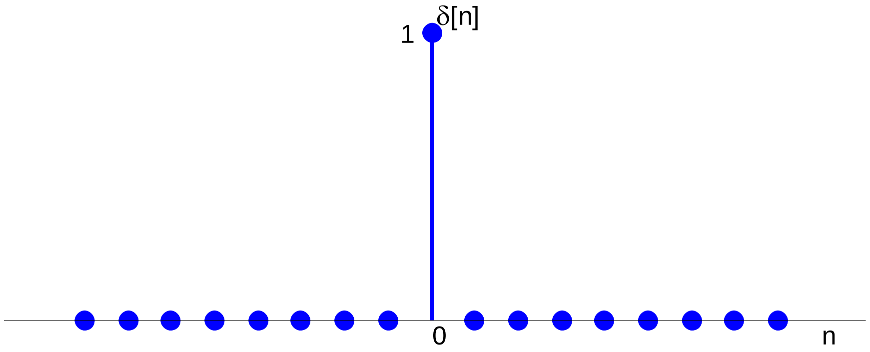
1.4 Unit Step & Unit Impulse Functions

1.4.1 The Discrete-Time Unit Impulse and Unit-Step Sequences

Unit Impulse Function:

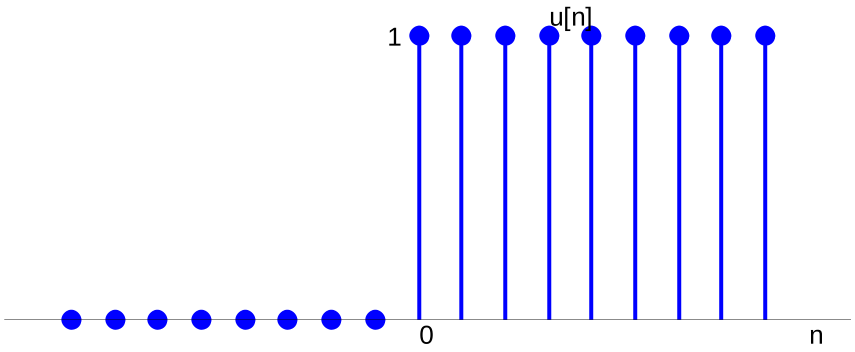
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



Unit Step & Unit Impulse

Unit Step Function:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



There is a close relationship between the discrete-time unit impulse and unit step signals. The discrete-time unit impulse can be written as the first-difference of the discrete-time unit step

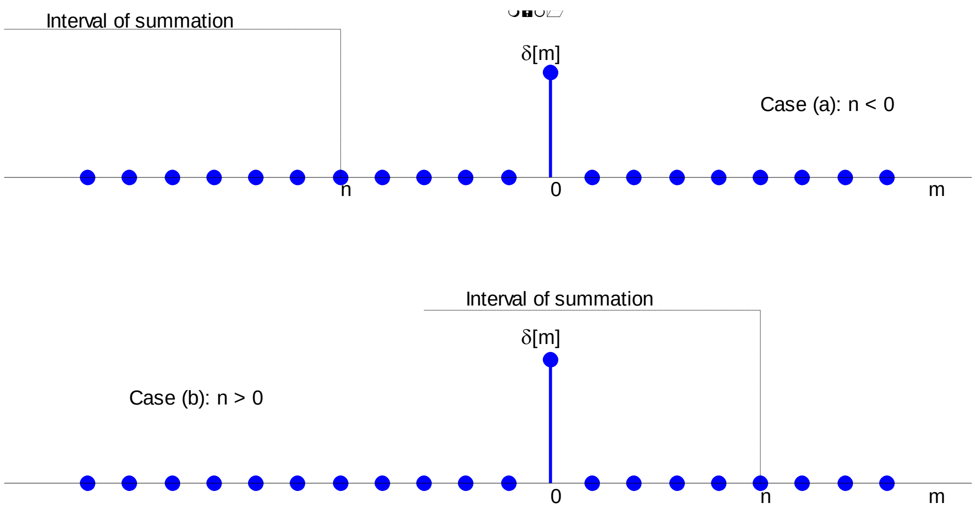
$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the running sum of the unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Unit Step & Unit Impulse - contd.

0 for $n < 0$ and 1 for $n \geq 0$



The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since $\delta[n]$ is non-zero (and equal to 1) only for $n = 0$, therefore

$$x[n]\delta[n] = x[0]\delta[n]$$

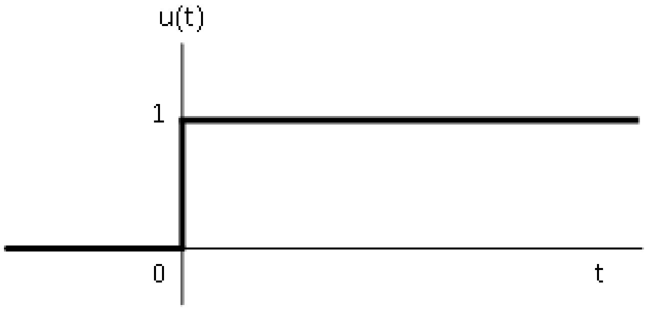
More generally, if we consider a unit impulse $\delta[n - n_0]$ at $n = n_0$, then

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$

1.4.2 The Continuous-Time Unit Impulse and Unit-Step Sequences

The continuous-time unit step function, denoted by $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



The unit step can be written as the running integral of the unit impulse,

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

The unit impulse in the continuous-time can be written as the first derivative of the unit step in continuous time

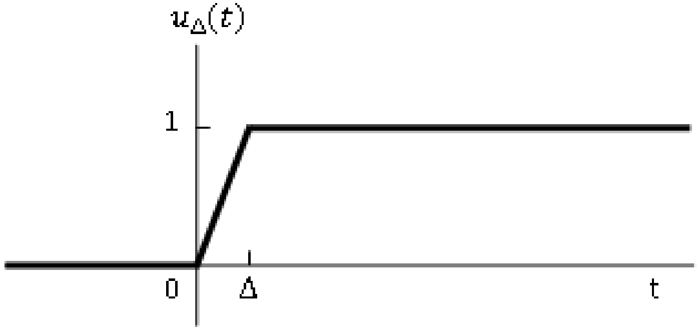
$$\delta(t) = \frac{du(t)}{dt}$$

The Continuous-Time Unit Impulse and Unit-Step Sequences

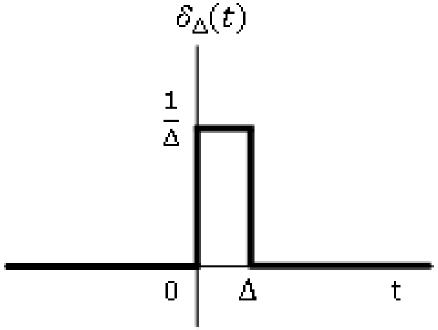
We notice that u(t) is discontinuous at t=0 (and consequently cannot be differentiated at t=0), therefore, there is some formal difficulty with this equation in the previous slide.

Therefore, we interpret equation by considering an approximation to the unit step $u_\Delta(t)$ in which the function rises from 0 to 1 in a short time interval of length Δ. The step function $u(t)$ can be considered as an idealization of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. More formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \rightarrow 0$.

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$



Continuous-time approximation to the unit step function, $u_\Delta(t)$

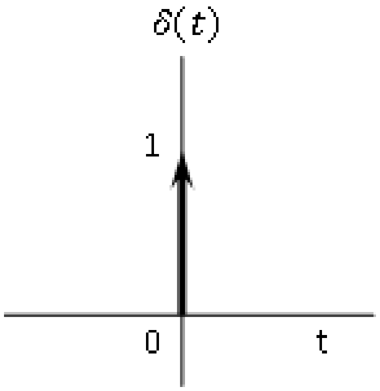


Derivative of $u_\Delta(t)$

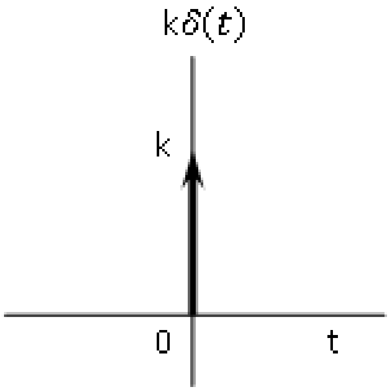
Continuous-Time Unit Impulse

It could be noticed that $\delta_\Delta(t)$ is short pulse of duration Δ and with unit area for any value of Δ. If we gradually decrease the value of Δ, the pulse will become narrower and the height will increase (to maintain the area to unity). Therefore, in the limiting case, we can write

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



Continuous-time unit impulse



Continuous-time scaled impulse

$$\int_{\tau=-\infty}^{\tau=t} k\delta(\tau)d\tau = ku(t)$$

1.12

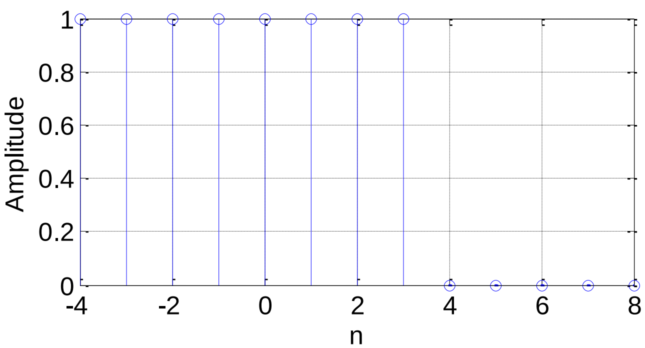
Workout - (11)

For $x[n]=1-\sum_{k=3}^{\infty}\delta[n-1-k]$ Determine the values of M and n_0 so that $x[n]=u[M_n-n_0]$

$$x[n]=1-\left(\delta[n-1-3]+\delta[n-1-4]+\delta[n-1-5]+\ldots+\delta[n-1-\infty]\right)$$
$$=1-\left(\delta[n-4]+\delta[n-5]+\delta[n-6]+\ldots+\delta[n-\infty]\right)$$

$$x[-4]=1-\left(\delta[-8]+\delta[-9]+\delta[-10]+\ldots\right)=1-(0+0+0+\ldots)=1$$
$$x[0]=1-\left(\delta[-4]+\delta[-5]+\delta[-6]+\ldots\right)=1-(0+0+0+\ldots)=1$$
$$x[4]=1-\left(\delta[0]+\delta[-1]+\delta[-2]+\ldots\right)=1-(1+0+0+\ldots)=0$$
$$x[5]=1-\left(\delta[1]+\delta[0]+\delta[-1]+\ldots\right)=1-(0+1+0+\ldots)=0$$

...



u[n] is shifted by +3, and then reflected.

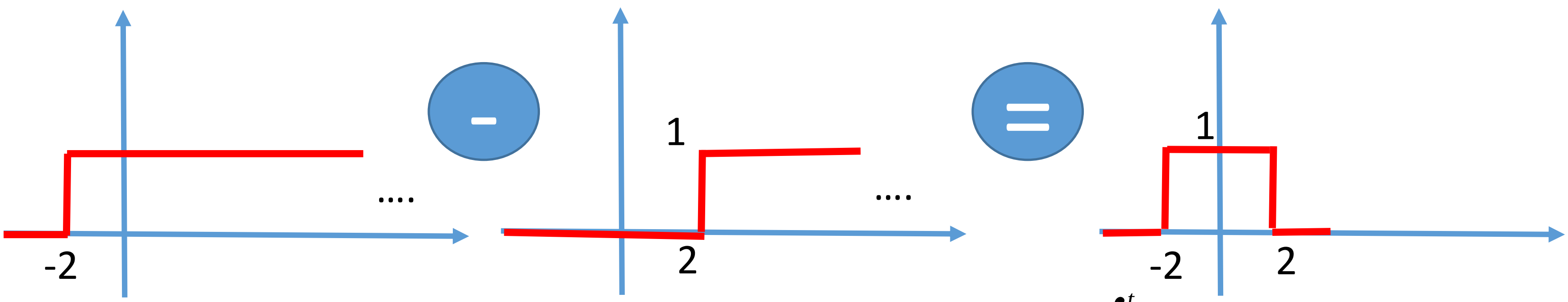
$$x[n]=u[M_n-n_0]=u[-n+3]$$

Therefore, M_n = -1 and n_0 = -3

Workout - (12)

Given $x(t) = \delta(t+2) - \delta(t-2)$ Calculate E_∞ for $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} \big(\delta(\tau+2) - \delta(\tau-2)\big)\,d\tau$$



$$\int_{-\infty}^{t} \big(\delta(\tau+2)\big)\,d\tau \qquad \int_{-\infty}^{t} \big(\delta(\tau-2)\big)\,d\tau \qquad \int_{-\infty}^{t} \big(\delta(\tau+2) - \delta(\tau-2)\big)\,d\tau$$

$$E_\infty = \int_{-2}^{2} |1|^2\,dt = 2 - (-2) = 4$$

1.14

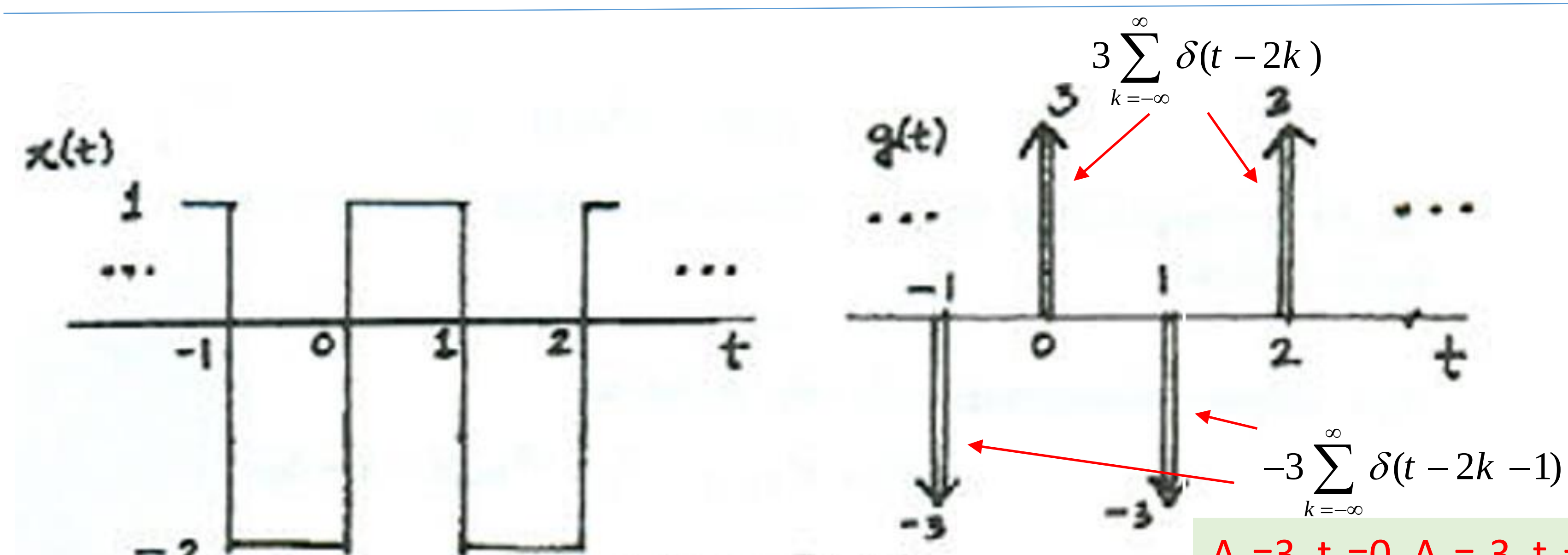
Workout - (13)

Consider a periodic signal with period T = 2: $x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -2, & 1 < t < 2 \end{cases}$

The derivative of this signal is related to the “impulse train” with period T = 2: $g(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k)$

It can be shown that $\dfrac{dx(t)}{dt} = A_1 g(t-t_1) + A_2 g(t-t_2)$

Determine the values of A_1, t_1, A_2 and t_2.

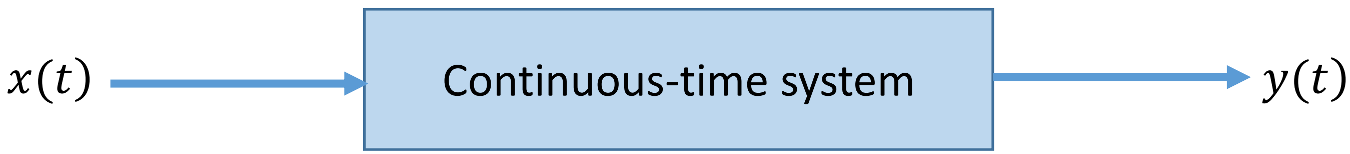


$A_1=3$, $t_1=0$, $A_2=-3$, $t_2=1$

1.5 Continuous-Time Discrete-Time Systems

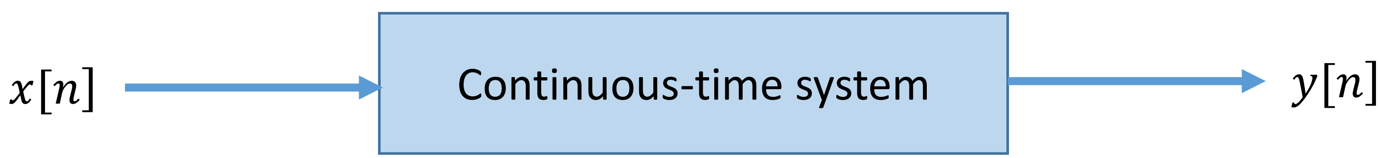
A **continuous-time system** is a system in which continuous-time input signals are applied and result in continuous-time output signals. The input-output relation of such systems can be represented by the notation:

$$x(t) \rightarrow y(t)$$

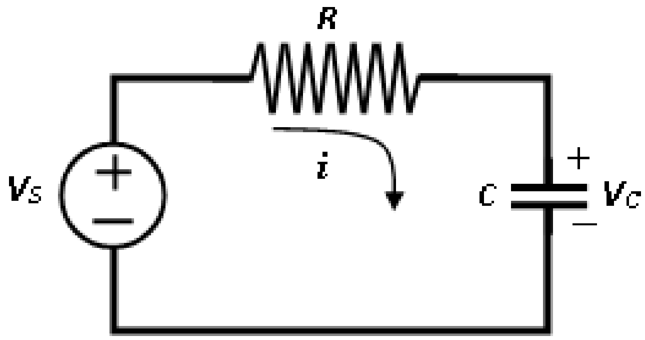


A **discrete-time system** is a system in which discrete-time input signals are applied and result in discrete-time output signals. The input-output relation of such systems can be represented by the notation:

$$x[n] \rightarrow y[n]$$



Simple Example of Systems



A simple ***RC*** circuit with source $\boldsymbol{V_s}$ and capacitor voltage $\boldsymbol{V_c}$

$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c(t)}{dt}$$

The differential equation giving a relationship between the input $v_s(t)$ and the output $v_c(t)$,

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

Simple Example of Systems - contd.

An automobile responding to an applied force ***f*** from the engine and to a retarding force ***ρv*** proportional to the automobile's velocity ***v***.

Here we regard the force $f(t)$ as the input and velocity $v(t)$ as the output. If we let m denote the mass of the automobile and $m\rho v$ the resistance due to friction, then equating acceleration i.e. the time derivative of velocity, with net force divided by mass, we get

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)] \quad \Rightarrow \quad \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

Example 1: Systems

As a simple example of a discrete-time system, consider a simple model for the balance in a bank account from month to month. Specifically, let $y[n]$ denote the balance at the end of the n-th month, and suppose that $y[n]$ evolves from month to month according to the equation

$$y[n] = y[n-1] + 0.01y[n-1] + x[n] = 1.01y[n-1] + x[n]$$

Or,

$$y[n] - 1.01y[n-1] = x[n]$$

Where $x[n]$ represents the net deposit (i.e., deposits minus withdrawals) during the month and the term $1.01y[n-1]$ is the 1% profit each month.

Example 2: Systems

As a second example, consider the digital simulation of the differential equation in which we resolve the time into discrete intervals of length Δ and approximate the derivative $dv(t)/dt$ at $t = n\Delta$ by the first backward difference, i.e.,

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

In this case, if we let

$$v[n] = v(n\Delta) \quad \text{and} \quad f[n] = f(n\Delta)$$

We obtain the following discrete-time model relating the sampled signals $f[n]$ and $v[n]$:

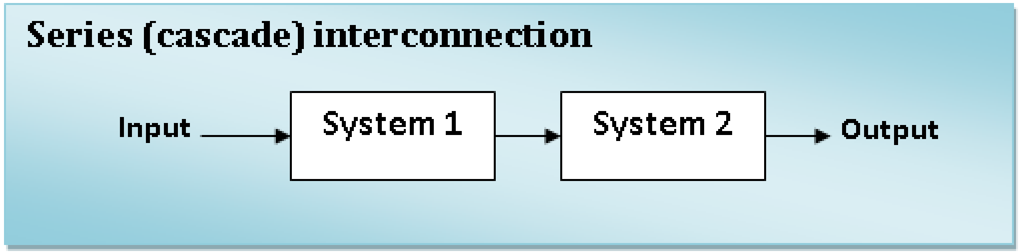
$$v[n] - \frac{m}{(m+\rho\Delta)} v[n-1] = \frac{\Delta}{(m+\rho\Delta)} f[n]$$

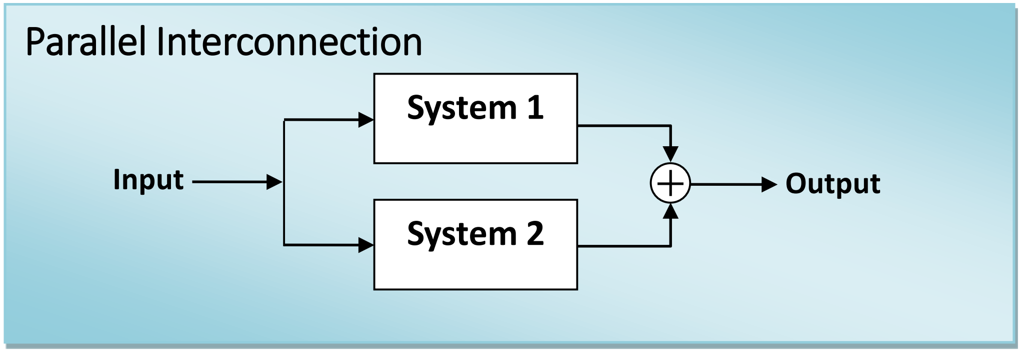
Interconnection of Systems

Many real systems are built as interconnections of several subsystems.

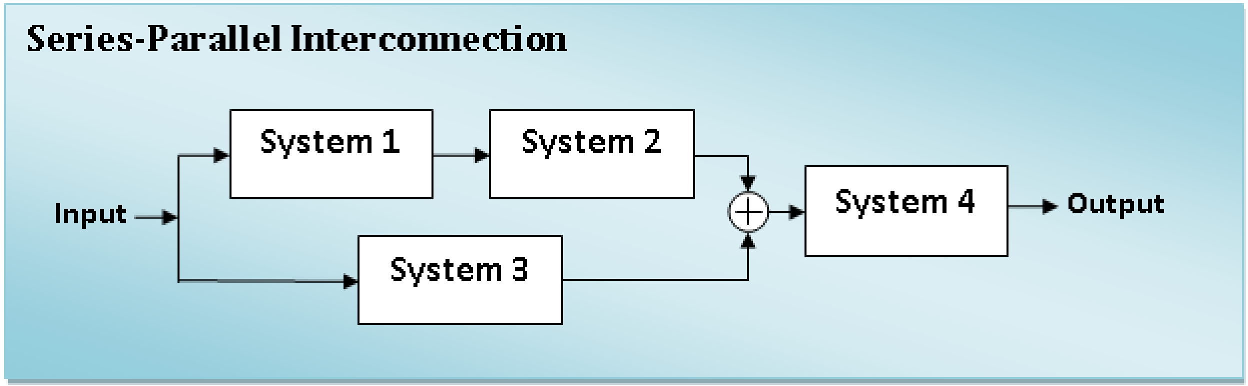
For example, a modern digital telephone system involves the interconnections of a microphone receiver, audio to digital converter, a transmitter, a receiver, a digital to audio convertor and one or more speakers (apart from several other sub-systems).

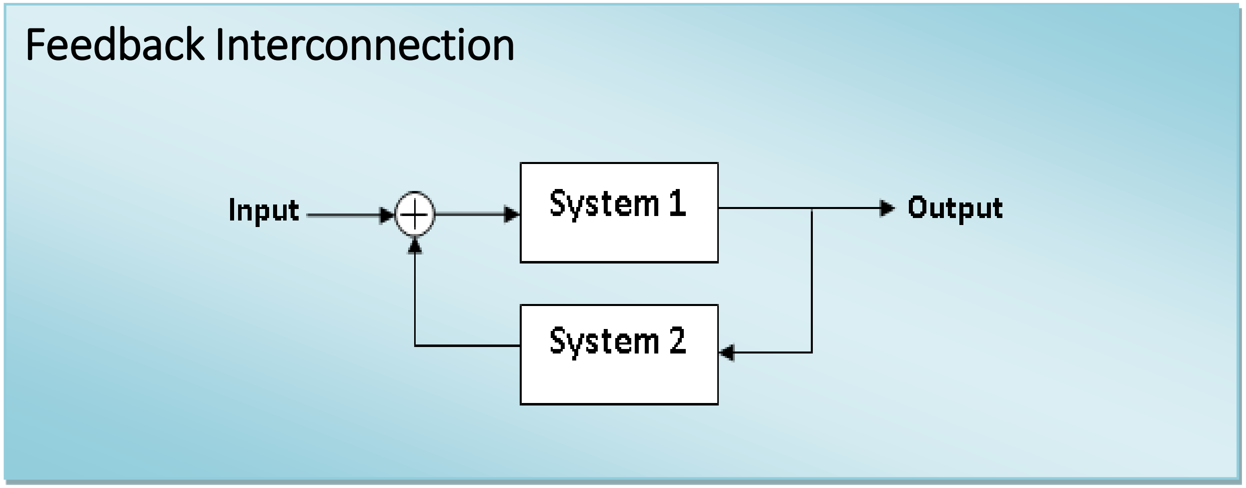
There are several basic system interconnections that are encountered more frequently:





Interconnection of Systems - contd.





1.6 Basic System Properties

1.6.1 Systems with and without memory

A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent on the input at only that same time.

For example, the system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2$$

is memoryless, as the value of $y[n]$ at any particular time n_0 depends on the value of $x[n]$ only at that time, i.e. $x[n_0]$.

As a particular case, a resistor can be considered as a memoryless system: with the input $x(t)$ taken as the current and with voltage taken as the output $y(t)$, the input-output relationship for a resistor is,

$$y(t) = Rx(t)$$

where R is the resistance.

Systems with and without memory

Another particular, simple memoryless system is the ***identity system***, whose output is identical with the input. That is, the input-output relationship for the continuous-time identity system is

$$y(t) = x(t)$$

and the corresponding relationship in discrete-time is

$$y[n] = x[n]$$

An example of a discrete-time system with memory is an ***accumulator*** or ***summer***

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

and a second example is a delay

$$y[n] = x[n-1]$$

Systems with memory

A capacitor is an example of a continuous-time system with memory; since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

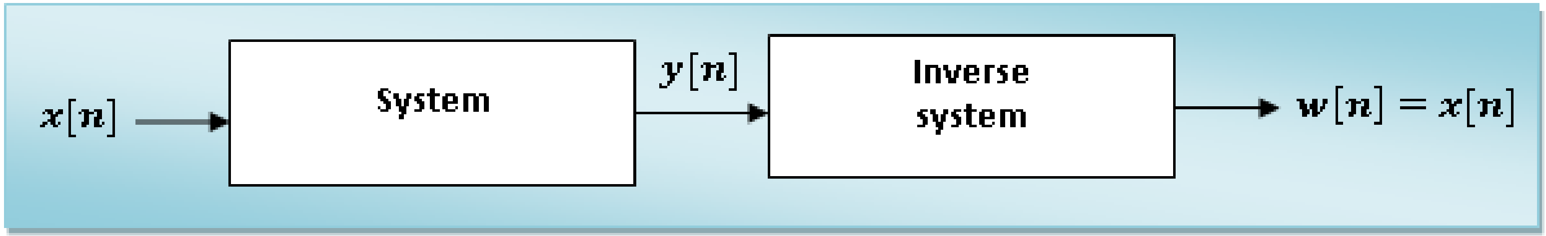
where C is the capacitance

Examples:

- Delay
- Accumulator
- Storage of energy
- Memory dependent on the future values of the input and the output

Invertibility and Inverse Systems

A system is said to be invertible if distinct inputs lead to distinct outputs.



Concept of an inverse system for a general invertible system

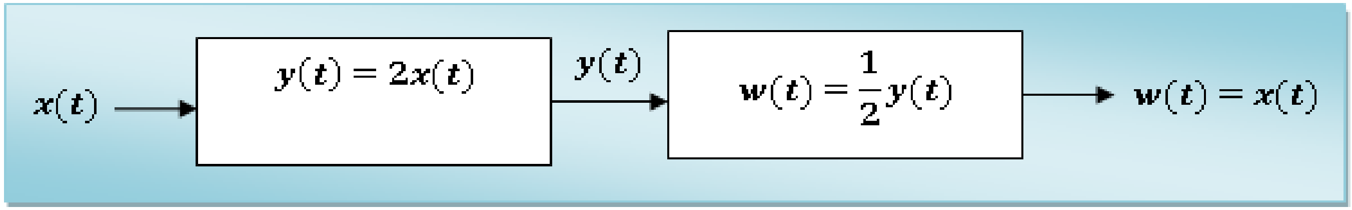
If a system is invertible, than an inverse system exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system.

System

$$y(t) = 2x(t)$$

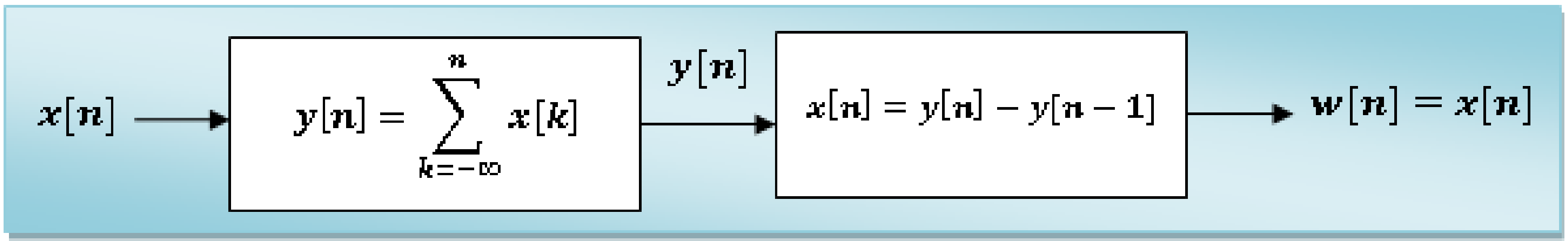
Inverse System

$$x(t) = \frac{1}{2}y(t)$$



Example: Invertible Systems

Accumulator



Example: Non-invertible Systems

- A system that produces a zero output sequence for any input sequence.

$$y(t) = 0$$

- A system where the output is the square of the input.

$$y(t) = x^2(t)$$

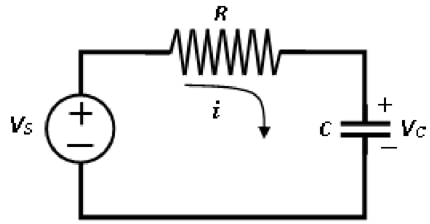
Because, t = ±n will produce the same output.

The concept of invertibility is important in many applications. One particular example is that of systems used for encoding in a variety of communication systems.

Causality

A system is causal if the output at any time depends on values of the input at only the present and past times.

Such systems are also referred to as ***non-anticipative***, as the system output does not anticipate future values of the input.



- The RC circuit (see figure) is causal, since the capacitor voltage depends only on the present and past values of the source voltage.
- The motion of an automobile is causal, as it does not anticipate future actions of the driver.
- The systems described by these equations are also causal.
- All memoryless systems are causal, since the output responds only to the current value of the input.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n-1]$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

Causality - contd.

The following systems are not causal:

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

Causality is not an essential consideration in applications where the independent variable is not time, such as in image processing.

- In processing data that have been collected previously, as often is the case with speech, geophysical or meteorological signals etc., we are by no means constrained to causal processing.
- An example of noncausal averaging system is

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]$$

Causality - contd.

Consider the following system:

$$y[n] = x[-n]$$

Checking for the negative time, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input. Hence the system is not causal.

Consider the following system:

$$y(t) = x(t)\cos(t+1)$$

In this system, the output at any time equals the input at that same time multiplied with a number that fluctuate with time. Specifically, we can re-write

$$y(t) = x(t)\mathrm{g}(\mathrm{t})$$

where $\mathrm{g}(\mathrm{t})$ is a time-varying function, namely $\mathrm{g}(\mathrm{t}) = \cos(\mathrm{t}+1)$. Thus, only the current value of the input influences the current value of the output, and we conclude that this system is causal (and, also memoryless).

Stability

A system is said to be *stable* if a small input leads to a response that does not diverge.

There are several examples of stable systems. “Stability of physical systems generally results from the presence of mechanisms that dissipate energy”.

For example, in the ***RC*** circuit shown before, the resistor dissipates energy and this circuit is a stable system.

More specifically,

If the input to a *stable system* is bounded (i.e., if its magnitude does not grow without bounds), then the output must also be bounded, and therefore cannot diverge.

Stability - contd.

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k]$$

If the input $x[n]$ to the system is bounded (say by a number $\boldsymbol{B}$), for all values of n, then according to Equation above, the output $y[n]$ of the system is also bounded by $\boldsymbol{B}$. This is because the output $y[n]$ is the average of a finite set of values of the input. Therefore, the output $y[n]$ is bounded and the system is stable.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

This systems sums all of the past values of the input rather than just a finite set of values, and the system is unstable, since the sun can grow even if the input $x[n]$ is bounded.

Example: Stability

Suppose we suspect that a particular system is unstable, then a useful strategy is to look for a specific bounded input that leads to an unbounded output for that system.

$$S_1: \ y(t) = tx(t)$$

$$S_2: \ y(t) = e^{x(t)}$$

Now, for system S_1, a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded: since no matter what finite constant input we pick, $|y(t)|$ will exceed that constant for some t. Therefore, the system S_1 is unstable.

For system S_2, let us the input $x(t)$ be bounded by a positive number B, i.e.

$$|x(t) < B| \longrightarrow \quad -B < x(t) < B \qquad \text{for all } t.$$

Using the definition of S_2, we can write $\quad e^{-B} < |y(t)| < e^{B}$

The system S_2 is therefore, stable.

Time Invariance

A system is said to be *time invariant* if a time shift in the input signal leads to an identical time shift in the output signal.

if $y[n]$ is the output of a discrete-time, time-invariant system when $x[n]$ is the input, then $y[n - n_0]$ is the output when $x[n - n_0]$ is applied as an input.

In continuous-time when $y(t)$ is the output corresponding to the input $x(t)$, a time-invariant system will have $y(t - t_0)$ as the output when $x(t - t_0)$ is the input.

Consider now the discrete-time system defined by $y[n] = n\, x[n]$

Suppose, we consider the input signal $x_1[n] = \delta[n]$, which yields an output $y_1[n] = 0$ (since $n\delta[n] = 0$).

However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$. Thus, while $x_2[n]$ is a shifted version of $x_1[n]$, $y_2[n]$ is not a shifted version of $y_1[n]$.

Time Invariance - contd.

Consider the continuous-time system defined by $y(t) = \sin[x(t)]$

To check this system is time invariant, we must determine whether the time-invariance property holds for any input and any time shift t_0. Thus, let $x(t)$ be an arbitrary input to this system, and let

$$y_1(t) = \sin[x_1(t)]$$

to be the corresponding output. Then, consider a second input obtained by shifting $x_1(t)$ in time

$$x_2(t) = x_1(t + t_0)$$

The corresponding output to this new input

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t + t_0)]$$

$$y_1(t + t_0) = \sin[x_1(t + t_0)]$$

We see that $y_2(t) = y_1(t + t_0)$, and therefore, the system is time invariant.

Linearity

A linear system, in continuous-time or discrete-time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition – that is, the weighted sum – of the responses of the system to each of those signals.

Let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the response of a continuous-time system to an input $x_2(t)$. Then the system is linear if,

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant.

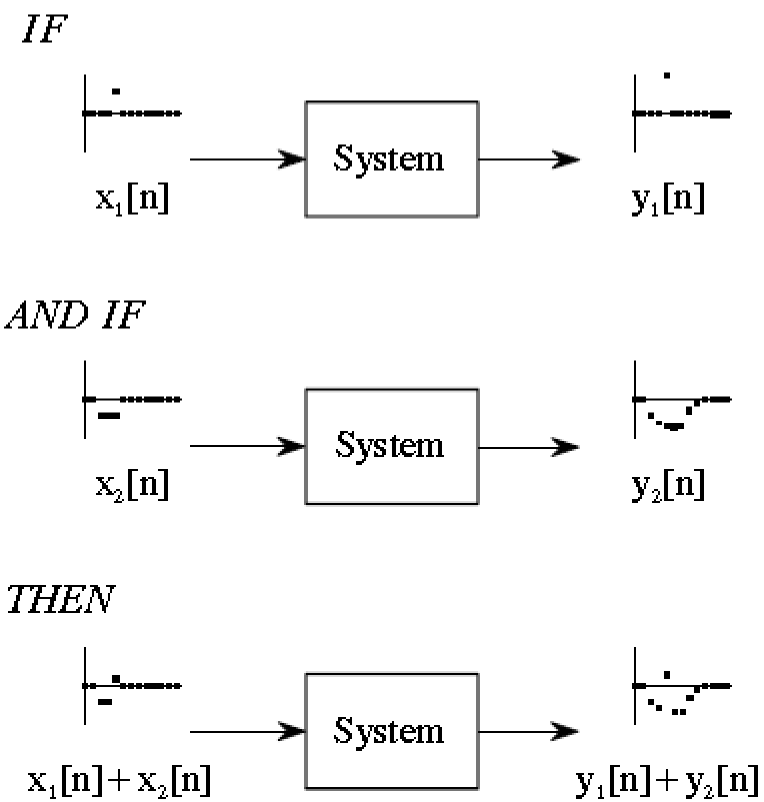
The first of these two properties is called the ***additivity*** property and the second is known as the ***scaling*** or ***homogeneity*** property.

The two properties defining a linear system can be combined into a single statement:

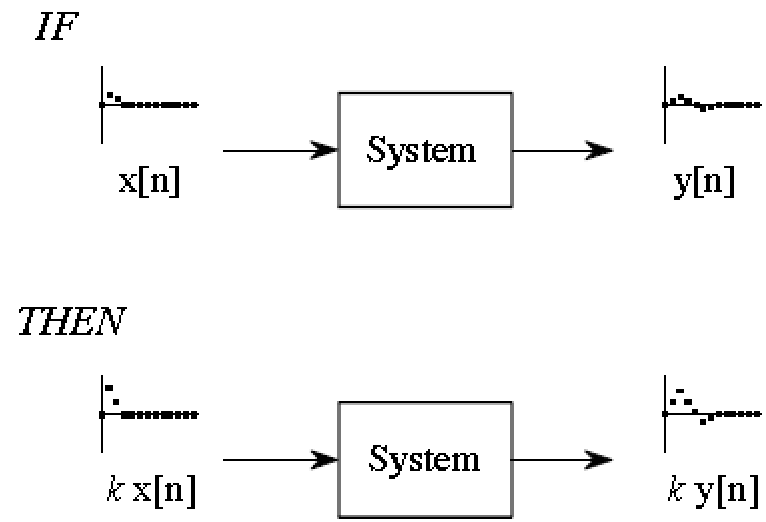
$$x_1(t) + bx_2(t) = ay_1(t) + by_2(t)$$

Linearity - contd.

Additivity



Homogeneity



Example: Linearity

Consider a system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = tx(t)$$

To determine whether or not S is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$,

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output $y_3(t)$ may be expressed as:

$$y_3(t) = tx_3(t) = t(ax_1(t) + bx_2(t)) = atx_1(t) + btx_2(t)$$

$$= ay_1(t) + by_2(t)$$

We conclude that the system S is **linear.**

Example: Linearity - contd.

Let us now consider another system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x^2(t)$$

Like the previous example, to determine whether or not S is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$,

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$

$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output $y_3(t)$ may be expressed as:

$$y_3(t) = x_3^2(t) = \left(ax_1(t) + bx_2(t)\right)^2 = a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t)$$

$$= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t)$$

Not Linear

Example: Linearity - contd.

While checking the linearity of a system, it is important to keep in mind that the system must satisfy both the additivity and homogeneity properties and that the signals as well the scaling constants are allowed to be complex.

Consider: $y[n] = \mathcal{Re}\{x[n]\}$

Additive √	Homogeneity ×

Let us assume, $x_1[n] = r[n] + js[n]$

where $r[n]$ and $s[n]$ are the real and imaginary parts of the complex signal $x[n]$, and that the corresponding output is given by $y_1[n] = r[n]$

Now we consider the scaling of the complex input with a complex number say $a = j$, i.e.

$$x_2[n] = jx_1[n] = j(r[n] + js[n]) = jr[n] - s[n]$$

Therefore, the corresponding output $y_2[n]$ is given by $y_2[n] = \mathcal{Re}\{x_2[n]\} = -s[n]$

which is not equal to the scaled version of the $y_1[n]$: $ay_1[n] = jr[n]$

We conclude that the system violates the homogeneity property, therefore it is not linear.

Example: Linearity - contd.

Find whether the following system is linear or not.

$$y[n] = 2x[n] + 3$$

If $x_1[n]$ = 2 and $x_2[n]$ = 3, then

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3 = 7$$
$$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3 = 9$$

However, the response to $x_3[n] = x_1[n]+x_2[n] = 5$

$$y_3[n] = 2\{x_1[n] + x_2[n]\} + 3 = 13 \neq y_1[n] + y_2[n]$$

Not linear

Another way:

For a linear system, an input which is zero for all time results in an output which is zero for all time.

If x[n] = 0, y[n] = 3, which is not zero. So, the above system is not linear.

Workout - (14)

A continuous-time signal $\boldsymbol{x(t)}$ is depicted below in Figure (1).

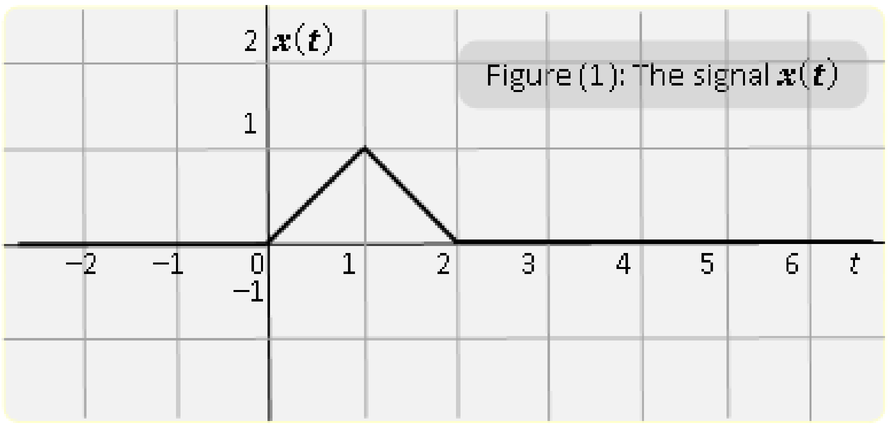
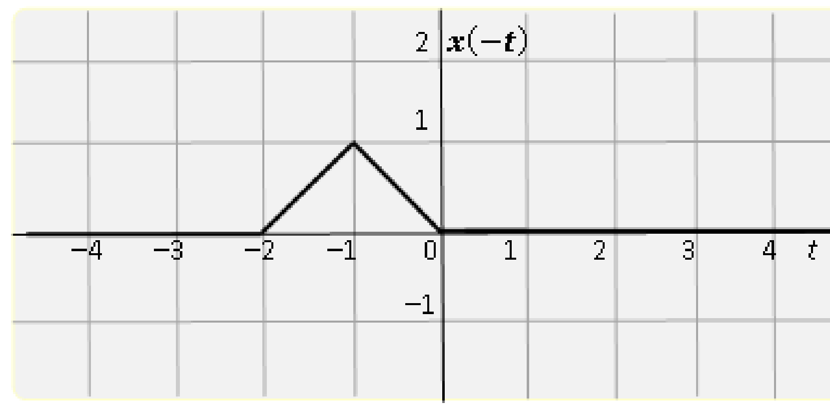


Figure (1): The signal $x(t)$



Draw and label the following signals:

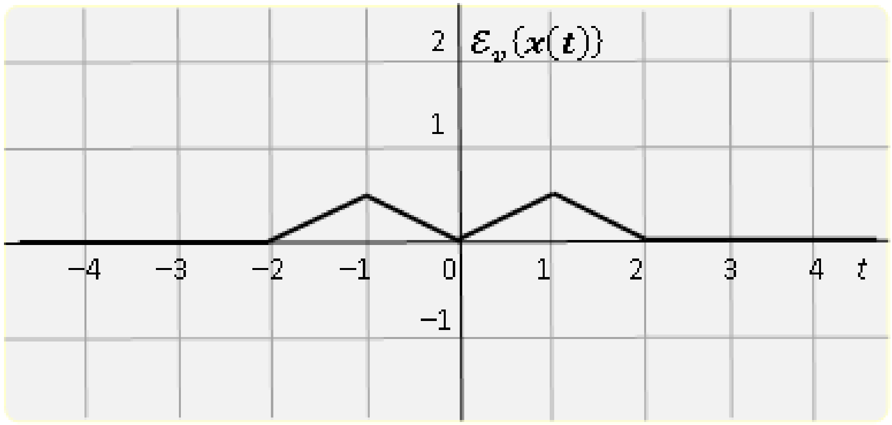
(1) $\boldsymbol{x(-t)}$,

(2) Even component of $\boldsymbol{x(t)}$: $\boldsymbol{\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]}$

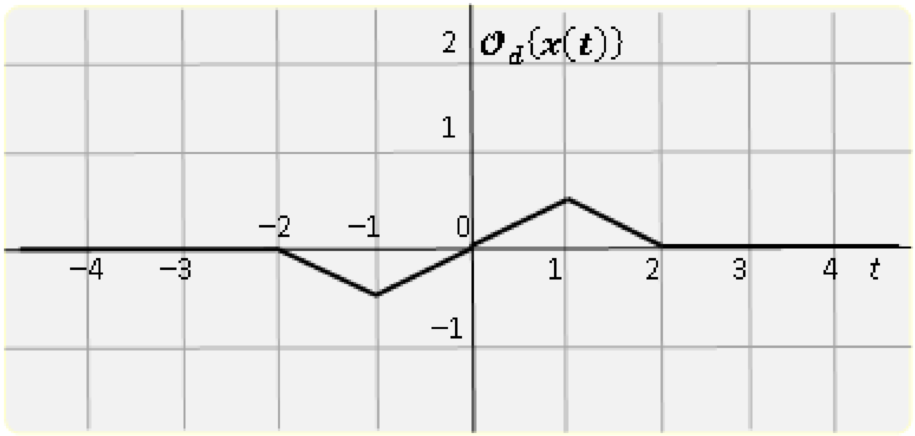
Odd component of $\boldsymbol{x(t)}$: $\boldsymbol{\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]}$

Workout - (14) - contd.

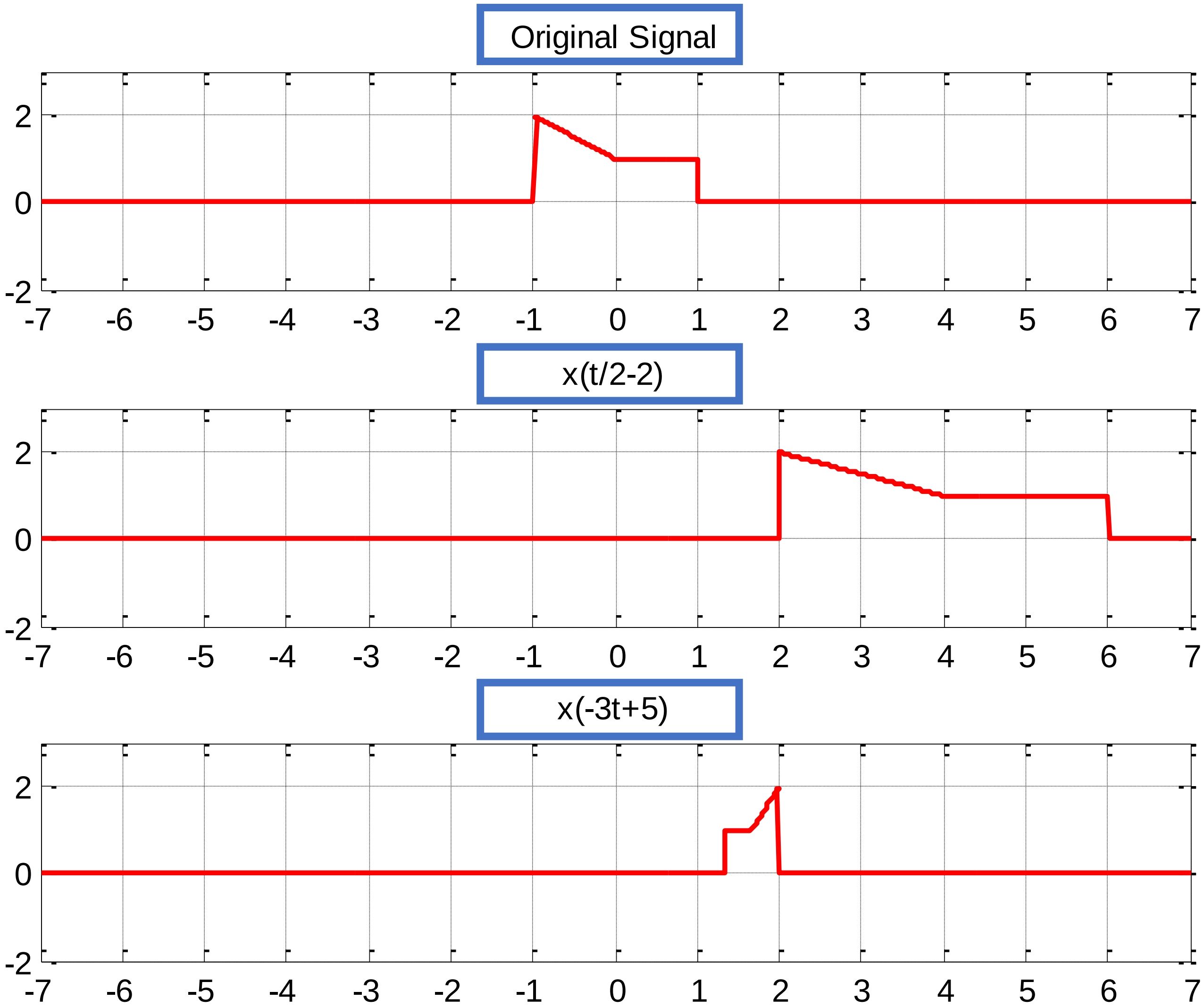
Even component of $x(t)$: $\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$



Odd component of $x(t)$: $\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$



Workout - (15)

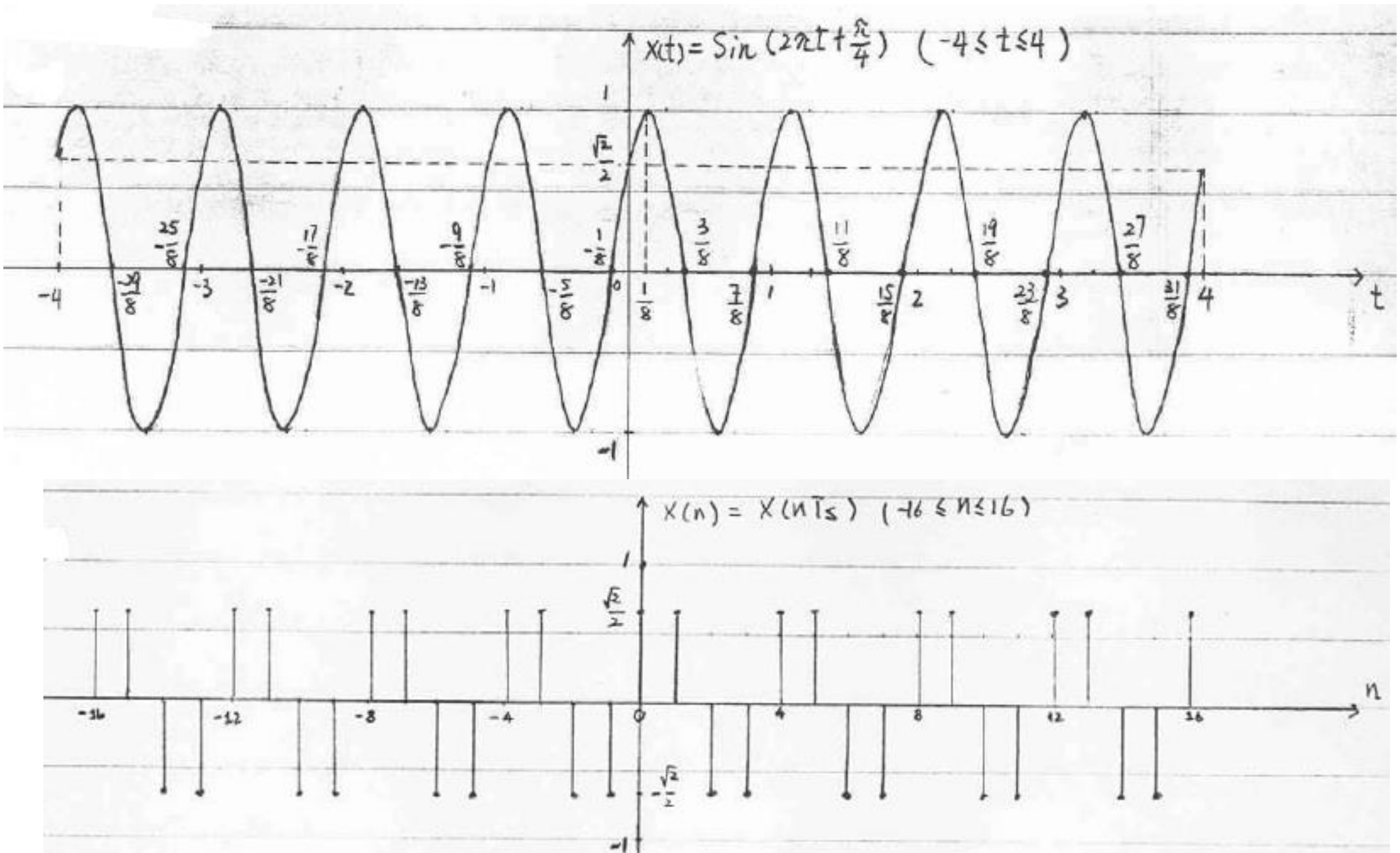


First, shift right by 2 samples. The first transition is now shifted to (-1+2 =) 1. Then each position is multiplied by 2. So, 1 becomes (1x2 =) 2.

First, shift left by 5 samples. The first transition is now shifted to (-1-5 =) -6. Then each position is multiplied by 1/3. So, -6 becomes (-6x1/3=) -2. Then, reflect. So -2 becomes 2.

Workout – (16)

Drawing Sinusoids $x(t) = \sin\left(2\pi t + \frac{\pi}{4}\right), \quad -4 \le t \le 4$



Acknowledgement

The slides are prepared based on the following textbook:

- Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab, *Signals & Systems*, 2nd Edition, Prentice-Hall, Inc., 1997.

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