KING SAUD UNIVERSITY
Math Department

Math106 Midterm2 Summer term Time : 1H30

Question 1 : (3+3+3)
Evaluate the following integrals

1. $\displaystyle\int \frac{1}{2x\sqrt{1-9x^2}}\ dx$

   $u = 3x \Rightarrow du = 3\ dx$ and $x = \frac{u}{3}$

   $$\int \frac{1}{2x\sqrt{1-9x^2}}\ dx = \frac{1}{3}\int \frac{1}{\frac{2}{3}u\sqrt{1-u^2}}\ du$$
   $$= \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}\ du = -\frac{1}{2}sech^{-1}u + C$$
   $$= -\frac{1}{2}sech^{-1}3x + C$$

2. $\displaystyle\int 4x\cos(2-3x)\ dx$

   $u = 4x \quad \rightarrow \quad du = 4dx$

   $dv = \cos(2-3x)\ dx \quad \rightarrow \quad v = -\frac{1}{3}\sin(2-3x)$

   $$\int 4x\cos(2-3x)\ dx = (4x)\left(-\frac{1}{3}\sin(2-3x)\right) - \int -\frac{4}{3}\sin(2-3x)\ dx$$
   $$= -\frac{4}{3}x\sin(2-3x) + \frac{4}{3}\int \sin(2-3x)\ dx$$

   $$\int 4x\cos(2-3x)\ dx = -\frac{4}{3}x\sin(2-3x) + \frac{4}{9}\cos(2-3x) + c$$

3. $\displaystyle\int \sec^9 x\tan^5 x\,dx$

   $$\int \sec^9 x\tan^5 x\,dx = \int \sec^8 x\tan^4 x\ \tan x\sec x\,dx$$
   $$= \int \sec^8 x\left(\sec^2 x - 1\right)^2\ \tan x\sec x\,dx \qquad u = \sec x$$
   $$= \int u^8\left(u^2-1\right)^2 du$$
   $$= \int u^{12} - 2u^{10} + u^8\,du$$
   $$= \frac{1}{13}\sec^{13}x - \frac{2}{11}\sec^{11}x + \frac{1}{9}\sec^9 x + c$$

Question 2 : (3+3+3)

1. Evaluate the following integral.
$\int \frac{x+1}{x\,(x^2+4)}\,dx$
$\frac{x+1}{x\,(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow x+1 = Ax^2 + 4A + Bx^2 + Cx$
$A+B=0,\ C=1,$ and $4A=1 \Rightarrow A=\frac{1}{4},\ B=\frac{-1}{4}$ and $C=1$

$$\begin{aligned}\int \frac{x+1}{x\,(x^2+4)}\,dx &= \int \frac{1}{4x} - \frac{x}{4\,(x^2+4)} + \frac{1}{x^2+4}\,dx\\ &\int \frac{1}{4x} - \int \frac{x}{4\,(x^2+4)} + \int \frac{1}{x^2+4}\,dx\\ &= \frac{1}{4}\ln|x| + \frac{1}{8}\ln\left|x^2+4\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c\end{aligned}$$

2. Determine if each of the following integral is convergent or divergent. If it is convergent find its value.
$a)\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx,$ $\qquad b)\int_{0}^{3} \frac{1}{\sqrt{3-x}}\,dx$

$$\begin{aligned}a)\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_{t}^{0}\\ &= \lim_{t\to-\infty}\left(-2\sqrt{3}+2\sqrt{3-t}\right)\\ &= -2\sqrt{3}+\infty\\ &= \infty\end{aligned}$$

So, the limit is infinite and so this integral is divergent.

$$\begin{aligned}b)\int_{0}^{3} \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to3^-}\int_{0}^{t} \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to3^-}\left(-2\sqrt{3-x}\right)\Big|_{0}^{t}\\ &= \lim_{t\to3^-}\left(2\sqrt{3}-2\sqrt{3-t}\right)\\ &= 2\sqrt{3}\end{aligned}$$

The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$.

Question 3 : (3+4)

1. Evaluate the following limit. $\lim\limits_{x\to 0^+} x\ln x$

   $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{1/x}$

   $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{1/x} = \lim\limits_{x\to 0^+} \frac{1/x}{-1/x^2}$

   $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim\limits_{x\to 0^+} (-x) = 0$

2. Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$

$$2x^2 + 10 = 4x + 16$$
$$2x^2 - 4x - 6 = 0$$
$$2(x+1)(x-3) = 0$$

   So the intersection points $x = -1$ and $x = 3$

$$\begin{aligned} A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\ &= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx \\ &= \int_{-1}^{3} -2x^2 + 4x + 6\, dx \\ &= \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Big|_{-1}^{3} \\ &= \frac{64}{3} \end{aligned}$$