د/ برهان

الواجب المنزلي الثاني 209 ريض

س ١ لتكن f دالة معرفة بـ

$$f(x) = \begin{cases} x \ ; & 0 \le x < 1 \\ 1 \ ; & 1 \le x \le 2 \\ 0 \ ; & x < 0 \text{ أو } x > 2 \end{cases}$$

(أ) ارسم الدالة f على $\mathbb{R}$.

(ب) أوجد تكامل فورييه للدالة f.

(ج) برهن باستخدام تكامل فورييه عند $x=1$، صحة العلاقة التالية

$$\int_0^\infty \left(\frac{1-\cos\alpha}{\alpha^2} + \frac{\sin\alpha}{\alpha} \right) d\alpha = \pi$$

(مع العلم $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$)

س ٢ بيّن أن

$$f(x,y) = xy \ln\left(\frac{1}{x} - \frac{1}{y}\right)$$

حيث $x>0$ و $y>0$

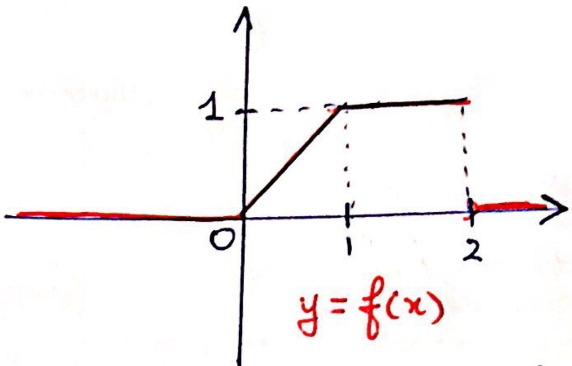
تحقق المعادلة التالية: $x^2 \dfrac{\partial f}{\partial x} + y^2 \dfrac{\partial f}{\partial y} = (x+y) f$

س ٣ حل المعادلة التفاضلية التالية

$$y \ln x \, dx = \left(\frac{y+1}{x}\right)^2 dy$$

تصحيح الواجب المنزلي الثاني (د ابرهان)

س ١ (أ)



ب) f مستمرة على $\mathbb{R}\setminus\{2\}$ فان نستطيع f

بتكامل فورية لكل $x \neq 2$

$$f(x) = \frac{1}{\pi}\int_0^{+\infty}\left[A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right] d\alpha$$

حيث:

$$A(\alpha) = \int_{\mathbb{R}} f(x)\cos(\alpha x)\,dx$$

$$A(\alpha) = \int_0^1 x\cos(\alpha x)\,dx + \int_1^2 \cos(\alpha x)\,dx$$

- $\int_0^1 x\cos(\alpha x)\,dx = \frac{1}{\alpha}\left[x\sin(\alpha x)\right]_0^1 - \frac{1}{\alpha}\int_0^1 \sin(\alpha x)\,dx = \frac{1}{\alpha}\left[\sin\alpha\right] + \frac{1}{\alpha^2}\left[\cos(\alpha x)\right]_0^1$

$u(x) = x$, $v'(x) = \cos(\alpha x)$ $\Rightarrow$ $u'(x) = 1$, $v(x) = \frac{\sin(\alpha x)}{\alpha}$

- $\int_1^2 \cos(\alpha x)\,dx = \frac{1}{\alpha}\left[\sin(\alpha x)\right]_1^2 = \frac{1}{\alpha}\left[\sin(2\alpha) - \sin\alpha\right]$

$$A(\alpha) = \cancel{\frac{\sin\alpha}{\alpha}} + \frac{\cos\alpha - 1}{\alpha^2} + \frac{\sin(2\alpha)}{\alpha} - \cancel{\frac{\sin\alpha}{\alpha}}.$$

$$\boxed{A(\alpha) = \frac{\cos\alpha - 1}{\alpha^2} + \frac{\sin(2\alpha)}{\alpha}}$$

$$B(\alpha) = \int_{\mathbb{R}} f(x)\sin(\alpha x)\,dx$$

$$B(\alpha) = \int_0^1 x\sin(\alpha x)\,dx + \int_1^2 \sin(\alpha x)\,dx$$

- $\int_0^1 x\sin(\alpha x)\,dx = -\frac{1}{\alpha}\left[x\cos(\alpha x)\right]_0^1 + \frac{1}{\alpha}\int_0^1 \cos(\alpha x)\,dx.$

$u(x) = x$, $v'(x) = \sin(\alpha x)$ $\Rightarrow$ $u'(x) = 1$, $v(x) = -\frac{\cos(\alpha x)}{\alpha}$

$$B(\alpha) = -\frac{1}{\alpha}\left[\cos\alpha - 0\right] + \frac{1}{\alpha^2}\left[\sin(\alpha x)\right]_0^1 - \frac{1}{\alpha}\left[\cos(\alpha x)\right]_1^2$$

$$= -\frac{\cos\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2} - \frac{1}{\alpha}\left[\cos(2\alpha) - \cos\alpha\right]$$

$$\boxed{B(\alpha) = \frac{-\cos(2\alpha)}{\alpha} + \frac{\sin\alpha}{\alpha^2}}$$

وبالتالي لكل $0 \le x < 1$

$$x = \frac{1}{\pi}\int_0^{+\infty} \left(\frac{\cos\alpha - 1}{\alpha^2} + \frac{\sin(2\alpha)}{\alpha}\right)\cos(\alpha x) + \left(\frac{-\cos(2\alpha)}{\alpha} + \frac{\sin\alpha}{\alpha^2}\right)\sin(\alpha x)\, d\alpha$$

ولكل $1 \le x < 2$

$$1 = \frac{1}{\pi}\int_0^{+\infty}\left[\left(\frac{\cos\alpha - 1}{\alpha^2} + \frac{\sin(2\alpha)}{\alpha}\right)\cos(\alpha x) + \left(\frac{-\cos(2\alpha)}{\alpha} + \frac{\sin\alpha}{\alpha^2}\right)\sin(\alpha x)\right] d\alpha$$

(ج) بما أن f متصلة عند $x=1$ فإن لدينا

$$1 = \frac{1}{\pi}\int_0^{+\infty}\left[\left(\frac{\cos\alpha - 1}{\alpha^2} + \frac{\sin(2\alpha)}{\alpha}\right)\cos\alpha + \left(\frac{-\cos 2\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2}\right)\sin\alpha\right] d\alpha$$

يعني

$$\pi = \int_0^{+\infty}\left[\frac{\cos^2\alpha - \cos\alpha + \sin^2\alpha}{\alpha^2} + \frac{\sin(2\alpha)\cos(\alpha) - \cos(2\alpha)\sin\alpha}{\alpha}\right] d\alpha$$

$$\pi = \int_0^{+\infty}\left[\frac{1-\cos\alpha}{\alpha^2} + \frac{\sin\alpha}{\alpha}\right] d\alpha$$

لأن $\sin(2\alpha)\cos\alpha - \cos(2\alpha)\sin\alpha = \sin\alpha$

س٢ لكل $x>0$ و $y>0$ ؛ $f(x,y) = xy\ln\left(\frac{1}{x} - \frac{1}{y}\right)$

- $\frac{\partial f}{\partial x}(x,y) = y\ln\left(\frac{1}{x} - \frac{1}{y}\right) + (xy)\frac{-1/x^2}{\frac{1}{x} - \frac{1}{y}}$

$$= y\ln\left(\frac{1}{x} - \frac{1}{y}\right) + \cancel{(xy)}\frac{\cancel{(xy)}(-1/x^2)}{y-x}$$

- $\frac{\partial f}{\partial y}(x,y) = x\ln\left(\frac{1}{x} - \frac{1}{y}\right) + xy\frac{1/y^2}{1/x - 1/y}$

$$= x\ln(1/x - 1/y) + (xy)\frac{(xy)\ 1/y^2}{\ldots}$$

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عندئذ

$$x^2\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}=x^2\left[y\ln\left(\frac{1}{x}-\frac{1}{y}\right)-\frac{1}{x^2(y-x)}\right]+y^2\left[x\ln\left(\frac{1}{x}-\frac{1}{y}\right)+\frac{1}{y^2(y-x)}\right]$$

$$x^2\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}=x^2y\ln\left(\frac{1}{x}-\frac{1}{y}\right)-\cancel{\frac{1}{y-x}}+y^2x\ln\left(\frac{1}{x}-\frac{1}{y}\right)+\cancel{\frac{1}{y-x}}$$

$$=(x+y)\underbrace{xy\ln\left(\frac{1}{x}-\frac{1}{y}\right)}$$

$$x^2\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}=(x+y)\quad f(x,y)$$

س ٣

$$y\ln x\,dx=\frac{(y+1)^2}{x^2}dy$$

$$\boxed{x^2\ln x\,dx=\frac{(y+1)^2}{y}dy}$$

وهي معادلة تفاضلية ذات متحولات منفصلة

بتكامل طرفي المعادلة،

$$\int x^2\ln x\,dx=\int\frac{(y+1)^2}{y}dy$$

تكامل بالتجزئة

$u'(x)=x^2 \Rightarrow u(x)=\frac{x^3}{3}$

$v(x)=\ln x \quad v'(x)=1/x$

$$\frac{x^3}{3}\ln x-\int\frac{x^3}{3}\cdot\frac{1}{x}dx=\int\frac{y^2+2y+1}{y}dy$$

$$\frac{x^3}{3}\ln x-\frac{1}{3}\int x^2dx=\int\left(y+2+\frac{1}{y}\right)dy$$

$$\frac{x^3}{3}\ln x-\frac{x^3}{9}=\frac{y^2}{2}+2y+\ln|y|+c$$

الحل العام للمعادلة التفاضلية هو:

$$\boxed{\frac{x^3}{3}\ln x-\frac{x^3}{9}-\frac{y^2}{2}-2y-\ln y=cst}$$