)$.

14. $$r(p) = p_3P\{\max(X_1, X_2) = 1 = \max(X_4, X_5)\}$$
$$+(1 - p_3)P\{\max(X_1X_4, X_2X_5) = 1\}$$
$$= p_3(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5)$$
$$+(1 - p_3)(p_1p_4 + p_2p_5 - p_1p_4p_2p_5).$$

15. (a) $\frac{7}{32} \leq \mathrm{r}\left[\frac{1}{2}\right] \leq 1 - \left[\frac{7}{8}\right]^3 = \frac{169}{512}$.
The exact value is $r(1/2) = 7/32$, which agrees with the minimal cut lower bound since the minimal cut sets $\{1\}$, $\{5\}$, $\{2,3,4\}$ do not overlap.

17. $$E[N^2] = E[N^2|N > 0]P\{N > 0\}$$
$$\geq (E[N|N > 0])^2P\{N > 0\},$$
since $E[X^2] \geq (E[X])^2$.
Thus,
$$E[N^2]P\{N > 0\} \geq (E[N|N > 0]P\{N > 0\})^2$$
$$= (E[N])^2.$$
Let N denote the number of minimal path sets having all of its components functioning. Then $r(p) = P\{N > 0\}$.

Similarly, if we define N as the number of minimal cut sets having all of its components failed, then $1 - r(p) = P\{N > 0\}$.

In both cases we can compute expressions for $E[N]$ and $E[N^2]$ by writing N as the sum of indicator (i.e., Bernoulli) random variables. Then we can use the inequality to derive bounds on $r(p)$.

18. (a) $\{3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$.
(b) $P\left\{\text{system life} > \frac{1}{2}\right\} = r\left[\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right]$.
Now $r(p) = p_1p_2p_3 + p_3p_4p_5 - p_1p_2p_3p_4p_5$, and so
$$P\left\{\text{system life} < \frac{1}{2}\right\} = 1 - \frac{1}{8} - \frac{1}{8} + \frac{1}{32}$$
$$= \frac{25}{32}.$$

19. $X_{(i)}$ is the system life of an $n - i + 1$ of n system each having the life distribution F. Hence, the result follows from Example 5e.

20. The densities are related as follows.
$$g(t) = a[\bar{F}(t)]^{a-1}f(t).$$
Therefore,
$$\lambda_C(t) = a[\bar{F}(t)]^{a-1}f(t)/[\bar{F}(t)]^a$$
$$= a\, f(t)/\bar{F}(t)$$
$$= a\, \lambda_F(t).$$

21. (a) (i), (ii), (iv) − (iv) because it is two-of-three.
(b) (i) because it is series, (ii) because it can be thought of as being a series arrangement of 1 and the parallel system of 2 and 3, which as $F_2 = F_3$ is IFR.
(c) (i) because it is series.

22. (a) $$F_t(a) = P\{X > t + a \mid X > t\}$$
$$= \frac{P\{X > t + a\}}{P\{X > t\}} = \frac{F(t + a)}{F(t)}.$$
(b) Suppose $\lambda(t)$ is increasing. Recall that
$$F(t) = e^{-\int_0^t \lambda(s)ds}.$$
Hence,
$\frac{F(t + a)}{F(t)} = e^{-\int_0^{t+a} \lambda(s)ds}$, which decreases in t since $\lambda(t)$ is increasing. To go the other way, suppose $F(t + a)/F(t)$ decreases in t. Now for a small
$$F(t + a)/F(t) = e^{-a\lambda(t)}.$$
Hence, $e^{-a\lambda(t)}$ must decrease in t and thus $\lambda(t)$ increases.

23. (a) $F(t) = \prod_{i=1}^{n} F_i(t)$

$$\lambda_F(t) = \frac{\frac{d}{dt}F(t)}{F(t)} = \frac{\sum_{j=1}^{n} F_j^{'}(t) \prod_{i \neq j} F_j(t)}{\prod_{i=1}^{n} F_i(t)}$$

$$= \frac{\sum_{j=1}^{n} F_j^{'}(t)}{F_j(t)}$$

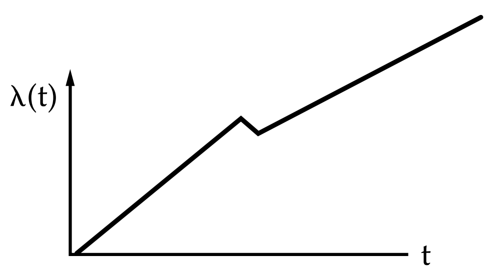
$$= \sum_{j=1}^{n} \lambda_j(t).$$

(b) $F_t(a) = P\{\text{additional life of } t\text{-year-old} > a\}$

$$= \frac{\prod_{1}^{n} F_i(t+a)}{F_i(t)},$$

where F_i is the life distribution for component i. The point being that as the system is series, it follows that knowing that it is alive at time t is equivalent to knowing that all components are alive at t.

24. It is easy to show that $\lambda(t)$ increasing implies that $\int_0^t \lambda(s)\, ds/t$ also increases. For instance, if we differentiate, we get $t\lambda(t) - \int_0^t \lambda(s)\, ds/t^2$, which is nonnegative since $\int_0^t \lambda(s)\, ds \le \int_0^t \lambda(t)\, dt = t\lambda(t)$. A counterexample is



25. For $x \ge \xi$,

$$1 - p = 1 - F(\xi) = 1 - F(x(\xi/x)) \ge [1 - F(x)]^{\xi/x}$$

since IFRA.

Hence,

$$1 - F(x) \le (1-p)^{x/\xi} = e^{-\theta x}.$$

For $x \le \xi$,

$$1 - F(x) = 1 - F(\xi(x/\xi)) \ge [1 - F(\xi)]^{x/\xi}$$

since IFRA.

Hence,

$$1 - F(x) \ge (1-p)^{x/\xi} = e^{-\theta x}.$$

26. Either use the hint in the text or the following, which does not assume a knowledge of concave functions.

To show: $h(y) \equiv \lambda^\alpha x^\alpha + (1-\lambda^\alpha)y^\alpha - (\lambda x + (1-\lambda)y)^\alpha \ge 0,$
$0 \le y \le x,$
where $0 \le \lambda \le 1,\ 0 \le \alpha \le 1.$

Note: $h(0) = 0$, assume $y > 0$, and let $g(y) = h(y)/y^a$

$$g(y) = \left[\frac{\lambda x}{y}\right]^\alpha + 1 - \lambda^\alpha - \left[\frac{\lambda x}{y} + 1 - \lambda\right]^\alpha.$$

Let $z = x/y$. Now $g(y) \ge 0\ \forall\ 0 < y < x \Leftrightarrow f(z) \ge 0\ \forall\ z \ge 1,$

where $f(z) = (\lambda z)^\alpha + 1 - \lambda^\alpha - (\lambda z + 1 - \lambda)^\alpha.$

Now $f(1) = 0$ and we prove result by showing that $f^{'}(z) \ge 0$ whenever $z > 1$. This follows since

$$f^{'}(z) = \alpha\lambda(\lambda z)^{\alpha-1} - \alpha\lambda(\lambda z + 1 - \lambda)^{\alpha-1}$$

$$f^{'}(z) \ge 0 \Leftrightarrow (\lambda z)^{\alpha-1} \ge (\lambda z + 1 - \lambda)^{\alpha-1}$$

$$\Leftrightarrow (\lambda z)^{1-\alpha} \le (\lambda z + 1 - \lambda)^{1-\alpha}$$

$$\Leftrightarrow \lambda z \le \lambda z + 1 - \lambda$$

$$\Leftrightarrow \lambda \le 1.$$

27. If $p > p_0$, then $p = {p_0}^\alpha$ for some $a \in (0,1)$. Hence,

$r(p) = r({p_0}^\alpha) \ge [r(p_0)]^\alpha = {p_0}^\alpha = p.$

If $p < p_0$, then $p_0 = p^\alpha$ for some $a \in (0,1)$. Hence,

$p^\alpha = p_0 = r(p_0) = r(p^\alpha) \ge [r(p)]^\alpha.$

28. (a) $F(t) = (1-t)\left[\frac{2-t}{2}\right], \quad 0 \le t \le 1$

$$E[\text{lifetime}] = \frac{1}{2}\int_0^1 (1-t)(2-t)\, dt = \frac{5}{12}.$$

(b) $F(t) = \begin{cases} 1 - t^2/2, & 0 \le t \le 1 \\ 1 - t\,/\,2, & 1 \le t \le 2 \end{cases}$

$$E[\text{lifetime}] = \frac{1}{2}\int_0^1 (2-t^2)\, dt + \frac{1}{2}\int_1^2 (2-t)\, dt$$

$$= \frac{13}{12}.$$

29. Let X denote the time until the first failure and let Y denote the time between the first and second failure. Hence, desired result is

$$EX + EY = \frac{1}{\mu_1 + \mu_2} + EY.$$

Now,

$$E[Y] = E[Y|\mu_1 \text{ component fails first}]\frac{\mu_1}{\mu_1+\mu_2}$$
$$+ E[Y|\mu_2 \text{ component fails first}]\frac{\mu_2}{\mu_1+\mu_2}$$
$$= \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1+\mu_2}.$$

30. $r(p) = p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4$
$-3p_1p_2p_3p_4$

$r(1 - F(t))$

$$= \begin{cases} 2(1-t)^2(1-t/2) + 2(1-t)(1-t/2)^2 \\ -3(1-t)^2(1-t/2)^2, & 0 \le t \le 1 \\ 0, & 1 \le t \le 2 \end{cases}$$

$$E[\text{lifetime}] = \int_0^1 \Big[2(1-t)^2(1-t/2)$$
$$+ 2(1-t)(1-t/2)^2$$
$$- 3(1-t)^2(1-t/2)^2\Big]dt$$
$$= \frac{31}{60}.$$

31. Use the remark following Equation (6.3).

32. Letting I_i equal 1 if $X_i > c^\alpha$ and letting it be 0 otherwise. Then,

$$E\left[\sum_{i=1}^n I_i\right] = \sum_{i=1}^n E[I_i] = \sum_{i=1}^n P\{X_i > c^\infty\}$$

33. The exact value can be obtained by conditioning on the ordering of the random variables. Let M denote the maximum, then with $A_{i,j,k}$ being the even that $X_i < X_j < X_k$, we have that

$$E[M] = \sum E[M|A_{i,\,j,\,k}]P(A_{i,\,j,\,k})$$

where the preceding sum is over all 6 possible permutations of $1, 2, 3$. This can now be evaluated by using

$$P(A_{i,\,j,\,k}) = \frac{\lambda_i}{\lambda_i + \lambda_j + \lambda_k}\frac{\lambda_j}{\lambda_j + \lambda_k}$$
$$E[M|A_{i,\,j,\,k}] = \frac{1}{\lambda_i + \lambda_j + \lambda_k} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k}$$

35. (a) It follows when $i = 1$ since $0 = (1-1)^n = 1 - \left[{n \atop 1}\right] + \left[{n \atop 2}\right] \cdots \pm \left[{n \atop n}\right]$. So assume it true for i and consider $i + 1$. We must show that

$$\begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n \\ i+1 \end{bmatrix} - \begin{bmatrix} n \\ i+2 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, is equivalent to

$$\begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is easily seen to be true.

(b) It is clearly true when $i = n$, so assume it for i. We must show that

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, reduces to

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is true.

Chapter 10

1. $X(s) + X(t) = 2X(s) + X(t) - X(s)$.

   Now $2X(s)$ is normal with mean 0 and variance $4s$ and $X(t) - X(s)$ is normal with mean 0 and variance $t - s$. As $X(s)$ and $X(t) - X(s)$ are independent, it follows that $X(s) + X(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

2. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s - t_1)$ given that $X(0) = 0$ and $X(t_2 - t_1) = B - A$, which by (1.4) is normal with mean $\dfrac{s - t_1}{t_2 - t_1}(B - A)$ and variance $\dfrac{(s - t_1)}{t_2 - t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \dfrac{(s - t_1)(B - A)}{t_2 - t_1}$ and variance $\dfrac{(s - t_1)(t_2 - s)}{t_2 - t_1}$.

3.
$$
\begin{aligned}
&E[X(t_1)X(t_2)X(t_3)] \\
&= E[E[X(t_1)X(t_2)X(t_3) \mid X(t_1), X(t_2)]] \\
&= E[X(t_1)X(t_2)E[X(t_3) \mid X(t_1), X(t_2)]] \\
&= E[X(t_1)X(t_2)X(t_2)] \\
&= E[E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \\
&= E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \qquad (*) \\
&= E[X(t_1)\{(t_2 - t_1) + X^2(t_1)\}] \\
&= E[X^3(t_1)] + (t_2 - t_1)E[X(t_1)] \\
&= 0
\end{aligned}
$$

   where the equality $(*)$ follows since given $X(t_1)$, $X(t_2)$ is normal with mean $X(t_1)$ and variance $t_2 - t_1$. Also, $E[X^3(t)] = 0$ since $X(t)$ is normal with mean 0.

4. (a)
$$
\begin{aligned}
P\{T_a < \infty\} &= \lim_{t\to\infty} P\{T_a \le t\} \\
&= \frac{2}{\sqrt{2r}} \int_0^\infty e^{-y^2/2} dy \quad \text{by (10.6)} \\
&= 2P\{N(0,1) > 0\} = 1.
\end{aligned}
$$

   Part (b) can be proven by using

   $$E[T_a] = \int_0^\infty P\{T_a > t\} dt$$

   in conjunction with (2.3).

5.
$$
\begin{aligned}
P\{T_1 < T_{-1} < T_2\} &= P\{\text{hit 1 before} - 1 \text{ before } 2\} \\
&= P\{\text{hit 1 before } -1\} \\
&\quad \times P\{\text{hit } -1 \text{ before } 2 \mid \text{hit 1 before } -1\} \\
&= \frac{1}{2} P\{\text{down 2 before up 1}\} \\
&= \frac{1}{2}\frac{1}{3} = \frac{1}{6}.
\end{aligned}
$$

   The next to last equality follows by looking at the Brownian motion when it first hits 1.

6. The probability of recovering your purchase price is the probability that a Brownian motion goes up c by time t. Hence the desired probability is

   $$1 - P\{\max_{0\le s\le t} X(s) \ge c\} = 1 - \frac{2}{\sqrt{2\pi t}} \int_{c/\sqrt{t}}^\infty e^{-y^2/2} dy$$

7. Let $M = \{\max_{t_1 \le s \le t_2} X(s) > x\}$. Condition on $X(t_1)$ to obtain

   $$P(M) = \int_{-\infty}^\infty P(M|X(t_1) = y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} dy$$

   Now, use that

   $$P(M|X(t_1) = y) = 1 \quad y \ge x$$

   and, for $y < x$

   $$
   \begin{aligned}
   P(M|X(t_1) = y) &= P\{\max_{0<s<t_2-t_1} X(s) > x - y\} \\
   &= 2P\{X(t_2 - t_1) > x - y\}
   \end{aligned}
   $$

8. (a) Let $X(t)$ denote the position at time t. Then

   $$X(t) = \sqrt{\Delta t} \sum_{i=1}^{[t/\Delta t]} X_i$$

where

$$X_i = \begin{cases} +1 & \text{if } i^{th} \text{ step is up} \\ -1 & \text{if } i^{th} \text{ step is down.} \end{cases}$$

As

$$E[X_1] = p - 1(1-p)$$
$$= 2p - 1$$
$$= \mu\sqrt{\Delta t}$$

and

$$Var(X_i) = E\left[X_i^2\right] - (E\left[X_i\right])^2$$
$$= 1 - \mu^2 \Delta t \quad \text{since } X_i^2 = 1$$

we obtain that

$$E[X(t)] = \sqrt{\Delta t}\left[\frac{t}{\Delta t}\right]\mu\sqrt{\Delta t}$$
$$\to \mu t \text{ as } \Delta t \to 0$$
$$Var(X(t)) = \Delta t\left[\frac{t}{\Delta t}\right](1 - \mu^2\Delta t)$$
$$\to t \text{ as } \Delta t \to 0.$$

(b) By the gambler's ruin problem the probability of going up A before going down B is

$$\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

when each step is either up 1 or down 1 with probabilities p and $q = 1 - p$. (This is the probability that a gambler starting with B will reach his goal of $A + B$ before going broke.) Now, when $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t}), q = 1 - p = \frac{1}{2}(1 - \mu\sqrt{\Delta t})$ and so $q/p = \frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}$. Hence, in this case the probability of going up $A/\sqrt{\Delta t}$ before going down $B/\sqrt{\Delta t}$ (we divide by $\sqrt{\Delta t}$ since each step is now of this size) is

$$(*) \qquad \frac{1-\left[\dfrac{1-\mu\quad\sqrt{\Delta t}}{1+\mu\quad\sqrt{\Delta t}}\right]^{B/\sqrt{\Delta t}}}{1-\left[\dfrac{1-\mu\quad\sqrt{\Delta t}}{1+\mu\quad\sqrt{\Delta t}}\right]^{(A+B/\sqrt{\Delta t})}}.$$

Now

$$\lim_{\Delta t\to 0}\left[\frac{1-\mu\quad\sqrt{\Delta t}}{1+\mu\quad\sqrt{\Delta t}}\right]^{1/\sqrt{\Delta t}} = \lim_{h\to 0}\left[\frac{1-\mu h}{1+\mu h}\right]^{1/h}$$
$$= \lim_{n\to\infty}\left[\frac{1-\dfrac{\mu}{n}}{1+\dfrac{\mu}{n}}\right]^n$$
$$\text{by } n = 1/h$$
$$= \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu}$$

where the last equality follows from

$$\lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n = e^x.$$

Hence the limiting value of $(*)$ as $\Delta t \to 0$ is

$$\frac{1-e^{-2\mu B}}{1-e^{-2\mu(A+B)}}.$$

11. Let $X(t)$ denote the value of the process at time $t = nh$. Let $X_i = 1$ if the i^{th} change results in the state value becoming larger, and let $X_i = 0$ otherwise. Then, with $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$

$$X(t) = X(0)u^{\sum_{i=1}^n X_i} d^{n-\sum_{i=1}^n X_i}$$
$$= X(0)d^n\left(\frac{u}{d}\right)^{\sum_{i=1}^n X_i}$$

Therefore,

$$\log\left(\frac{X(t)}{X(0)}\right) = n\log(d) + \sum_{i=1}^n X_i\log(u/d)$$
$$= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\sum_{i=1}^{t/h} X_i$$

By the central limit theorem, the preceding becomes a normal random variable as $h \to 0$. Moreover, because the X_i are independent, it is easy to see that the process has independent increments. Also,

$$E\left[\log\left(\frac{X(t)}{X(0)}\right)\right]$$
$$= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\,\frac{t}{h}\,\frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})$$
$$= \mu t$$

and

$$Var\left[\log\left(\frac{X(t)}{X(0)}\right)\right] = 4\sigma^2 h\frac{t}{h}p(1-p)$$
$$\to \sigma^2 t$$

where the preceding used that $p \to 1/2$ as $h \to 0$.

12. If we purchase x units of the stock any y of the option then the value of our holdings at time 1 is

$$\text{value} = \begin{cases} 150x + 25y & \text{if price is 150} \\ 25x & \text{if price is 25} \end{cases}.$$

So if

$$150x + 25y = 25x, \text{ or } y = -5x$$

then the value of our holdings is $25x$ no matter what the price is at time 1. Since the cost of purchasing x units of the stock and $-5x$ units of options is $50x - 5xc$ it follows that our profit from such a purchase is

$$25x - 50x + 5xc = x(5c - 25).$$

(a) If $c = 5$ then there is no sure win.

(b) Sell $|x|$ units of the stock and buy $-5|x|$ units of options will realize a profit of $5|x|$ no matter what the price of the stock is at time 1. (That is, buy x units of the stock and $-5x$ units of the options for $x < 0$.)

(c) Buying x units of the stock and $-5x$ units of options will realize a positive profit of $25x$ when $x > 0$.

(d) Any probability vector $(p, 1 - p)$ on $(150, 25)$, the possible prices at time 1, under which buying the stock is a fair bet satisfies the following

$$50 = p(150) + (1 - p)(25),$$

or

$$p = 1/5.$$

That is, (1/5, 4/5) is the only probability vector that makes buying the stock a fair bet. Thus, in order for there to be no arbitrage possibility, the price of an option must be a fair bet under this probability vector. This means that the cost c must satisfy

$$c = 25(1/5) = 5.$$

13. If the outcome is i then our total winnings are

$$x_i o_i - \sum_{j \neq i} x_j = \frac{o_i(1 + o_i)^{-1} - \sum_{j \neq i} (1 + o_j)^{-1}}{1 - \sum_k (1 + o_k)^{-1}}$$

$$= \frac{(1 + o_i)(1 + o_i)^{-1} - \sum_j (1 + o_j)^{-1}}{1 - \sum_k (1 + o_k)^{-1}}$$

$$= 1.$$

14. Purchasing the stock will be a fair bet under probabilities $(p_1,\ p_2,\ 1 - p_1 - p_2)$ on (50, 100, 200), the set of possible prices at time 1, if

$$100 = 50p_1 + 100p_2 + 200(1 - p_1 - p_2),$$

or equivalently, if

$$3p_1 + 2p_2 = 2.$$

(a) The option bet is also fair if the probabilities also satisfy

$$c = 80(1 - p_1 - p_2).$$

Solving this and the equation $3p_1 + 2p_2 = 2$ for p_1 and p_2 gives the solution

$$p_1 = c/40,\ p_2 = (80 - 3c)/80,$$

$$1 - p_1 - p_2 = c/80.$$

Hence, no arbitrage is possible as long as these p_i all lie between 0 and 1. However, this will be the case if and only if

$$80 \geq 3c.$$

(b) In this case, the option bet is also fair if

$$c = 20p_2 + 120(1 - p_1 - p_2).$$

Solving in conjunction with the equation $3p_1 + 2p_2 = 2$ gives the solution

$$p_1 = (c - 20)/30,\ p_2 = (40 - c)/20,$$

$$1 - p_1 - p_2 = (c - 20)/60.$$

These will all be between 0 and 1 if and only if

$$20 \leq c \leq 40.$$

15. The parameters of this problem are

$$\sigma = .05, \quad \sigma = 1, \quad x_o = 100, \quad t = 10.$$

(a) If $K = 100$ then from Equation (4.4)

$$b = [.5 - 5 - \log(100/100)]/\sqrt{10}$$

$$= -4.5\sqrt{10} = -1.423$$

and

$$c = 100\phi(\sqrt{10} - 1.423) - 100e^{-.5}\phi(-1.423)$$

$$= 100\phi(1.739) - 100e^{-.5}[1 - \phi(1.423)]$$

$$= 91.2.$$

The other parts follow similarly.

16. Taking expectations of the defining equation of a Martingale yields that

$E[Y(s)] = E[E[Y(t)/Y(u),\ 0 \le u \le s]] = E[Y(t)].$

That is, $E[Y(t)]$ is constant and so is equal to $E[Y(0)]$.

17. $E\,[B(t)|B(u),\ 0 \le u \le s]$

$= E[B(s) + B(t) - B(s)|B(u),\ 0 \le u \le s]$

$= E[B(s)|B(u),\ 0 \le u \le s]$

$+ E[B(t) - B(s)|B(u), 0 \le u \le s]$

$= B(s) + E[B(t) - B(s)]$ by independent increments

$= B(s).$

18. $E[B^2(t)|B(u),\ 0 \le u \le s] = E[B^2(t)|B(s)],$

where the above follows by using independent increments as was done in Problem 17. Since the conditional distribution of $B(t)$ given $B(s)$ is normal with mean $B(s)$ and variance $t - s$ it follows that

$E[B^2(t)|B(s)] = B^2(s) + t - s.$

Hence,

$E[B^2(t) - t|B(u),\ 0 \le u \le s] = B^2(s) - s.$

Therefore, the conditional expected value of $B^2(t) - t$, given all the values of $B(u),\ 0 \le u \le s$, depends only on the value of $B^2(s)$. From this it intuitively follows that the conditional expectation given the squares of the values up to time s is also $B^2(s) - s$. A formal argument is obtained by conditioning on the values $B(u),\ 0 \le u \le s$ and using the above. This gives

$E[B^2(t) - t|B^2(u),\ 0 \le u \le s]$

$= E\,[E[B^2(t) - t|B(u),\ 0 \le u \le s]|B^2(u),$

$0 \le u \le s]$

$= E[B^2(s) - s|B^2(u),\ 0 \le u \le s]$

$= B^2(s) - s$

which proves that $\{B^2(t) - t,\ t \ge 0\}$ is a Martingale. By letting $t = 0$, we see that

$E[B^2(t) - t] = E[B^2(0)] = 0.$

19. Since knowing the value of $Y(t)$ is equivalent to knowing $B(t)$ we have that

$E[Y(t)|Y(u),\ 0 \le u \le s]$

$= e^{-c^2t/2}E[e^{cB(t)}|B(u),\ 0 \le u \le s]$

$= e^{-c^2t/2}E[e^{cB(t)}|B(s)]$

Now, given $B(s)$, the conditional distribution of $B(t)$ is normal with mean $B(s)$ and variance $t - s$. Using the formula for the moment generating function of a normal random variable we see that

$e^{-c^2t/2}E[e^{cB(t)}|B(s)]$

$= e^{-c^2t/2}e^{cB(s)+(t-s)c^2/2}$

$= e^{-c^2s/2}e^{cB(s)}$

$= Y(s).$

Thus, $\{Y(t)\}$ is a Martingale.

$E[Y(t)] = E[Y(0)] = 1.$

20. By the Martingale stopping theorem

$E[B(T)] = E[B(0)] = 0.$

However, $B(T) = 2 - 4T$ and so

$2 - 4E[T] = 0$

or, $E[T] = 1/2.$

21. By the Martingale stopping theorem

$E[B(T)] = E[B(0)] = 0.$

But, $B(T) = (x - \mu T)/\sigma$ and so

$E[(x - \mu T)/\sigma] = 0,$

or

$E[T] = x/\mu.$

22. (a) It follows from the results of Problem 19 and the Martingale stopping theorem that

$E[\exp\{cB(T) - c^2T/2\}]$

$= E[\exp\{cB(0)\}] = 1.$

Since $B(T) = [X(T) - \mu T]/\sigma$ part (a) follows.

(b) This follows from part (a) since

$-2\mu[X(T) - \mu T]/\sigma^2 - (2\mu/\sigma)^2T/2$

$= -2\mu X(T)/\sigma^2.$

(c) Since T is the first time the process hits A or $-B$ it follows that

$$X(T) = \begin{cases} A, & \text{with probability } p \\ -B, & \text{with probability } 1-p \end{cases}.$$

Hence, we see that

$$1 = E[e^{-2\mu X(T)/\sigma^2}] = pe^{-2\mu A/\sigma^2} + (1-p)e^{2\mu B/\sigma^2}$$

and so

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}.$$

23. By the Martingale stopping theorem we have that

$$E[B(T)] = E[B(0)] = 0.$$

Since, $B(T) = [X(T) - \mu T]/\sigma$ this gives the equality

$$E[X(T) - \mu T] = 0$$

or

$$E[X(T)] = \mu E[T].$$

Now

$$E[X(T)] = pA - (1-p)B$$

where, from part (c) of Problem 22,

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}.$$

Hence,

$$E[T] = \frac{A(1 - e^{2\mu B/\sigma^2}) - B(e^{-2\mu A/\sigma^2} - 1)}{\mu(e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2})}.$$

24. It follows from the Martingale stopping theorem and the result of Problem 18 that

$$E[B^2(T) - T] = 0,$$

where T is the stopping time given in this problem and $B(t) = [X(t) - \mu t]/\sigma$. Therefore,

$$E[(X(T) - \mu T)^2/\sigma^2 - T] = 0.$$

However, $X(T) = x$ and so the above gives that

$$E[(x - \mu T)^2] = \sigma^2 E[T].$$

But, from Problem 21, $E[T] = x/\mu$ and so the above is equivalent to

$$Var(\mu T) = \sigma^2 x/\mu$$

or

$$Var(T) = \sigma^2 x/\mu^3.$$

25. The means equal 0.

$$Var\left[\int_0^1 t\,dX(t)\right] = \int_0^1 t^2 dt = \frac{1}{3}$$

$$Var\left[\int_0^1 t^2 dX(t)\right] = \int_0^1 t^4 dt = \frac{1}{5}.$$

26. (a) Normal with mean and variance given by:

$$E[Y(t)] = tE[X(1/t)] = 0$$

$$Var(Y(t)) = t^2 Var[X(1/t)] = t^2/t = t.$$

(b)
$$\begin{aligned} Cov(Y(s), Y(t)) &= Cov(sX(1/s), tX(1/t)) \\ &= st\, Cov(X(1/s), X(1/t)) \\ &= st\frac{1}{t} \quad \text{when } s \le t \\ &= s \quad \text{when } s \le t. \end{aligned}$$

(c) Clearly $\{Y(t)\}$ is Gaussian. As it has the same mean and covariance function as the Brownian motion process (which is also Gaussian) it follows that it is also Brownian motion.

27. $E[X(a^2 t)/a] = \frac{1}{a} E[X(a^2 t)] = 0.$

For $s < t$,

$$\begin{aligned} Cov(Y(s), Y(t)) &= \frac{1}{a^2} Cov(X(a^2 s), X(a^2 t)) \\ &= \frac{1}{a^2} a^2 s = s. \end{aligned}$$

As $\{Y(t)\}$ is clearly Gaussian, the result follows.

28.
$$\begin{aligned} Cov(B(s) - \tfrac{s}{t}B(t),\ B(t)) &= Cov(B(s),\ B(t)) \\ &\quad - \tfrac{s}{t} Cov(B(t),\ B(t)) \\ &= s - \tfrac{s}{t} t = 0 \end{aligned}$$

29. $\{Y(t)\}$ is Gaussian with

$$E[Y(t)] = (t+1)E(Z[t/(t+1)]) = 0;$$

and for $s \le t$

$$\begin{aligned} &Cov(Y(s), Y(t)) \\ &= (s+1)(t+1)\, Cov\left[Z\left[\frac{s}{s+1}\right],\quad Z\left[\frac{t}{t+1}\right]\right] \\ &= (s+1)(t+1)\frac{s}{s+1}\left[1 - \frac{t}{t+1}\right] \quad (*) \\ &= s \end{aligned}$$

where $(*)$ follows since $Cov(Z(s), Z(t)) = s(1-t)$. Hence, $\{Y(t)\}$ is Brownian motion since it is also Gaussian and has the same mean and covariance function (which uniquely determines the distribution of a Gaussian process).

30. For $s < 1$

$$Cov[X(t), X(t+s)]$$
$$= Cov[N(t+1) - N(t), N(t+s+1) - N(t+s)]$$
$$= Cov(N(t+1), N(t+s+1) - N(t+s))$$
$$-Cov(N(t), N(t+s+1) - N(t+s))$$
$$= Cov(N(t+1), N(t+s+1) - N(t+s)) \quad (*)$$

where the equality $(*)$ follows since $N(t)$ is independent of $N(t+s+1) - N(t+s)$. Now, for $s \leq t$,

$$Cov(N(s), N(t)) = Cov(N(s), N(s) + N(t) - N(s))$$
$$= Cov(N(s), N(s))$$
$$= \lambda s.$$

Hence, from $(*)$ we obtain that, when $s < 1$,

$$Cov(X(t), X(t+s)) = Cov(N(t+1), N(t+s+1))$$
$$-Cov(N(t+1), N(t+s))$$
$$= \lambda(t+1) - \lambda(t+s)$$
$$= \lambda(1-s)$$

When $s \geq 1$, $N(t+1) - N(t)$ and $N(t+s+1) - N(t+s)$ are, by the independent increments property, independent and so their covariance is 0.

31. (a) Starting at any time t the continuation of the Poisson process remains a Poisson process with rate λ.

(b) $E[Y(t)Y(t+s)]$

$$= \int_0^\infty E[Y(t)Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y} dy$$
$$= \int_0^\infty yE[Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y} dy$$
$$+ \int_s^\infty y(y-s)\lambda e^{-\lambda y} dy$$
$$= \int_0^s y\frac{1}{\lambda}\lambda e^{-\lambda y} dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y} dy$$

where the above used that

$$E[Y(t)Y(t+s)|Y(t) = y]$$
$$= \begin{cases} yE(Y(t+s)) = \dfrac{y}{\lambda} & \text{if } y < s \\ y(y-s) & \text{if } y > s \end{cases}.$$

Hence,

$$Cov(Y(t), Y(t+s))$$
$$= \int_0^s ye^{-y\lambda} dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y} dy - \frac{1}{\lambda^2}.$$

32. (a) $Var(X(t+s) - X(t))$

$$= Cov(X(t+s) - X(t), X(t+s) - X(t))$$
$$= R(0) - R(s) - R(s) + R(0)$$
$$= 2R(0) - 2R(s).$$

(b) $Cov(Y(t), Y(t+s))$

$$= Cov(X(t+1) - X(t), X(t+s+1)$$
$$- X(t+s))$$
$$= R_x(s) - R_x(s-1) - R_x(s+1) + R_x(s)$$
$$= 2R_x(s) - R_x(s-1) - R_x(s+1), \quad s \geq 1.$$

33. $Cov(X(t), X(t+s))$

$$= Cov(Y_1 \cos wt + Y_2 \sin wt,$$
$$Y_1 \cos w(t+s) + Y_2 \sin w(t+s))$$
$$= \cos wt \cos w(t+s) + \sin wt \sin w(t+s)$$
$$= \cos(w(t+s) - wt)$$
$$= \cos ws.$$

Chapter 11

1. (a) Let u be a random number. If $\sum_{j=1}^{i-1} P_j < u \le \sum_{j=1}^{i} P_j$ then simulate from F_i.

   $\left(\text{In the above } \sum_{j=1}^{i-1} P_j \equiv 0 \text{ when } i = 1.\right)$

   (b) Note that

   $$F(x) = \frac{1}{3}F_1(X) + \frac{2}{3}F_2(x)$$

   where

   $$F_1(x) = 1 - e^{-2x}, \quad 0 < x < \infty,$$

   $$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x \end{cases}.$$

   Hence, using (a), let U_1, U_2, U_3 be random numbers and set

   $$X = \begin{cases} \dfrac{-\log U_2}{2} & \text{if } U_1 < 1/3 \\ U_3 & \text{if } U_1 > 1/3 \end{cases}.$$

   The above uses the fact that $\dfrac{-\log U_2}{2}$ is exponential with rate 2.

2. Simulate the appropriate number of geometrics and sum them.

3. If a random sample of size n is chosen from a set of $N + M$ items of which N are acceptable then X the number of acceptable items in the sample, is such that

   $$P\{X = k\} = \begin{bmatrix} N \\ k \end{bmatrix} \begin{bmatrix} M \\ n-k \end{bmatrix} \Bigg/ \begin{bmatrix} N+M \\ k \end{bmatrix}.$$

   To simulate X note that if

   $$I_j = \begin{cases} 1 & \text{if the } j^{th} \text{ section is acceptable} \\ 0 & \text{otherwise} \end{cases}$$

   then

   $P\{I_j = 1 | I_1, \ldots, I_{j-1}\} = \dfrac{N - \sum_1^{j-1} I_i}{N + M - (j-1)}$. Hence, we can simulate $I_1, \ldots, I_n$ by generating random numbers $U_1, \ldots, U_n$ and then setting

   $$I_j = \begin{cases} 1 & \text{if } U_j < \dfrac{N - \sum_1^{j-1} I_i}{N + M - (j-1)} \\ 0 & \text{otherwise} \end{cases}$$

   $X = \sum_{j=1}^{n} I_j$ has the desired distribution.

   Another way is to let

   $$X_j = \begin{cases} 1 & \text{the } j^{th} \text{ acceptable item is in the sample} \\ 0 & \text{otherwise} \end{cases}$$

   and then simulate $X_1, \ldots, X_N$ by generating random numbers $U_1, \ldots, U_N$ and then setting

   $$X_j = \begin{cases} 1 & \text{if } U_j < \dfrac{N - \sum_{i=1}^{j-1} I_i}{N + M - (j-1)} \\ 0 & \text{otherwise.} \end{cases}$$

   $X = \sum_{j=1}^{N} X_j$ then has the desired distribution.

   The former method is preferable when $n \le N$ and the latter when $N \le n$.

4. $$\frac{\partial R}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial R}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

   $$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left[\frac{y}{x}\right]^2} \left[\frac{-y}{x^2}\right] = \frac{-y}{x^2 + y^2},$$

   $$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left[\frac{y}{x}\right]^2} \left[\frac{1}{x}\right] = \frac{x}{x^2 + y^2}$$

Hence, the Jacobian of the transformation is

$$J = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}.$$

The joint density of R, θ is thus

$$\begin{aligned} f_{R,\theta}(s,\theta) &= s f_{X,Y}\left[\sqrt{x^2+y^2}, \tan^{-1} y/x\right] \\ &= \frac{s}{\pi r^2} \\ &= \frac{1}{2\pi} \cdot \frac{2s}{r^2}, \quad 0 < \theta < 2\pi, \quad 0 < s < r. \end{aligned}$$

Hence, R and θ are independent with

$$f_R(s) = \frac{2s}{r^2}, \quad 0 < s < r,$$

$$f_\theta(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi.$$

As $F_R(s) = \dfrac{2s}{r^2}$ and so $F_R^{-1}(U) = \sqrt{r^2 U} = r\sqrt{U}$, it follows that we can generate R, θ by letting U_1 and U_2 be random numbers and then setting $R = r\sqrt{U_1}$ and $\theta = 2rU_2$.

(b) It is clear that the accepted point is uniformly distributed in the desired circle. Since

$$P\left\{Z_1^2 + Z_2^2 \le r^2\right\} = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}$$

it follows that the number of iterations needed (or equivalently that one-half the number of random numbers needed) is geometric with mean $\pi/4$.

7. Use the rejection method with $g(x) = 1$. Differentiating $f(x)/g(x)$ and equating to 0, gives the 2 roots 1/2 and 1. As $f(.5) = 30/16 > f(1) = 0$, we see that $c = 30/16$, and so the algorithm is

Step 1: Generate random numbers U_1 and U_2.

Step 2: If $U_2 \le 16(U_1^2 - 2U_1^3 + U_1^4)$, set $X = U_1$. Otherwise return to Step 1.

8. (a) With $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}$

and $g(x) = \dfrac{\lambda e^{-\lambda x/n}}{n}$,

$$f(x)/g(x) = \frac{n(\lambda x)^{n-1} e^{-\lambda x(1-1/n)}}{(n-1)!}.$$

Differentiating this ratio and equating to 0 yields the equation

$$(n-1)x^{n-2} = x^{n-1}\lambda(1-1/n)$$

or $\quad x = n/\lambda. \quad$ Therefore,

$$c = \max[f(x)/g(x)] = \frac{n^n e^{-(n-1)}}{(n-1)!}.$$

(b) By Stirling's approximation

$$(n-1)! \approx (n-1)^{n-1/2} e^{-(n-1)} (2\pi)^{1/2},$$

and so

$$\begin{aligned} & n^n e^{-(n-1)}/(n-1) \\ &\approx (2\pi)^{-1/2}\left[\frac{n}{n-1}\right]^n (n-1)^{1/2} \\ &= \frac{[(n-1)/2\pi]^{1/2}}{(1-1/n)^n} \\ &\approx e[(n-1)/2\pi]^{1/2} \end{aligned}$$

since $(1-1/n)^n \approx e^{-1}$.

(c) Since

$$f(x)/cg(x) = e^{-\lambda x(1-1/n)}(\lambda x)^{n-1}\frac{e^{n-1}}{n^{n-1}},$$

the procedure is

Step 1: Generate Y, an exponential with rate λ/n and a random number U.

Step 2: If $U \le f(Y)/cg(Y)$, set $X = Y$. Otherwise return to Step 1.

The inequality in Step 2 is equivalent, upon taking logs, to

$$\begin{aligned} \log U \le\ & n - 1 - \lambda Y(1-1/n) \\ & + (n-1)\log(\lambda Y) - (n-1)\log n \end{aligned}$$

or

$$\begin{aligned} -\log U \ge\ & (n-1)\lambda Y/n + 1 - n \\ & - (n-1)\log(\lambda Y/n). \end{aligned}$$

Now, $Y_1 = -\log U$ is exponential with rate 1, and $Y_2 = \lambda Y/n$ is also exponential with rate 1. Hence, the algorithm can be written as given in part (c).

(d) Upon acceptance, the amount by which Y_1 exceeds $(n-1)\{Y_2 - \log(Y_2) - 1\}$ is exponential with rate 1.

10. Whenever i is the chosen value that satisfies Lemma 4.1 name the resultant $\underline{Q}$ as $Q^{(i)}$.

12. Let

$$I_j = \begin{cases} 1 & \text{if } X_i = j \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

then

$$D = \sum_{j=1}^{n} I_j$$

and so

$$E[D] = \sum_j = 1^n E[I_j] = \sum_{j=1}^{n} \left[1 - \left[\frac{n-1}{n}\right]^k\right]$$
$$= n\left[1 - \left[\frac{n-1}{n}\right]^k\right]$$
$$\approx n\left[1 - 1 + \frac{k}{n} - \frac{k(k-1)}{2n^2}\right].$$

13. $P\{X = i\} = P\{Y = i | U \leq P_Y/CQ_Y\}$

$$= \frac{P\{Y = i, U \leq P_Y/CQ_Y\}}{K}$$
$$= \frac{Q_i P\{U \leq P_Y/CQ_Y | Y = i\}}{K}$$
$$= \frac{Q_i P_i/CQ_i}{K}$$
$$= \frac{P_i}{CK}$$

where $K = P\{U \leq P_Y/CQ_Y\}$. Since the above is a probability mass function it follows that $KC = 1$.

14. (a) By induction we show that

$(^*)P\{X > k\} = (1 - \lambda(1)) \cdots (1 - \lambda(k)).$

The above is obvious for $k = 1$ and so assume it true. Now

$P\{X > k + 1\}$

$= P\{X > k + 1 | X > k\}P\{X > k\}$

$= (1 - \lambda(k + 1))P\{X > k\}$

which proves (*). Now

$P\{X = n\}$

$= P\{X = n | X > n - 1\}P\{X > n - 1\}$

$= \lambda(n)P\{X > n - 1\}$

and the result follows from (*).

(b) $P\{X = n\}$

$= P\{U_1 > \lambda(1), U_2 > \lambda(2), \ldots, U_{n-1} > \lambda(n-1), U_n \leq \lambda(n)\}$

$= (1 - \lambda(1))(1 - \lambda(2)) \cdots (1 - \lambda(n-1))\lambda(n).$

(c) Since $\lambda(n) \equiv p$ it sets

$X = \min\{n : U \leq p\}.$

That is if each trial is a success with probability p then it stops at the first success.

(d) Given that $X \geq n$, then

$$P\{X = n | X > n\} = P\frac{\lambda(n)}{p} = \lambda(n)$$

15. Use $2\mu = X$.

16. (b) Let I_j denote the index of the j^{th} smallest X_i.

17. (a) Generate the $X_{(i)}$ sequentially using that given $X_{(1)}, \ldots, X_{(i-1)}$ the conditional distribution of $X_{(i)}$ will have failure rate function $\lambda_i(t)$ given by

$$\lambda_i(t) = \begin{cases} 0 & t < X_{(i-1)} \\ & , X_{(0)} \equiv 0. \\ (n - i + 1)\lambda(t) & t > X_{(i-1)} \end{cases}$$

(b) This follows since F is an increasing function the density of $U_{(i)}$ is

$$f_{(i)}(t) = \frac{n!}{(i-1)!(n-i)}(F(t))^{i-1} \times (F(t))^{n-i} f(t)$$
$$= \frac{n!}{(i-1)!(n-i)} t^{i-1}(1-t)^{n-i},$$
$$0 < t < 1$$

which shows that $U_{(i)}$ is beta.

(c) Interpret Y_i as the i^{th} interarrival time of a Poisson process. Now given $Y_1 + \cdots + Y_{n+1} = t$, the time of the $(n+1)^{st}$ event, it follows that the first n event times are distributed as the ordered values of n uniform $(0, t)$ random variables. Hence,

$$\frac{Y_1 + \cdots + Y_i}{Y_1 + \cdots + Y_{n+i}}, \quad i = 1, \ldots, n$$

will have the same distribution as $U_{(1)}, \ldots, U_{(n)}$.

(d) $f_{U_{(1)},\ldots|U_{(n)}}(y_1,\ldots,y_{n-1}|y_n)$

$$= \frac{f(y_1,\ldots,y_n)}{f_{U_{(n)}}(y_n)}$$

$$= \frac{n!}{ny^{n-1}}$$

$$= \frac{(n-1)!}{y^{n-1}}, 0 < y_1 < \cdots < y_{n-1} < y$$

where the above used that

$$F_{U_{(n)}}(y) = P\{\max U_i \leq y\} = y^n$$

and so

$$F_{U_{(n)}}(y) = ny^{n-1}.$$

(e) Follows from (d) and the fact that if $F(y) = y^n$ then $F^{-1}(U) = U^{1/n}$.

18. Consider a set of n machines each of which independently functions for an exponential time with rate 1. Then W_1, the time of the first failure, is exponential with rate n. Also given W_{i-1}, the time of the i^{th} failure, the additional time until the next failure is exponential with rate $n-(i-1)$.

20. Since the interarrival distribution is geometric, it follows that independent of when renewals prior to k occurred there will be a renewal at k with probability p. Hence, by symmetry, all subsets of k points are equally likely to be chosen.

21. $P_{m+1}\{i_1,\ldots,i_{k-1},m+1\}$

$$= \sum_{\substack{j\leq m \\ j\neq i_1,\ldots,i_{k-1}}} P_m\{i_1,\ldots,i_{k-1},j\}\frac{k}{m+1}\frac{1}{k}$$

$$= (m-(k-1))\frac{1}{\begin{bmatrix} m \\ k \end{bmatrix}}\frac{1}{m+1}\frac{1}{\begin{bmatrix} m+1 \\ k \end{bmatrix}}.$$

25. See Problem 4.

27. First suppose $n = 2$.

$$Var(\lambda X_1 + (1-\lambda)X_2) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2.$$

The derivative of the above is $2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2$ and equating to 0 yields that

$$\lambda = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \frac{1/\sigma_1^2}{1/\sigma_1^2+1/\sigma_2^2}.$$

Now suppose the result is true for $n-1$. Then

$$Var\left[\sum_{i=1}^{n}\lambda_i X_i\right] = Var\left[\sum_{i=1}^{n-1}\lambda_i X_i\right] + Var(\lambda_n X_n)$$

$$= (1-\lambda_n)^2\, Var\left[\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}X_i\right]$$

$$+ \lambda_n^2\, Var\, X_n.$$

Now by the inductive hypothesis for fixed λ_n the above is minimized when

$$(^*)\ \frac{\lambda_i}{1-\lambda_n} = \frac{1/\sigma_i^2}{\sum_{j=1}^{n-1} 1/\sigma_j^2}, \quad i = 1,\ldots,n-1.$$

Hence, we now need choose λ_n so as to minimize

$$(1-\lambda_n)^2\frac{1}{\sum_{j=1}^{n-1} 1/\sigma_j^2} + \lambda_n^2\,\sigma_n^2.$$

Calculus yields that this occurs when

$$\lambda_n = \frac{1}{1+\sigma_n^2\sum_{j=1}^{n-1} 1/\sigma_j^2} = \frac{1/\sigma_n^2}{\sum_{j=1}^{n} 1/\sigma_j^2}.$$

Substitution into (*) now gives the result.

28. (a) $E[I] = P\{Y < g(X)\}$

$$= \int_0^1 P\{Y < g(X)|X = x\}dx$$

since $X = U_1$

$$= \int_0^1 \frac{g(x)}{b}dx$$

since Y is uniform $(0, b)$.

(b) $Var(bI) = b^2 Var(I)$

$$= b^2(E[I] - E^2[I]) \text{ since } I \text{ is Bernoulli}$$

$$= b\int_0^1 g(x)dx - \left[\int_0^1 g(x)dx\right]^2.$$

On the other hand
$Var\, g(U) = E[g^2(U)] - E^2[g(U)]$

$$= \int_0^1 g^2(x)dx - \left[\int_0^1 g(x)dx\right]^2$$

$$\leq \int_0^1 bg(x)dx - \left[\int_0^1 g(x)dx\right]^2$$

since $g(x) \leq b$
$= Var(bI)$.

29. Use Hint.

30. In the following, the quantities C_i do not depend on x.

$$f_t(x) = C_1 e^{tx} e^{-(x-\mu)^2/(2\sigma)}$$
$$= C_2 \exp\{-(x^2 - (2\mu + 2t\sigma^2)x)/(2\sigma)\}$$
$$= C_3 \exp\{-(x - (\mu + t\sigma^2))^2/(2\sigma)\}$$

31. Since $E[W_n|D_n] = D_n + \mu$, it follows that to estimate $E[W_n]$ we should use $D_n + \mu$. Since $E[D_n|W_n] \neq W_n - \mu$, the reverse is not true and so we should use the simulated data to determine D_n and then use this as an estimate of $E[D_n]$.

32. $Var[(X+Y)/2]$

$$= \frac{1}{4}[Var(X) + Var(Y) + 2Cov(X, Y)]$$
$$= \frac{Var(X) + Cov(X, Y)}{2}.$$

Now it is always true that

$$\frac{Cov(V, W)}{\sqrt{Var(V)Var(W)}} \leq 1$$

and so when X and Y have the same distribution

$Cov(X, Y) \leq Var(X)$.

33. (a) $E[X^2] \leq E[aX] = aE[X]$

(b) $Var(X) = E[X^2] - E^2[X] \leq aE[X] - E^2[X]$

(c) From (b) we have that

$$Var(X) \leq a^2 \left(\frac{E[X]}{a}\right)$$
$$\left(1 - \frac{E[X]}{a}\right) \leq a^2 \max_{0<p<1} p(1-p) = a^2/4$$

34. Use the estimator $R + X_Q E[S]$. Let A be the amount of time the person in service at time t_0 has already spent in service. If $E[R|A]$ is easily computed, an even better estimator is $E[R|A] + X_Q E[S]$.