King Saud University
Department of Mathematics

1st Midterm Exam in Math151
Semester 1, 1439/1440 H.

Q1. (a) Construct the truth table of the proposition $(p \vee \neg q) \rightarrow (\neg q \leftrightarrow r)$. (3 pts)

(b) Without using truth tables, show that $p \rightarrow [(p \wedge q) \vee \neg q]$ is a tautology. (3 pts)

Q2. (a) Let a and b be real numbers. Prove by contraposition that if $a + 2b > 10$, then $a > 4$ or $b > 3$. (2 pts)

(b) Assume that $\sqrt{7}$ is irrational. Give a proof by contradiction to show that $\frac{2+\sqrt{7}}{3}$ is irrational. (2 pts)

Q3. (a) Prove that $3^{n-1} \geq 2^n + 1$ for all integers $n \geq 3$. (4 pts)

(b) Let $\{u_n\}$ be a sequence defined by the equations $u_1 = 0$, $u_2 = 1$ and $u_{n+1} = 3u_n - 2u_{n-1} - 1$ for $n = 2, 3, 4, \ldots$. Show that $u_n = n - 1$ for all $n \geq 1$. (4 pts)

Q4. (a) Let R be the relation from $A = \{0, 1, 2, 3, 4\}$ to $B = \{1, 2, 3, 4, 5\}$ defined as follows:

$$aRb \Longleftrightarrow a > b.$$

(i) List all ordered pairs of R. (2 pts)
(ii) Represent the relation R by a matrix. (1 pt)

(b) Let $S = \{(x, z), (y, x), (y, y), (z, y)\}$ be a relation on the set $C = \{x, y, z\}$.
(i) Find $S^{-1} \circ S$. (2 pts)
(ii) Find S^2. (2 pts)

a) $A : (p \vee \neg q) \rightarrow (\neg q \Leftrightarrow r)$

p	q	r	$\neg q$	$p \vee \neg q$	$\neg q \Leftrightarrow r$	A
T	T	T	F	T	F	F
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	F	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F
			0,5	0,5	1	1

b)

$p \rightarrow [(p \wedge q) \vee \neg q] \equiv$

0,5 $\neg p \vee [(p \wedge q) \vee \neg q] \equiv$

$(\neg p \vee \neg q) \vee (p \wedge q) \equiv \neg(p \wedge q) \vee (p \wedge q)$

$\equiv \neg A \vee A \equiv T$ 1,5

We suppose that $a \leq 4$ and $b \leq 3$

Then $a + 2b \leq 4 + 2 \times 3$

$a + 2b \leq 10$

So if $a + 2b > 10$ then $a > 4$ or $b > 3$

We suppose that $\frac{2+\sqrt{7}}{3}$ is rational. 1

put $x = \frac{2+\sqrt{7}}{3}$ then $3x = 2 + \sqrt{7}$ 0,5

$\sqrt{7} = 3x - 2 \in \mathbb{Q}$

as addition of 2 rational numbers

This is a contradiction with $\sqrt{7} \notin \mathbb{Q}$. 0,5

Q3) (a) $P(n) : 3^{n-1} \geq 2^n + 1$

Basis Step $n = 3$

$9 = 3^{3-1} \overset{?}{\geq} 2^3 + 1 = 9$ 1

So $P(3)$ is true

Induction step Let $k \geq 4$. We assume that $P(k)$ is true $(3^{k-1} \geq 2^k + 1)$

1 Now we prove that $P(k+1)$ remains true.

$3^k \overset{?}{\geq} 2^{k+1} + 1$

$3^k = 3 \cdot 3^{k-1} \geq 3 \cdot (2^k + 1)$

$\geq 3 \cdot 2^k + 3 \geq \boxed{2 \cdot 2^k + 1 = 2^{k+1} + 1}$

$3 \times 2^k \geq 2 \cdot 2^k$ 2

$3 \geq 1$

We deduce that $P(n)$ is true for $n \geq 3$

(b) Put $P(n)$: $U_n = n - 1$

Basis step : $n = 1$ | $n = 2$

$0 = U_1 \overset{?}{=} 1 - 1$ | $1 = U_2 \overset{?}{=} 2 - 1$ 1

So $P(1)$ & $P(2)$ are true.

Induction Step: Let $k \geq 3$. We assume that $P(3), \ldots, P(k)$ are all true.

and we prove that $P(k+1)$ remains true. 1

$U_{k+1} \overset{?}{=} (k+1) - 1 = k$

As $U_{k+1} = 3U_k - 2U_{k-1} - 1$ 0,5

Also $U_k = k - 1$ and $U_{k-1} = k - 2$ 0,5

because $P(k)$ and $P(k-1)$ are true.

So $U_{k+1} = 3(k-1) - 2(k-2) - 1$ 2

$= 3k - \cancel{3} - 2k + \cancel{4} - \cancel{1}$

$= k$ ✓

So $P(n)$ is true for $n \geq 1$.

Q4) (a) (i) $R = \{(2,1); (3,1); (3,2); (4,1); (4,2); (4,3)\}$ 2

(ii) $M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 & 0 \end{array}$ 1

(b) $S = \{(x,z); (y,x); (y,y); (z,y)\}$

$S^{-1} = \{(z,x); (x,y); (y,y); (y,z)\}$ 2

(i) $S^{-1} \circ S = \{(x,x); (y,y); (y,z); (z,y); (z,z)\}$

(ii) $S^2 = S \circ S = \{(x,y); (y,z); (y,x); (y,y); (z,x); (z,y)\}$ 2

$S = \{(x,z); (y,x); (y,y); (z,y)\}$

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