toc

9. Presentation of the Theory

In this section, we present the demonstration of many of the theorems discussed in the tutorial.

Theorem 9.1. *Let a and c be numbers, then*

$$\boxed{\lim_{x \to a} c = c.} \qquad \text{(Rule 1)}$$

Proof: Let $\epsilon > 0$. In reference to the definition of limit, the function under consideration is $f(x) = c$, and $L = c$. We want to choose a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

or,

$$0 < |x - a| < \delta \implies |c - c| < \epsilon \tag{1}$$

It is clear in this trivial situation that the condition $|c - c| < \epsilon$ will hold *no matter the choice of* $\delta > 0$; therefore, choose $\delta = 1$. Thus, for that choice of δ, obviously, (1) holds. $\square$

Theorem 9.2. *For any number a,*

$$\boxed{\lim_{x \to a} x = a.} \tag{Rule 2}$$

Proof: Within the context of the definition of limit of a function, $f(x) = x$ and $L = a$. Let $\epsilon > 0$. We want to find a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

or,

$$0 < |x - a| < \delta \implies |x - a| < \epsilon \tag{2}$$

Towards that end, choose $\delta = \epsilon$, then

$$0 < |x - a| < \delta \implies |x - a| < \delta = \epsilon.$$

But this is exactly what we wanted to prove, (2). $\square$

Theorem 9.3. (Algebra of Limits Theorem) *Let f and g be functions and let a and c be number. Suppose*

$$\lim_{x \to a} f(x), \;\; \textit{and} \;\; \lim_{x \to a} g(x)$$

exist and are finite. Then,

(1) $\displaystyle\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$;
(2) $\displaystyle\lim_{x \to a}(cf(x)) = c \lim_{x \to a} f(x)$;
(3) $\displaystyle\lim_{x \to a}(f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$;
(4) $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, *provided,* $\displaystyle\lim_{x \to a} g(x) \neq 0$.

Throughout the proofs below, let $L = \lim_{x \to a} f(x)$ and $M = \lim_{x \to a} g(x)$.

Proof of (1). Let $\epsilon > 0$. We must find a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |(f(x) + g(x)) - (L + M)| < \epsilon.$$

Towards that end, there exists a $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\epsilon}{2}, \tag{3}$$

since $\lim_{x \to a} f(x) = L$. And there exists a $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{\epsilon}{2}, \tag{4}$$

Finally, define $\delta = \min\{\delta_1, \delta_2\}$. Note that

$$0 < |x - a| < \delta \implies 0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2 \tag{5}$$

Now suppose x is a number in the domains of both f and g such that

$$0 < |x - a| < \delta. \tag{6}$$

Then,

$$
\begin{aligned}
&|(f(x)+g(x))-(L+M)| \\
&\qquad = |(f(x)-L)+(g(x)-M)| \\
&\qquad \le |f(x)-L|+|g(x)-M| \quad \triangleleft \text{ Abs. (1)} \\
&\qquad < \frac{\epsilon}{2}+\frac{\epsilon}{2} \qquad (7) \\
&\qquad = \epsilon.
\end{aligned}
$$

Thus,

$$|(f(x)+g(x))-(L+M)| < \epsilon$$

Since we are assuming (6), the inequalities on the right-hand side of (5). Since those inequalities are obtained, then the right-hand sides of (3) and (4) are true as well. This is were we obtained the inequality in (7).

We have shown that

$$0 < |x-a| < \delta \implies |(f(x)+g(x))-(L+M)| < \epsilon.$$

This is the definition of $\lim_{x \to a}(f(x)+g(x)) = L+M$. $\square$

Proof of (2). Let $\epsilon > 0$. We need to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |(cf(x)) - cL| < \epsilon. \tag{8}$$

Now, since $\lim_{x \to a} f(x) = L$, and $\epsilon > 0$ has been given, from the definition of limit, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \frac{\epsilon}{1 + |c|}. \tag{9}$$

Now suppose,

$$0 < |x - a| < \delta$$

then,

$$\begin{aligned} |(cf(x)) - cL| &= |c||f(x) - L| && \triangleleft \text{ Abs. (3)} \\ &< |c|\frac{\epsilon}{1 + |c|} \\ &= \frac{|c|}{1 + |c|}\epsilon \\ &\leq \epsilon. \end{aligned}$$

Thus,

$$|(cf(x)) - cL| < \epsilon$$

Note: We utilized the fact that $\dfrac{|c|}{1+|c|} \leq 1$, for any number $c \in \mathbb{R}$. The denominator, $1+|c|$, was chosen instead of $|c|$ to account for the possibility that c might be zero.

Thus, we have shown that for the $\delta > 0$ produced in (9), (8) is obtained. $\square$

Proof of (3). Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)g(x) - LM| < \epsilon. \tag{10}$$

Consider the following series of manipulations:

$$\begin{aligned}
&|f(x)g(x) - LM| \\
&\qquad = |f(x)g(x) - f(x)M + f(x)M - LM| \\
&\qquad \leq |f(x)g(x) - f(x)M| + |f(x)M - LM| && \triangleleft \text{ Abs. (1)} \\
&\qquad = |f(x)||g(x) - M| + |M||f(x) - L|. && \triangleleft \text{ Abs. (3)}
\end{aligned} \tag{11}$$

We can make the second term in (11) "small" since $\lim_{x\to a} f(x) = L$:

Choose $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - L| < \frac{\epsilon}{2(1 + |M|}. \tag{12}$$

Now for the problem of the first term of (11). We can make the factor $|g(x) - M|$ as small as we wish since $\lim_{x\to a} g(x) = M$ but we have to make sure that the smallest of this factor is not counter-balanced by the factor $|f(x)|$. (That factor *might* be large — we have to make sure that it is not.) To that end, for $0 < |x - a| < \delta_1$, we have, by (12),

$$|f(x) - L| < \frac{\epsilon}{2(1 + |M|)}$$

then,

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \leq |f(x) - L| + |L| \qquad \triangleleft \text{ Abs. (1)} \\ &\leq \frac{\epsilon}{2(1 + |M| + |L|)} \end{aligned}$$

For the purpose of convenience, let $C = \dfrac{\epsilon}{2(1+|M|+|L|)}$. Thus, there is a constant, C, such that

$$0 < |x-a| < \delta_1 \implies |f(x)| < C. \tag{13}$$

Continuing our quest for the ultimate $\delta > 0$ for which (10) is valid, choose $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |g(x) - M| < \frac{\epsilon}{2(1+C)}. \tag{14}$$

This is possible since $\lim\limits_{x \to a} g(x) = M$.

Finally, choose the ultimate δ-value as

$$\delta = \min\{\delta_1, \delta_2\}, \tag{15}$$

and suppose $0 < |x-a| < \delta$. Then $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ form (15) and, as a consequence, the inequalities in (12), (14) and (13) are valid. Thus, for $0 < |x-a| < \delta$, we have,

$$\begin{aligned}
&|f(x)g(x) - LM| \\
&\qquad \le |f(x)||g(x) - M| + |M||f(x) - L| && \triangleleft\ (11) \\
&\qquad < C\frac{\epsilon}{2(1+C)} + |M|\frac{\epsilon}{2(1+|M|)} && \triangleleft\ (13),\ (14),\ (12) \\
&\qquad < \frac{C}{1+C}\frac{\epsilon}{2} + \frac{|M|}{1+|M|}\frac{\epsilon}{2} \\
&\qquad \le \epsilon.
\end{aligned}$$

Thus, we have shown that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)g(x) - LM| < \epsilon.$$

This completes the proof of this part. $\square$

Proof of (4). Let $\epsilon > 0$. From the definition, we want to find a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| < \epsilon,$$

here, we are assuming $M \neq 0$.

Towards this end, consider the following:

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &= \left| \frac{f(x)M - g(x)L}{Mg(x)} \right| \\
&= \left| \frac{(f(x)M - LM) + (LM - g(x)L)}{Mg(x)} \right| \\
&= \left| \frac{M(f(x) - L) + L(M - g(x))}{Mg(x)} \right| \\
&\leq \left| \frac{M(f(x) - L)}{Mg(x)} \right| + \left| \frac{L(M - g(x))}{Mg(x)} \right| && \triangleleft \text{ Abs. (1)} \\
&= \frac{1}{|g(x)|} |f(x) - L| + \frac{|L|}{|Mg(x)|} |g(x) - M| && \triangleleft \text{ Abs. (3)}
\end{aligned}$$

Do you see where I am going with this calculation? To summarize, we have shown that

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| \leq \frac{1}{|g(x)|} |f(x) - L| + \frac{|L|}{|Mg(x)|} |g(x) - M| \tag{16}$$

Now, for some details. The $|g(x)|$ in the denominator is bothersome. Since $\lim_{x\to a} g(x) = M \neq 0$, there is a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |g(x) - M| < \frac{M}{2} \qquad \triangleleft \text{ definition}$$

Now, for $0 < |x-a| < \delta_1$,

$$\begin{aligned} |g(x)| &= |M - (M - g(x))| \\ &\geq |M| - |M - g(x)| \qquad \triangleleft \text{ Abs. (2)} \\ &\geq |M| - \frac{M}{2} \\ &\geq \frac{M}{2}. \end{aligned}$$

This establishes the inequality,

$$\begin{aligned} 0 < |x-a| < \delta_1 &\implies |g(x)| \geq \frac{M}{2} \\ &\implies \frac{1}{|g(x)|} \leq \frac{2}{M} \end{aligned} \tag{18}$$

Let's put this back into the inequality (16) to get

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &\le \frac{1}{|g(x)|}|f(x) - L| + \frac{|L|}{|Mg(x)|}|g(x) - M| \\ &\le \frac{2}{|M|}|f(x) - L| + \frac{2|L|}{M^2}|g(x) - M| \end{aligned} \tag{20}$$

Now for our final calculations! Since $\lim_{x \to a} f(x) = L$, there is a $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |f(x) - L| < \frac{\epsilon|M|}{4}. \tag{21}$$

Since $\lim_{x \to a} g(x) = M$, there is a $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \implies |g(x) - M| < \frac{\epsilon M^2}{4(1 + |L|)}. \tag{22}$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Consequently, if x satisfies the inequality $0 < |x - a| < \delta$, then all three of the inequalities (18), (21), and (22) hold. Thus, if we take x to satisfy,

$$0 < |x - a| < \delta,$$

then, from (20), we have

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &\le \frac{2}{|M|}|f(x) - L| + \frac{2|L|}{M^2}|g(x) - M| \\
&\le \frac{2}{|M|}\frac{\epsilon|M|}{4} + \frac{2|L|}{M^2}\frac{\epsilon M^2}{4(1+|L|)} \qquad \triangleleft \text{ from (21) and (22)} \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

That does it! We have found the $\delta > 0$ that "works." $\square$

Theorem 9.4. (Continuity of Power Functions) *Let* $a \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *Then*

$$\boxed{\lim_{x \to a} x^n = a^n.} \tag{23}$$

Proof: By Theorem 9.2, we have $\lim\limits_{x \to a} x = a$. Therefore, by Theorem 9.3 (3),

$$\lim_{x \to a} x^2 = \lim_{x \to a} xx = \lim_{x \to a} x \lim_{x \to a} x = aa = a^2.$$

Similarly, by Theorem 9.3 (3),

$$\lim_{x \to a} x^2 = \lim_{x \to a} x^2 x = \lim_{x \to a} x^2 \lim_{x \to a} x = a^2 a = a^3.$$

The formal mechanism for finishing the proof in *Mathemtical Induction*: Suppose we have shown that

$$\lim_{x \to a} x^{n-1} = a^{n-1},$$

then, from Theorem 9.3 (3) and Theorem 9.2,

$$\lim_{x \to a} x^n = \lim_{x \to a} x^{n-1} x = \lim_{x \to a} x^{n-1} \lim_{x \to a} x = a^{n-1} a = a^n.$$

By the *Principle of Mathematical Induction*, we have proved the theorem. $\square$

Theorem 9.5. (Continuity of Polynomial Functions) *Let p be a polynomial and $a \in \mathbb{R}$. Then*

$$\boxed{\lim_{x \to a} p(x) = p(a).} \tag{24}$$

Proof: Let $p(x)$ be a polynomial of degree $n \in \mathbb{N}$. This means that the functional form of p is

$$p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_{n-1} x^{n-1} + b_n x^n,$$

for some set of coefficients b_0, b_1, b_2, ... , b_n.

Notice that $p(a) b_0 + b_1 a + b_2 a^2 + b_3 a^3 + \cdots + b_{n-1} a^{n-1} + b_n a^n$.

Now apply Theorem 9.3 (1), Theorem 9.3 (2) as well as Theorem 9.4 to obtain

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_{n-1}x^{n-1} + b_n x^n) \\
&= \lim_{x\to a} b_0 + \lim_{x\to a} b_1 x + \lim_{x\to a} b_2 x^2 + \cdots + \lim_{x\to a} b_n x^n && \triangleleft \text{ (1)} \\
&= b_0 + b_1 \lim_{x\to a} x + b_2 \lim_{x\to a} x^2 + \cdots + b_n \lim_{x\to a} x^n && \triangleleft \text{ (2) Rule 1} \\
&= b_0 + b_1 a + b_2 a^2 + \cdots + b_n a^n && \triangleleft \text{ Thm 9.4} \\
&= p(a) \quad \square
\end{aligned}$$

Theorem 9.6. (Continuity of Rational Functions) *Let f be a rational function, and let $a \in \operatorname{Dom}(f)$. Then*

$$\lim_{x\to a} f(x) = f(a). \tag{25}$$

Proof: A rational function is a quotient of two polynomials. Let p and q be two polynomials such that

$$f(x) = \frac{p(x)}{q(x)}.$$

The natural domain of f is given by

$$\mathrm{Dom}(f) = \{\, x \in \mathbb{R} \mid g(x) \neq 0 \,\}$$

Let $a \in \mathrm{Dom}(f)$; thus, $g(a) \neq 0$. This observation is important as we are about to cite Theorem 9.3 (4). In that theorem, we require the limit of the denominator to be nonzero. Read on.

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \frac{p(x)}{q(x)} \\
&= \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} && \triangleleft \text{ Thm 9.3 (4)} \\
&= \frac{p(a)}{q(a)} && \triangleleft \text{ Thm 9.5}
\end{aligned}$$

Thus,

$$\lim_{x \to a} f(x) = \frac{p(a)}{q(a)} = f(a).$$

This is equation (25) that was asserted in the theorem. □

Theorem 9.7. *Let f and g be functions that are compatible for composition, let $a \in \mathbb{R}$. Suppose,*

(1) $\lim_{x \to a} g(x)$ *exists, let* $b = \lim_{x \to a} g(x)$*;*
(2) $b \in \mathrm{Dom}(f)$*, and* $\lim_{y \to b} f(y) = f(b)$ *exists.*

Then

$$\lim_{x \to a} f(g(x)), \text{ exists}$$

and,

$$\lim_{x \to a} f(g(x)) = f(b),$$

or,

$$\boxed{\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).} \tag{26}$$

Proof: Let $\epsilon > 0$. We want to find a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(b)| < \epsilon. \tag{27}$$

To this end, since $\epsilon > 0$ has been given, and it is assumed that $\lim_{y \to b} f(y) = f(b)$, there is a $\gamma > 0$ such that

$$0 < |y - b| < \gamma \implies |f(y) - f(b)| < \epsilon. \tag{28}$$

Now, we have a number $\gamma > 0$ defined, since we are assuming that $\lim_{x \to a} g(x) = b$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |g(x) - b| < \gamma. \tag{29}$$

Now we claim that the number, $\delta > 0$, produced in the last paragraph is the δ-value we seek. Indeed, suppose

$$0 < |x - a| < \delta$$

Then, from (29),

$$|g(x) - b| < \gamma.$$

But now, this inequality implies, by (28), that

$$|f(g(x)) - f(b)| < \epsilon.$$

This proves the theorem! $\square$

Theorem 9.8. (Continuity of the Root Function) *Let $n \in \mathbb{N}$. Define $f(x) = \sqrt[n]{x}$, for $a \in \mathrm{Dom}(f)$. Then*

$$\lim_{x \to a} f(x) = f(a)$$

or,

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \tag{30}$$

Proof: The key to this proof is the observation that $f(x) = \sqrt[n]{x}$ is an increasing function; i.e.,

$$x_1, x_2 \in \mathrm{Dom}(f) \text{ and } x_1 < x_2 \implies f(x_1) < f(x_2)$$

or,

$$x_1, x_2 \in \mathrm{Dom}(f) \text{ and } x_1 < x_2 \implies \sqrt[n]{x_1} < \sqrt[n]{x_2}$$

Let $a \in \mathrm{Dom}(f)$. Show $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$.

Case I: $a = 0$. Let $\epsilon > 0$, choose $\delta = \epsilon^n$. Then it is obvious that, since $\sqrt[n]{0} = 0$,

$$x \in \mathrm{Dom}(f) \text{ and } |x| < \delta = \epsilon^n \implies |\sqrt[n]{x}| < \epsilon$$

This is the trivial case.

Case II: $a > 0$. Let $\epsilon > 0$. We can assume ϵ is so small that $\sqrt[n]{a} - \epsilon > 0$. (Why?) Then,

$$0 < \sqrt[n]{a} - \epsilon < \sqrt[n]{a} < \sqrt[n]{a} + \epsilon \implies 0 < (\sqrt[n]{a} - \epsilon)^n < a < (\sqrt[n]{a} + \epsilon)^n$$

This is because the function $x \mapsto x^n$ is increasing on the interval $(0, \infty)$; i.e., $0 < x_1 < x_2 \implies x_1^n < x_2^n$.

Define the $\delta > 0$ we are looking for as

$$\delta := \min\{\, a - (\sqrt[n]{a} - \epsilon)^n, (\sqrt[n]{a} + \epsilon)^n - a \,\}$$

Now suppose $0 < |x - a| < \delta$. Notice that such an x must necessarily belong to the domain of f. (Why?) Then

$$\begin{aligned} 0 < |x - a| < \delta &\implies a - \delta < x < a + \delta \\ &\implies (\sqrt[n]{a} - \epsilon)^n < a - \delta < x < a + \delta < (\sqrt[n]{a} + \epsilon)^n \end{aligned} \tag{32}$$

Thus, we have,

$$0 < |x - a| < \delta \implies (\sqrt[n]{a} - \epsilon)^n < x < (\sqrt[n]{a} + \epsilon)^n.$$

Now we use the increasing property of $\sqrt[n]{n}$:

$$\begin{aligned} 0 < |x - a| < \delta &\implies (\sqrt[n]{a} - \epsilon)^n < x < (\sqrt[n]{a} + \epsilon)^n \\ &\implies \sqrt[n]{(\sqrt[n]{a} - \epsilon^n)} < \sqrt[n]{x} < \sqrt[n]{(\sqrt[n]{a} + \epsilon^n)} \\ &\implies \sqrt[n]{a} - \epsilon < \sqrt[n]{x} < \sqrt[n]{a} + \epsilon \\ &\implies |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon. \end{aligned}$$

Thus, we have shown that

$$0 < |x - a| < \delta \implies |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon \tag{33}$$

This proves that $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ in this case.

Case III: $a < 0$. This case is only present with n, the root being extracted, is an *odd integer*. The proof is left to the reader — it is similar to *Case II*. The student need only study the previous case, make appropriate changes in the steps. $\square$

Proof Notes: How can we assume that ϵ is such that $\sqrt[n]{a} - \epsilon > 0$? If ϵ does not satisfy this inequality, then choose another ϵ-value, say ϵ_1 that does satisfy the desired inequality. Now use ϵ_1 throughout the rest of the proof instead of ϵ. We would finished the proof with the declaration that

$$0 < |x - a| < \delta \implies |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon_1,$$

but since $\epsilon_1 < \epsilon$, we would have had

$$0 < |x - a| < \delta \implies |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon_1 < \epsilon,$$

which is (33). That being the case, we might as well assume at the beginning that $a - \epsilon > 0$ and avoid the introduction of ϵ_1.

■ If you go to the trouble of studying the proof given in *Case II*, it would appear that the only property of $\sqrt[n]{x}$ used in the proof was that it was *increasing*. (I used that fact that $x \mapsto x^n$ is increasing as well.) That being the case, can this proof be modified to argue that *any* increasing function f has the property that $\lim_{x \to a} f(x) = f(a)$?

■ Having read the previous paragraph, now consider the function

$$f(x) = \begin{cases} x & x < 0 \\ 1 + x & x \geq 0 \end{cases}$$

This function is strictly increasing over $\mathbb{R}$, but $\lim_{x \to 0} f(x) \neq f(0)$ since the two-sided limit *does not exist*?

■ Can you resolve the seeming contradiction between the two paragraphs? e-mail me with your thoughts.

Exercise 9.1. Draw the graph of $f(x) = \sqrt[n]{x}$, and use it to illustrate the main idea of the proof of *Case II* of Theorem 9.8.

Exercise 9.2. Let $n \in \mathbb{N}$ be odd, and let $a \in \mathbb{R}$ be negative. Prove

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

Corollary 9.9. *Suppose* $\lim_{x \to a} g(x)$ *exists, then*

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}, \tag{34}$$

provided that the number $b := \lim_{x \to a} g(x)$ *is within the domain of the* n^{th}*-root function.*

Proof: All the heavy lifting has been done. We apply Theorem 9.7 with the function f in that theorem as $f(x) = \sqrt[n]{x}$ and the function g in that theorem, the function g in this corollary! Now the point of Theorem 9.8 was that it is a verification of condition (2). Of course, condition (1) is apart of the assumptions of this corollary. Therefore, we can conclude by Theorem 9.7 that

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)},$$

which is (34). $\square$

Theorem 9.10. (Continuity of Algebraic Functions) *Let f be an algebraic function, and let $a \in \mathrm{Dom}(f)$. Then*

$$\boxed{\lim_{x \to a} f(x) = f(a).}$$

Proof: An algebraic function is constructed by sums, differences, products, quotients and compositions with functions of the form:

$$y = c \qquad y = x^n \qquad y = \sqrt[m]{x}.$$

These are constant functions, power functions, and root functions, respectively. For all three, we have shown the property:

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x^n = a^n \qquad \lim_{x \to a} \sqrt[m]{x} = \sqrt[m]{a}.$$

The latter case is true provided a belongs to the domain of the m^{th}-root functions. (See theorems: 9.1, 9.4, and 9.8.)

These observations, combined with the Algebra of Limits Theorem, which concerns sums, differences, products, and quotients of functions whose limits exist, and with Theorem 9.7, which concerns compositions of functions whose limits exist, allows us to make the assertion of the theorem. $\square$

Theorem 9.11. *Let g, f, and h be functions and a, $L \in \mathbb{R}$. Suppose there is some $\delta_0 > 0$ such that*

$$g(x) \leq f(x) \leq h(x) \qquad |x - a| < \delta_0, \tag{35}$$

and,

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\boxed{\lim_{x \to a} f(x) = L.}$$

Proof: Let $\epsilon > 0$.

Since $\lim_{x \to a} g(x) = L$, there is a $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |g(x) - L| < \epsilon.$$

This inequality implies,

$$L - \epsilon < g(x) \text{ whenever, } 0 < |x - a| < \delta_1 \tag{36}$$

(Do you know why?)

Now, since $\lim_{x \to a} h(x) = L$, there is a $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |h(x) - L| < \epsilon.$$

This implies

$$h(x) < L + \epsilon \text{ whenever, } 0 < |x - a| < \delta_2 \tag{37}$$

(Do you know why?)

Finally, choose $\delta = \min\{\, \delta_0, \delta_1, \delta_2 \,\}$, then

$$0 < |x - a| < \delta$$

implies,

$$\underbrace{L - \epsilon <}_{(36)} \underbrace{g(x) \le f(x) \le h(x)}_{(35)} \underbrace{< L + \epsilon}_{(37)}$$

But this implies,

$$|f(x) - L| < \epsilon, \text{ whenever } 0 < |x - a| < \delta.$$

which is what we wanted to prove. $\square$

Theorem 9.12. *The following limits are obtained.*

$$\lim_{x \to 0} \sin(x) = 0 \qquad \lim_{x \to 0} \cos(x) = 1. \tag{38}$$

Proof: Under construction.

Theorem 9.13. *The following limits are obtained.*

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0. \tag{39}$$

Proof: Under construction.

Appendix

Properties of Absolute Value. Let a, b, $c \in \mathbb{R}$, then each of the following inequalities are obtained:

1. $|a+b| \leq |a| + |b|$.
2. $|a-b| \geq |a| - |b|$.
3. $|ab| = |a||b|$.

Solutions to Exercises

9.1. Draw a graph that represents f. Designate a point on the horizontal axis as a, then mark off the corresponding point on the vertical axis — label this point $\sqrt[n]{a}$.

Now mark off a little distance, ϵ, equidistant above and below the point labeled $\sqrt[n]{a}$. These are the points $\sqrt[n]{a} - \epsilon$ and $\sqrt[n]{a} + \epsilon$. Label them so.

Now starting at each of the points $\sqrt[n]{a} - \epsilon$ and $\sqrt[n]{a} + \epsilon$, move horizontally until you hit the graph of f, now move vertically downward until you intersect the x-axis. The two points obtained in this way are $(\sqrt[n]{a} - \epsilon)^n$ and $(\sqrt[n]{a} + \epsilon)^n$, respectively. These two points have the number a between them.

The quantity δ is the shortest distance between a and each of the two points constructed in the previous paragraph. Now mark off a δ-distance on either side of a. Do you see that this δ interval lies completely inside the larger interval? (Because δ is the shortest distance to the endpoints.) Now it is clear, at least geometrically, the validity of the inequality (32). Can you show the inequality (32) algebraically?

Assuming you understand my verbal (bit-tell?) instructions, do you see why

$$0 < |x - a| < \delta \implies |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon?$$

Exercise Notes: Review my question in the *Proof Notes* above. Does this graphical construction give you additional insight into answering the question?

Exercise 9.1. ■

9.2. I said — it is left to the reader! $\mathfrak{DS}$

Study the proof of *Case II* of Theorem 9.8 and make appropriate changes to correspond to $a < 0$.

Exercise 9.2. ■