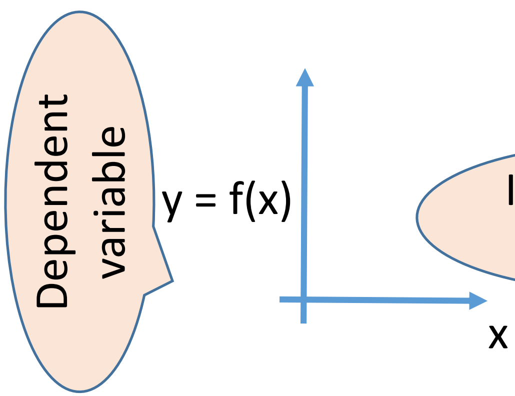
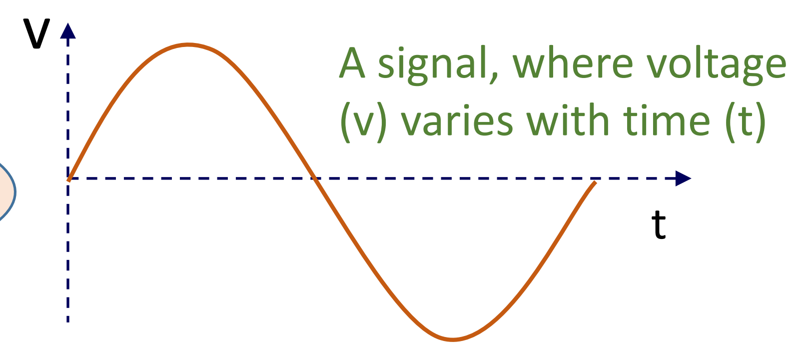
Signals and Systems: Introduction

What is a signal?

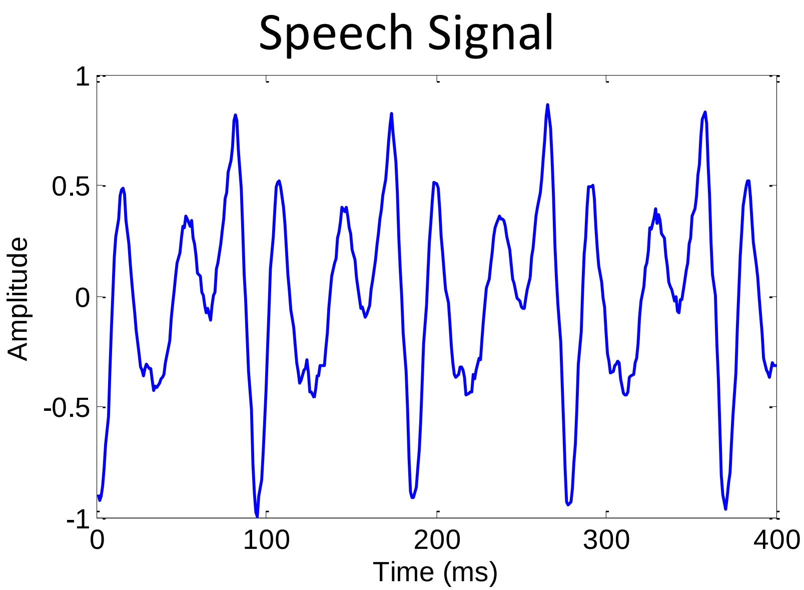
- Signals may describe a wide variety of physical phenomena.
- The information in a signal is contained in a pattern of variations of some form.
- A signal is represented mathematically as a function of one or more independent variables.

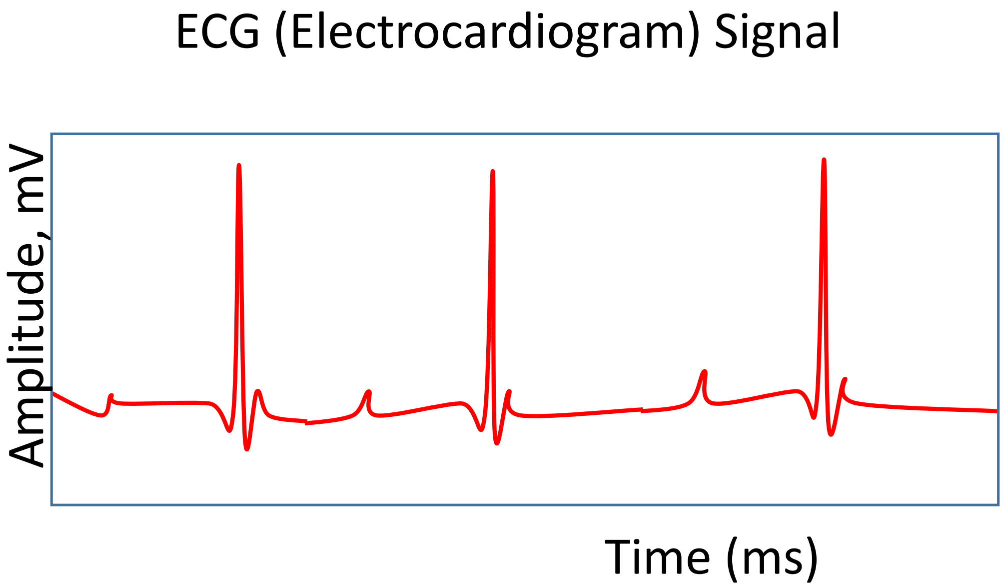




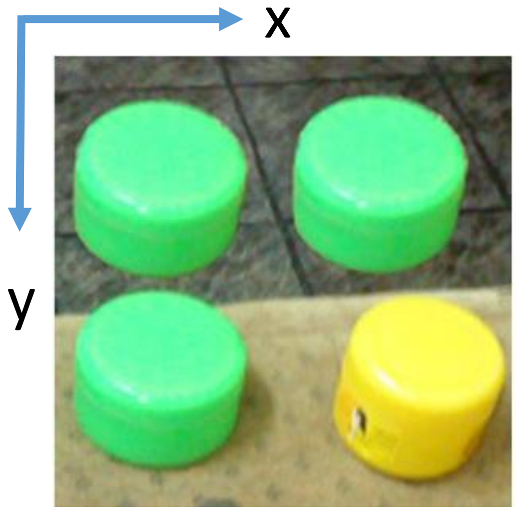
Examples of Signals - 1

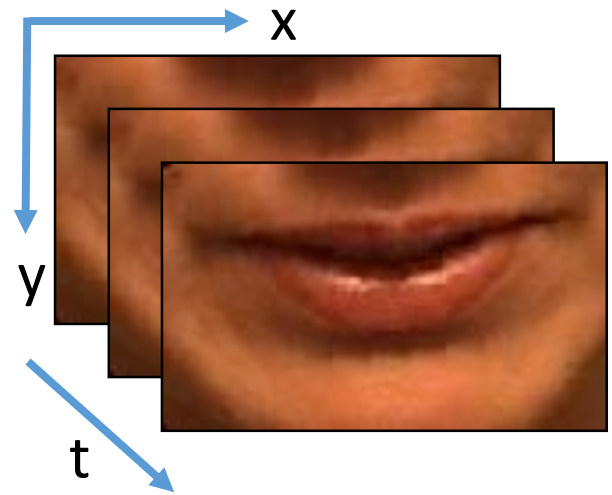
One dimensional signal, because there is only one independent variable, such as time.





Examples of Signals - 2





Intensity of the image at location (x, y) can be expressed as *I*(x, y). As there are two independent variables (x and y), the image is a two dimensional signal.

A video has three independent variables (x, y, and t (time)), therefore, it is a three dimensional signal. A video is a sequence of frames (images).

Two Basic Types of Signals

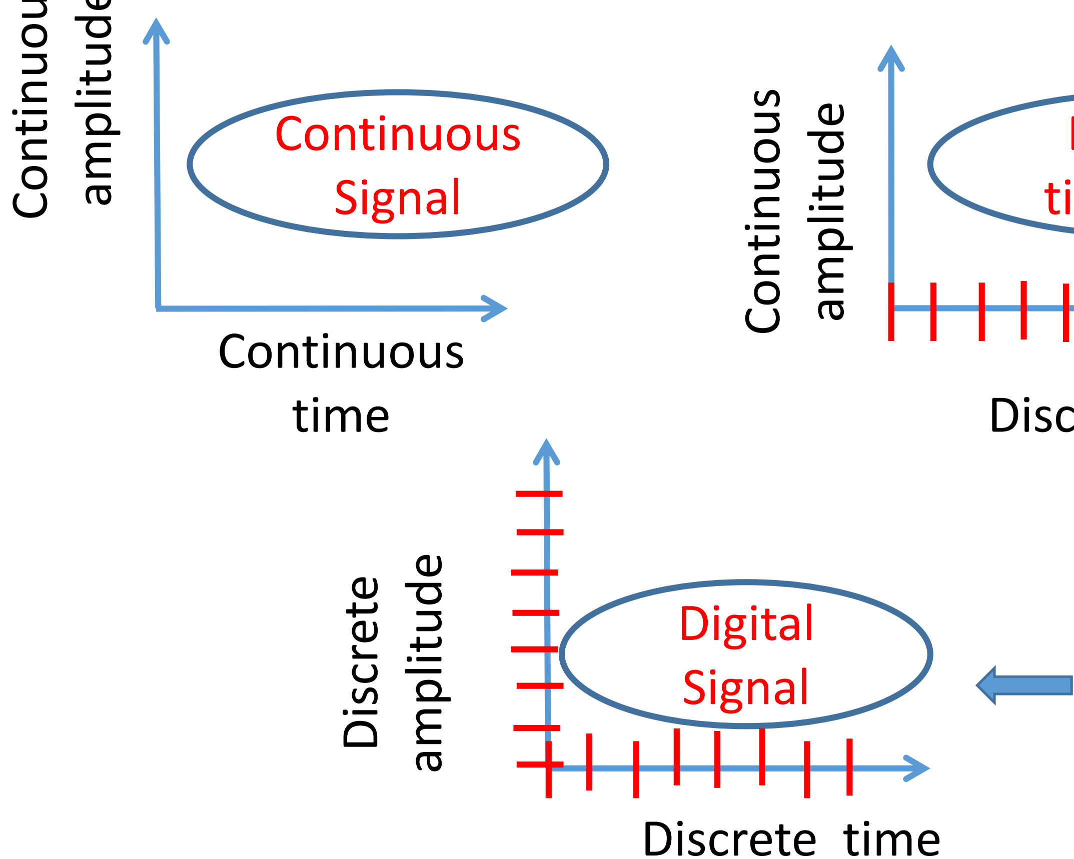
Continuous Signal

A continuous-time (CT) signal is one that is present at all instants in time or space, such as oscillating voltage signal.

Discrete-time Signal

A discrete-time (DT) signal is only present at discrete points in time or space. For example closing stock market average is a signal that changes only at discrete points in time (at the close of each day).

Continuous, discrete-time, & Digital Signals

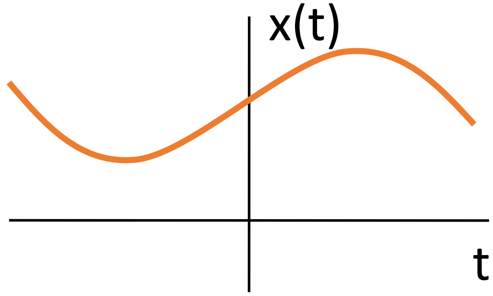


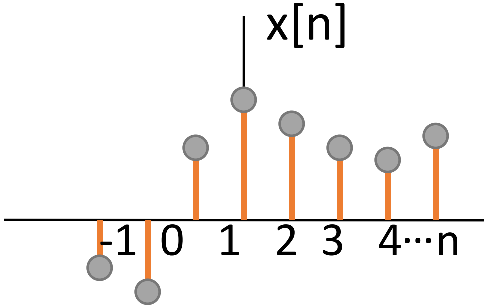
You will learn it in CEN 352.

Notation of Continuous and discrete-time Signals

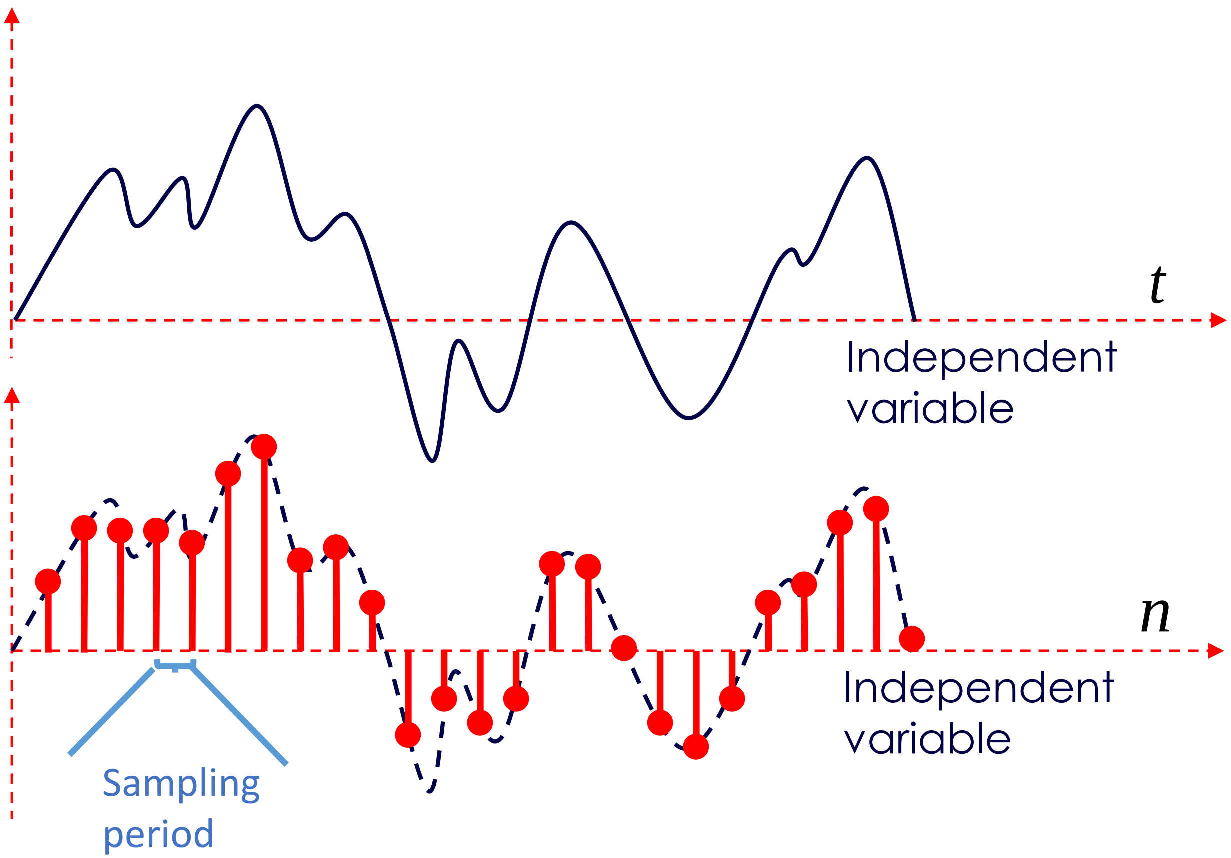
To distinguish between continuous-time and discrete-time signals, we will use

- The symbol ‘t’ to denote the continuous-time independent variable and
- ‘n’ to denote the discrete-time independent variable.
- We will enclose the independent variable in parentheses ‘(.)’ and for discrete-time signals, we will use brackets ‘[.]’





Continuous and discrete-time Signals



We can convert a continuous signal into a discrete-time signal by ‘sampling’.

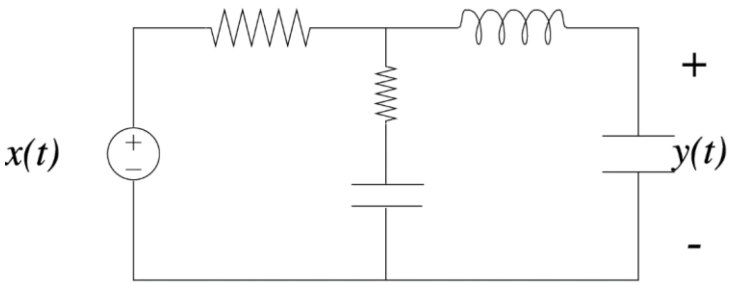
‘n’ is always an integer.

Systems

- A system is an abstraction of anything that takes an input signal, operates on it, and produces an output signal.
  - A system generally establishes a relationship between its input and its output.
  - Examples could be car, camera, etc.
- Systems that operate on continuous-time signal are known as continuous-time (CT) systems.
- Systems that operate on discrete-time signals are known as discrete-time (DT) systems.

Examples of Systems

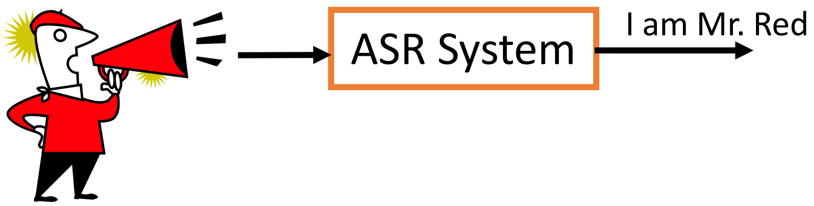
An RLC circuit



Courtesy of Prof. Alan S. Willsky

- What is the input signal?
  - *x(t) (the D.C. source)*
- What is the output signal?
  - *y(t) (the signal across capacitor)*
- What is the system?
  - *The whole RLC network*

Automatic speech recognition (ASR) system



Imaging system

Drill - 1

1. Most of the signals in this physical world is …………… (CT signals / DT signals). Choose the right one.
2. Mention four systems other than those mentioned in the slides.
3. Mention three signals other than those mentioned in the slides.
4. How can we convert a CT signal into a DT signal?
5. Can a system have multiple inputs and multiple outputs?
6. What do you mean by time-domain signal and spatial-domain signal?

MATLAB

- Matlab ® is a software tool for computation in science and engineering.
- Developed, published and trademarked by The MathWorks, Inc.
- Originally developed as a "Matrix Laboratory" but now used in applications in almost all areas of science and engineering.
- It has a rich collection of tool boxes covering basic mathematics, graphics, differential equations, electric/electronic circuits, partial differential equations, simulation problems, control systems, signal processing, image processing, statistics, symbolic computations, etc.
- http://www.mathworks.com/help/pdf_doc/matlab/getstart.pdf
- http://www.mathworks.com/academia/student_center/tutorials/launchpad.html

1.1.2 Signal Power and Energy

Continuous-time (CT) signal

- The total energy over the time internal $t_1 \le t \le t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 \, dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x.

- The time averaged power is given by $\dfrac{1}{(t_2 - t_1)} \displaystyle\int_{t_1}^{t_2} |x(t)|^2 dt$

Over an infinite time interval, i.e., for $-\infty < t < +\infty$

- Total energy: $$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Total averaged power: $$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

1.1.2 Signal Power and Energy

Discrete-time (DT) signal

- The total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \le n \le n_2$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

- The average power over the interval in this case is given by

$$\frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

Over an infinite time interval, i.e., for $-\infty < t < +\infty$

- Total energy:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

- Total averaged power:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

Three Important Cases

Case 1: Signals with finite total energy, i.e., $E_\infty < \infty$:

Such a signal must have zero average power. For example, in continuous case, if $E_\infty < \infty$, then

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

An example of a finite-energy signal is a signal that takes on the value of 1 for $0 \leq t \leq 1$ and 0 otherwise. In this case, $E_\infty = 1$ and $P_\infty = 0$.

Case 2: Signals with finite average power, i.e., $P_\infty < \infty$:

For example, consider the constant signal where $x[n] = 4$. This signal has infinite energy, as

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \lim_{N\to\infty} \sum_{n=-N}^{+N} 4^2 = \cdots + 16 + 16 + 16 \ldots$$

Three Important Cases - continued

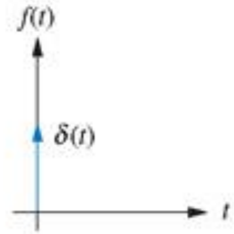
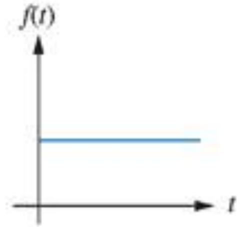
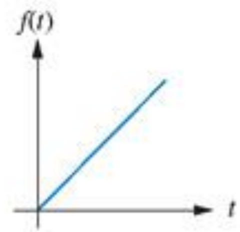
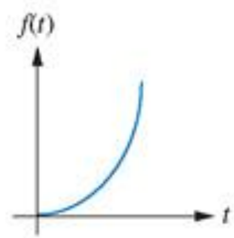
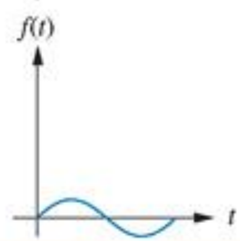
However, the total average power is finite,

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} 4^2$$

$$= \lim_{N\to\infty} \frac{16}{2N+1} \sum_{n=-N}^{+N} 1 = \lim_{N\to\infty} \frac{16(2N+1)}{2N+1} = \lim_{N\to\infty} 16 = 16$$

Case 3: Signals with neither E_∞ nor P_∞ finite:

A simple example of such a case could be $x(t) = t$. In this case both E_∞ and P_∞ are infinite

Input	Function	Description	Sketch
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$	
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$	
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere	
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere	
Sinusoid	$\sin \omega t$		

Some Frequently Used Signals

Example: Power and Energy

Problem 1: Find P_∞ and E_∞ for the signal, $x_1(t) = e^{-2t}u(t)$

Solution:

$$E_\infty = \int_{-\infty}^{\infty} |x_1(t)|^2\, dt = \int_{-\infty}^{\infty} |e^{-2t}u(t)|^2\, dt = \int_{0}^{\infty} |e^{-2t}|^2\, dt$$

$$= \int_{0}^{\infty} |e^{-4t}|\, dt = -\frac{1}{4}\left(-\frac{1}{e^{4(0)}} + \frac{1}{e^{4(\infty)}}\right) = -\frac{1}{4}(-1+0) = \frac{1}{4}$$

P_∞ is zero, because $E_\infty < \infty$

1.2 Transformations of the Independent Variable

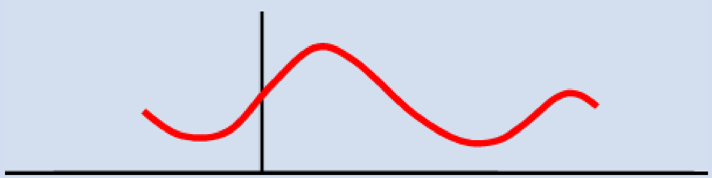
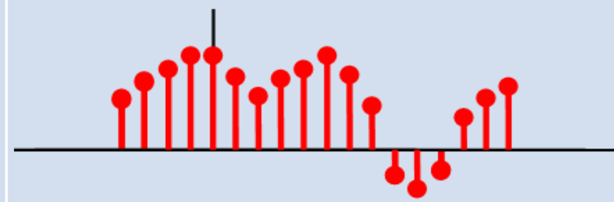
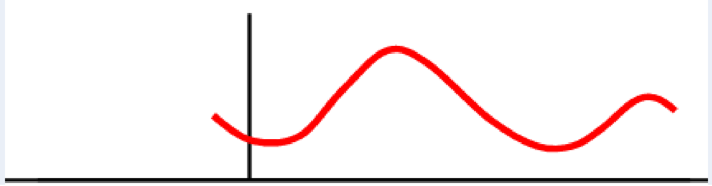
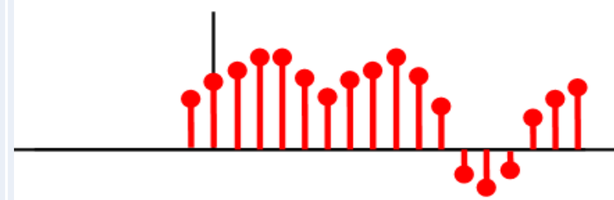
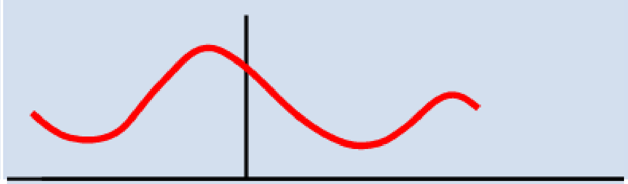
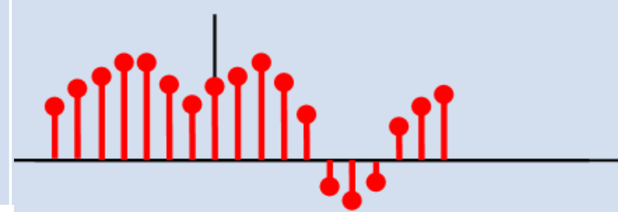
The ***transformation of a signal*** is one of the central concepts in the field of signals and systems.

We will focus on a very limited but important class of signal transformations that involves the *modifications of the independent variable, i.e., the time axis*.

(A) Time Shift

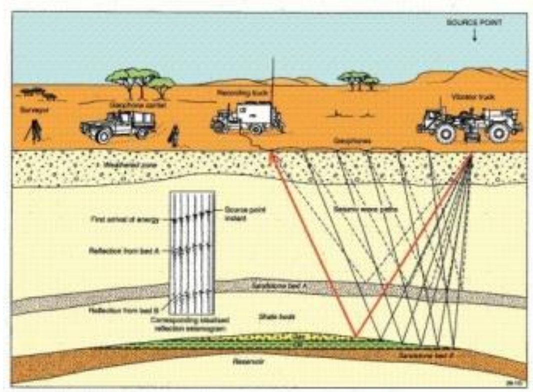
The original and the shifted signals are identical in shape, but are displaced or shifted along the time-axis with respect to each other. Signals could be termed as delayed or advanced in this case.

Time Shift

Signals	Continuous time	Discrete time
Original	$x(t)$ 	$x[n]$ 
Delayed	$x(t-t_0)$ 	$x[n-n_0]$ 
Advance	$x(t+t_0)$ 	$x[n+n_0]$ 

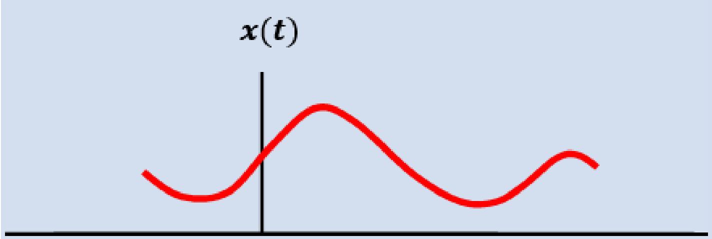
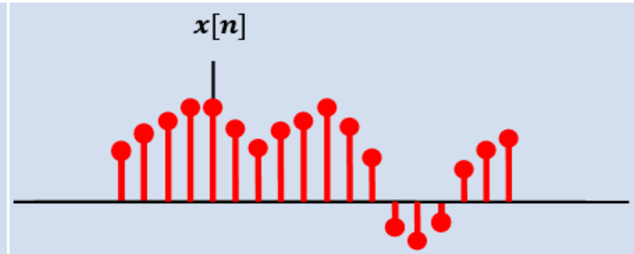
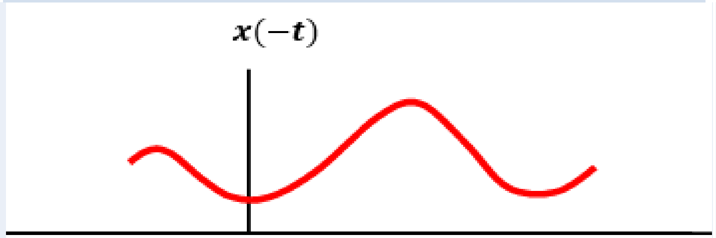
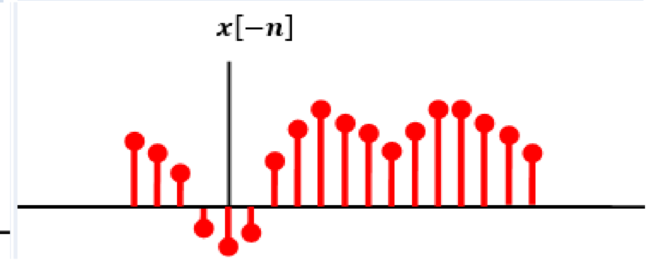
Time Shift - continued

Such signals arise in applications such as radar, sonar and seismic signal processing. Several receivers placed at different locations receive the time shifted signals due to the transmission time they take while passing through a medium (air, water or rock etc.).



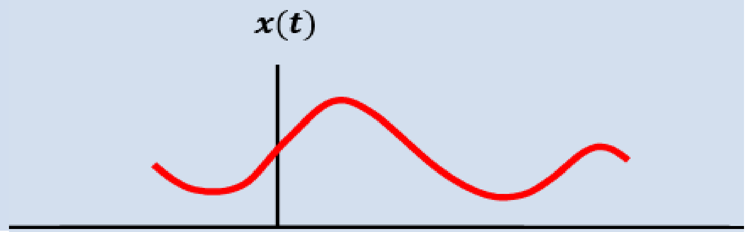
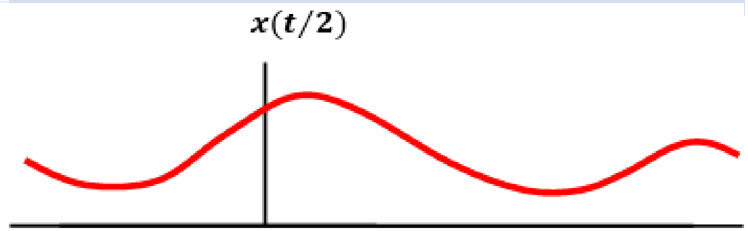
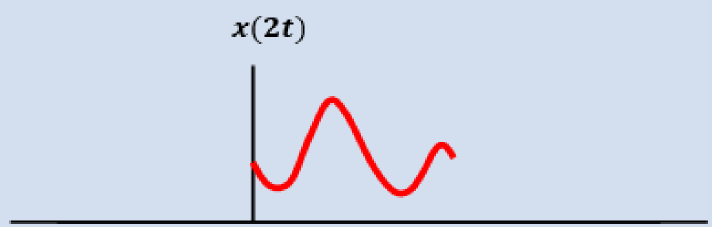
Time Reversal (Reflection)

In this case, the original signal is reflected about the time = 0. For example, if the original signal is some audio recording, then the time reversed signal would be the audio recording played backward.

Signals	Continuous time	Discrete time
Original		
Time reversed		

Time Scaling

In this case, if the original signal is $x(t)$, the time variable is multiplied with a constant to get a time-scaled signal, e.g., $x(2t)$, $x(5t)$, or $x(t/2)$. If we think of the signal $x(t)$ as audio recording, then $x(2t)$ is the audio recording played at twice the speed and $x(t/2)$ is the recording played at half of the speed.

Signals	**Continuous time**
Original	
Stretching	
Compressing	

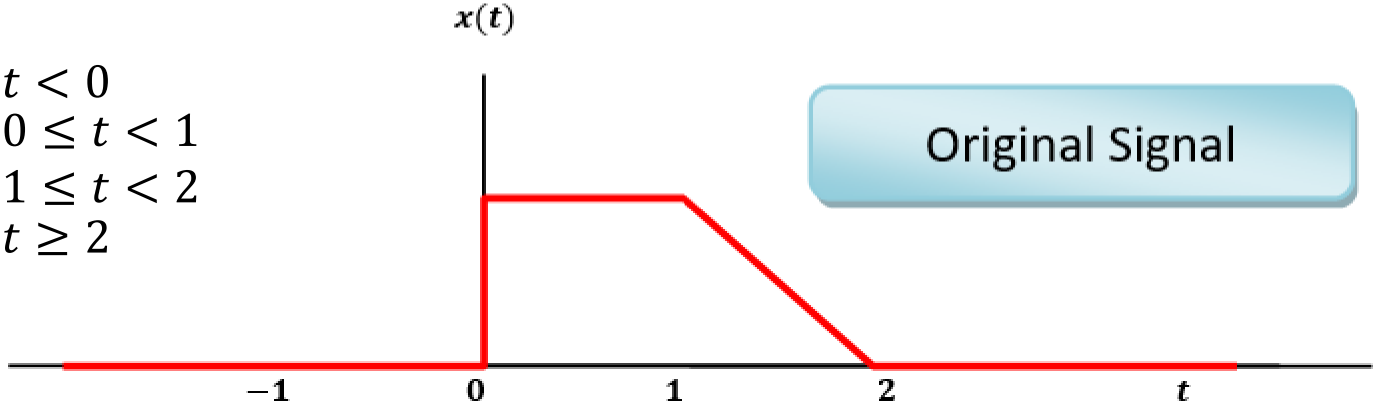
General Case of the Transformation of the Independent Variable

A general case for the transformation of independent variable is the one in which for the original signal $x(t)$ is changed to the form $x(\alpha t + \beta)$, where α and β are given numbers. It has the following effects on the original signal:

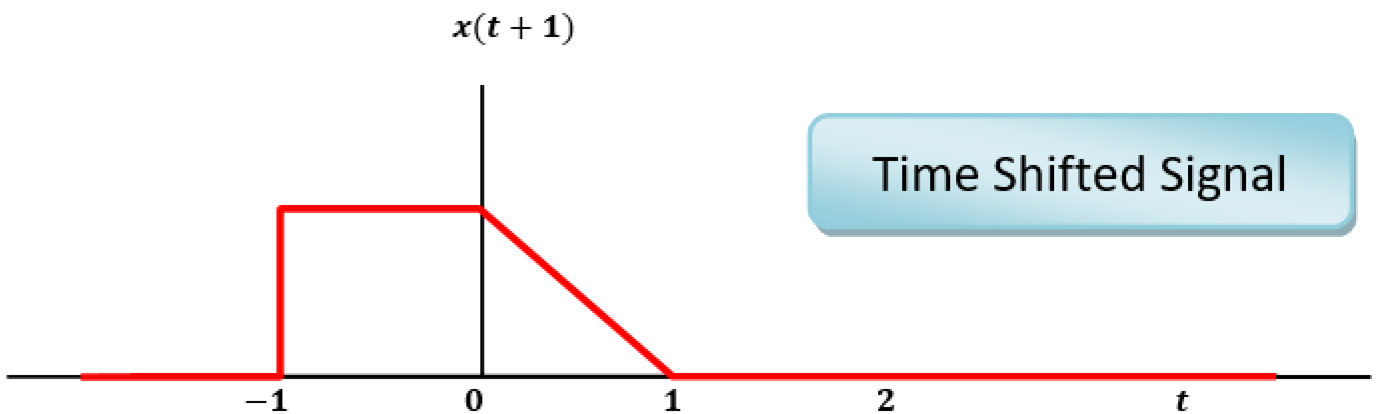
- The general shape of the signal is preserved.
- The signal is linearly stretched if $|\alpha| < 1$.
- The signal is linearly compressed if $|\alpha| > 1$.
- The signal is delayed (shifted in time) if $\beta < 0$.
- The signal is advanced (shifted in time) if $\beta > 0$.
- The signal is reversed in time (reflected) if $\alpha < 0$.

Example: Time Shift: (1)

$$x(t) = \begin{cases} 0 & if \; t < 0 \\ 1 & if \; 0 \le t < 1 \\ 2 - t & if \; 1 \le t < 2 \\ 0 & if \; t \ge 2 \end{cases}$$



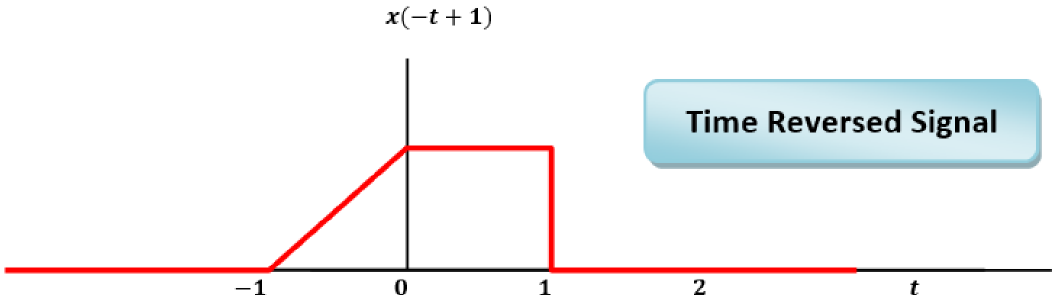
The signal $x(t+1)$ can be obtained by shifting the given signal to the left by one unit



Example: Time Shift: (2)

The signal $x(-t+1)$ can be obtained using the mathematical definition or figure of the original signal $x(t)$. If we use the mathematical definition, then making the following table could be useful.

t	$-t+1$	$x(-t+1)$
-2	3.0	0
-1.5	2.5	0
-1	2.0	0
-0.5	1.5	0.5
0	1.0	1
0.5	0.5	1
1	0.0	1
1.5	-0.5	0
2	-1.0	0
2.5	-1.5	0
3	-2.0	0



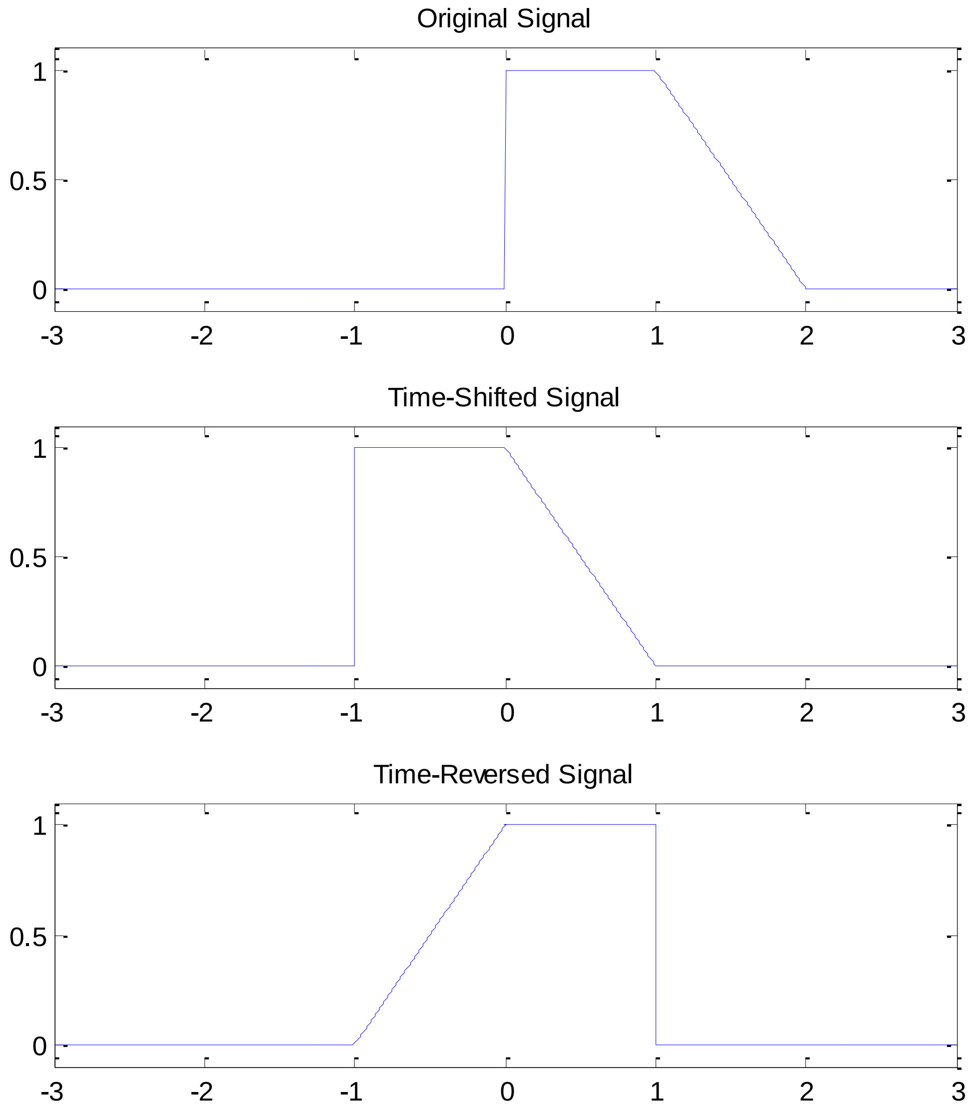
First plot x(t+1), then reflect.

MATLAB Drill - 1

In MATLAB®, the original signal can be written as an inline function. This function can then be used to plot the original signal, the shifted signal and the time-reversed signal using the following MATLAB® code.

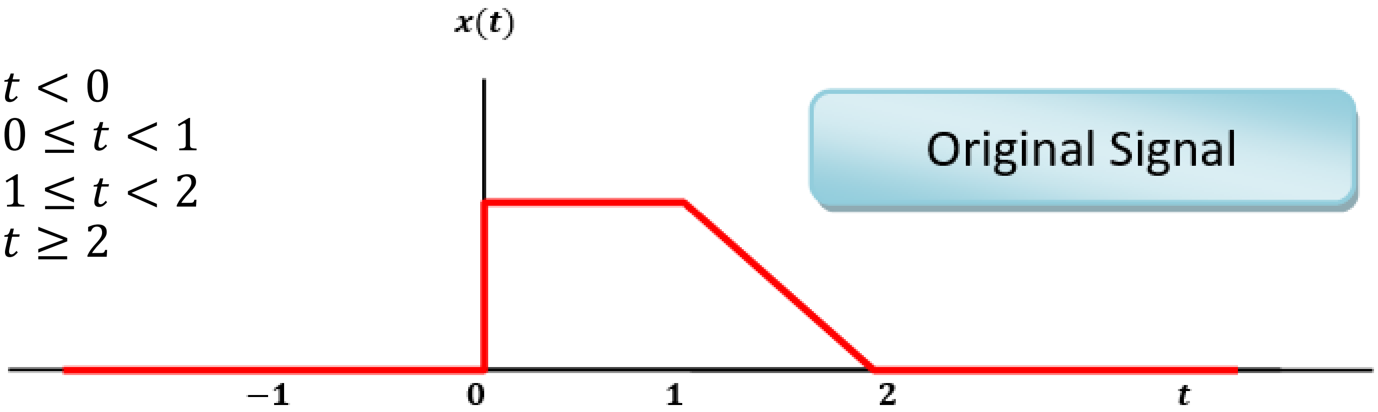
```
>>g = inline(' ((t>=0)&(t<1)) + (2-t).*((t>=1) & (t<2))','t');
>>t = -3:0.001:3;
>>subplot(3,1,1), plot(t, g(t)), axis([-3 3 -0.1 1.1]),
title('Original Signal')
>>subplot(3,1,2), plot(t, g(t+1)), axis([-3 3 -0.1 1.1]),
title('Time-Shifted Signal')
>>subplot(3,1,3),plot(t, g(-t+1)),axis([-3 3 -0.1 1.1]),
title('Time-Reversed Signal')
```

MATLAB Drill – 1: continued



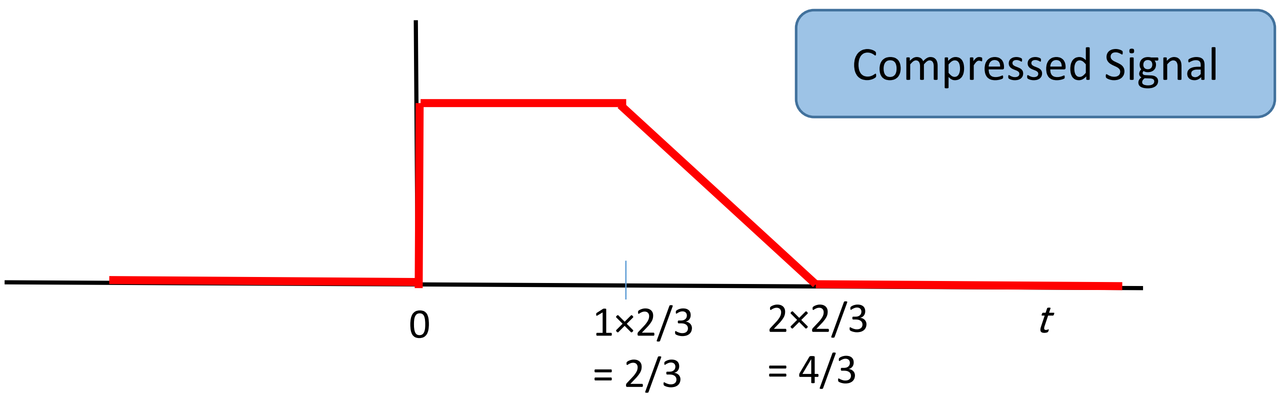
Example: Time Compression: (1)

$$x(t) = \begin{cases} 0 & if \; t < 0 \\ 1 & if \; 0 \leq t < 1 \\ 2 - t & if \; 1 \leq t < 2 \\ 0 & if \; t \geq 2 \end{cases}$$



Find $x\left(\frac{3}{2}t\right)$

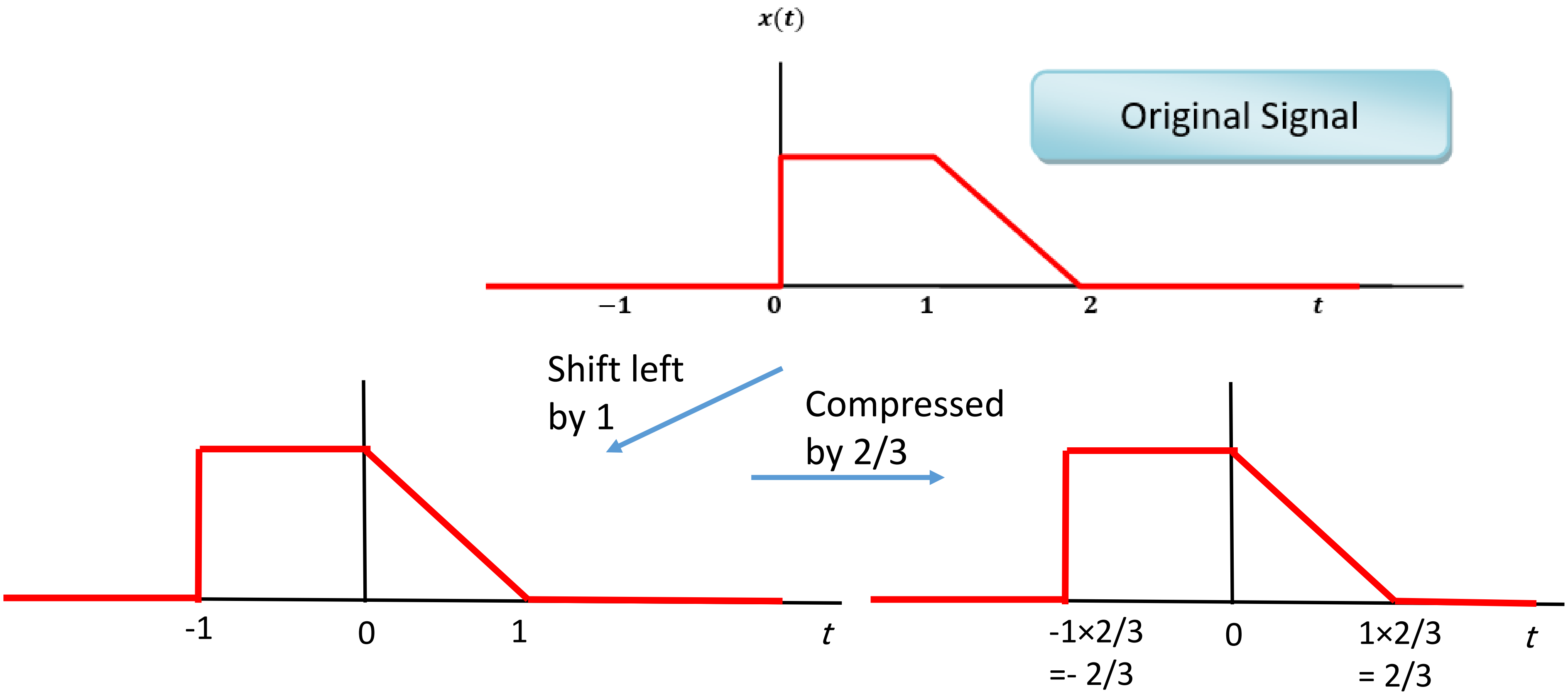
$x(\alpha t + \beta)$; $|\alpha| > 1$, so linear compression by a factor of 1 / (3/2) = 2/3



Example: Time Compression: (2)

Find $x\left(\frac{3}{2}t+1\right)$

Compressed by a factor of 2/3, and shift left by 1



Final Signal

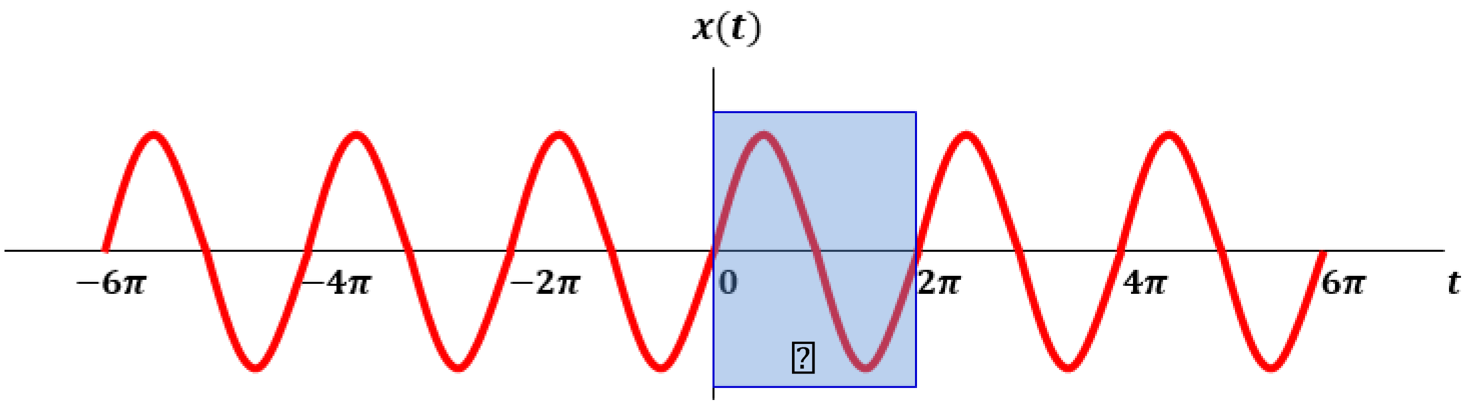
1.2.2 Periodic Signals

A periodic continuous-time signal $x(t)$ is defined as

$$x(t) = x(t + T)$$

where T is a positive number called the period.

A typical example is that of a sinusoidal signal $x(t) = \sin(t)$ for $-\infty < t < +\infty$.



For the above signal, the period is $T = 2$. It can be noticed that for any time t:

$$\boldsymbol{sin(t + 2\pi) = sin(t)} \qquad \boldsymbol{sin(t + m2\pi) = sin(t)}$$

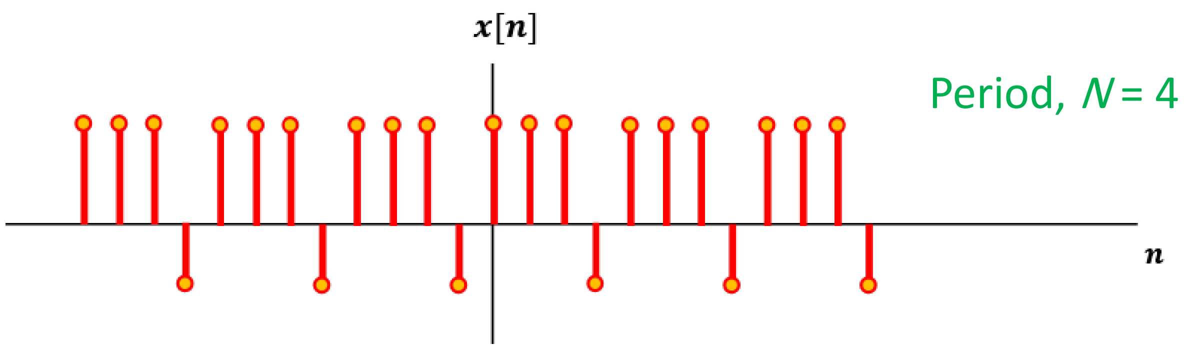
where m is a positive number.

Periodic Signals - continued

The ***fundamental period*** T_0 of $x(t)$ is the smallest positive value of T for which the equation $x(t) = x(t+T)$ holds.

A discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time-shift of N, i.e., if

$$x[n] = x[n+N] \quad \text{for all values of } n.$$



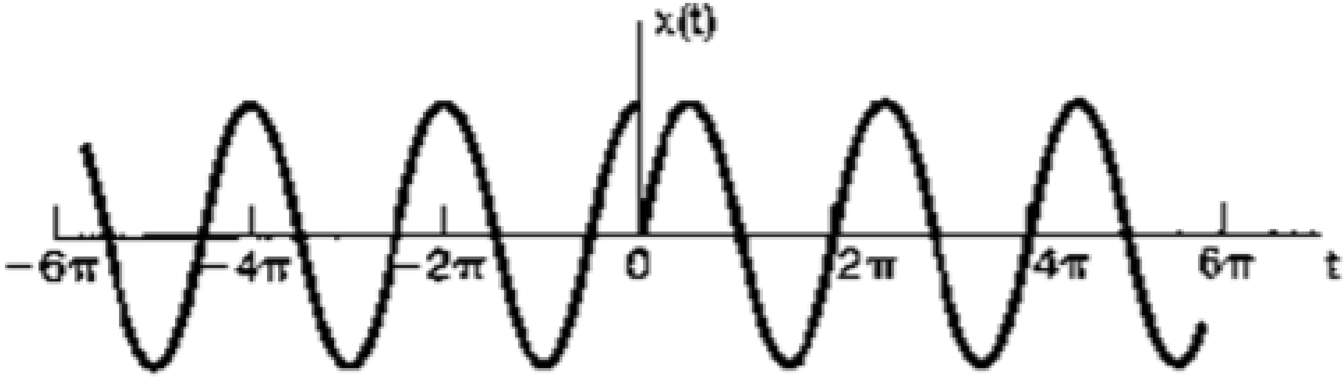
The ***fundamental period*** N_0 of $x[n]$ is the smallest positive value of N for which the equation $x[n] = x[n+N_0]$ holds.

Periodic Signals - Example

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$

Since, cos(2π+t) = cos(t) and sin(2π+t) = sin(t), considering t < 0 and t $\geq$ 0 separately, the signal repeats itself in every interval of 2π.

But, if we look at the following figure of x(t), we find there is a discontinuity at t = 0, which does not occur at any other time. Therefore, x(t) is not periodic.

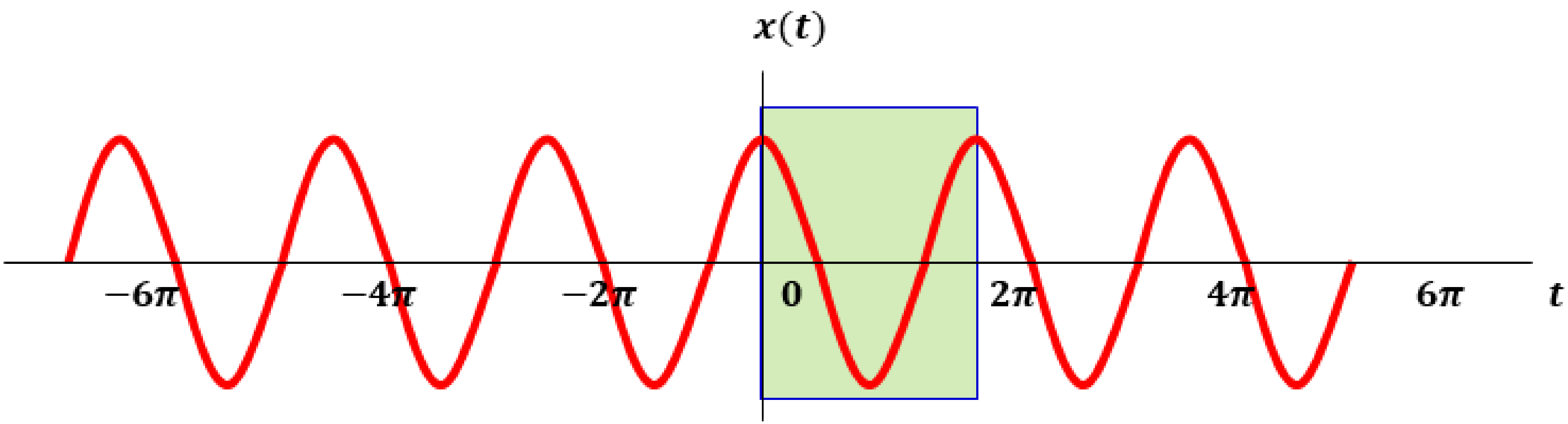


1.2.3 Even and Odd Signals

Even Signals

A signal $x(t)$ or $x[n]$ is defined as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin.

Even continuous-time Signal	$\boldsymbol{x(-t) = x(t)}$
Even Discrete-time Signal	$\boldsymbol{x[-n] = x[n]}$

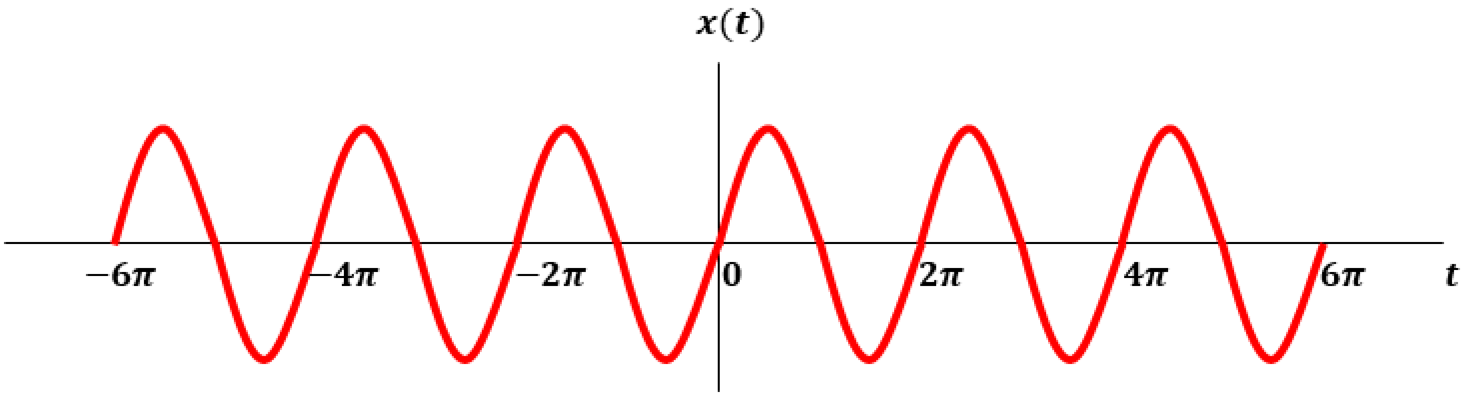


Odd Signals

A signal $x(t)$ or $x[n]$ is defined as an odd signal if,

Odd continuous-time Signal	$\boldsymbol{x(-t) = -x(t)}$
Odd Discrete-time Signal	$\boldsymbol{x[-n] = -x[n]}$

As a special case, the odd signal must be zero at $t = 0$ or $n = 0$.

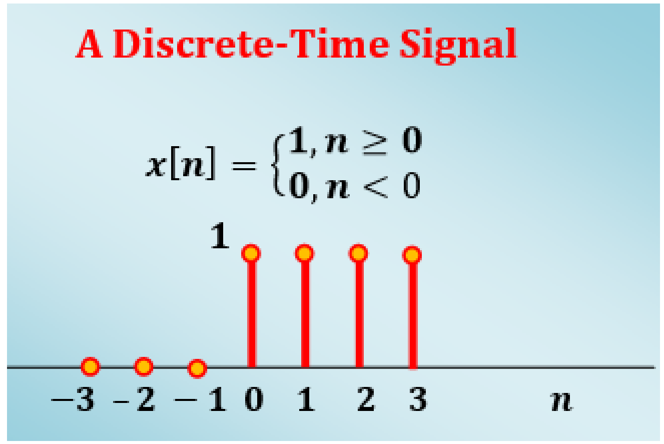


Decomposing a Signal into Even and Odd Parts

An important fact is that any signal (continuous-time or discrete-time) can be broken into a sum of two signals: even and odd.

Signal	Component	Mathematical Form
Continuous-time Signal $x(t)$	Even Part	$\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$
	Odd Part	$\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$
Discrete-time Signal $x[n]$	Even Part	$\mathcal{E}_v\{x[n]\} = \frac{1}{2}[x[n] + x[-n]]$
	Odd Part	$\mathcal{O}_d\{x[n]\} = \frac{1}{2}[x[n] - x[-n]]$

Decomposing a Signal into Even and Odd Parts Example



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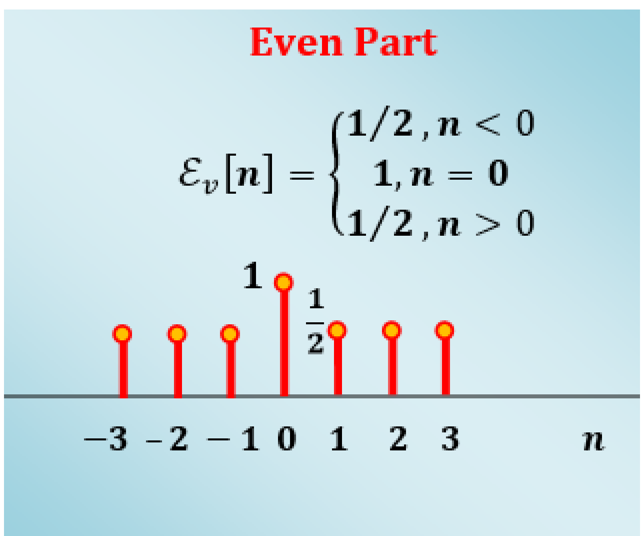

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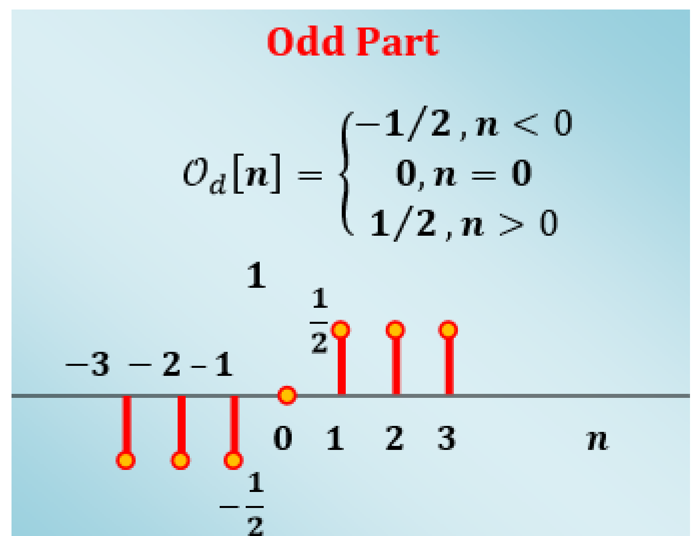


1.3 Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

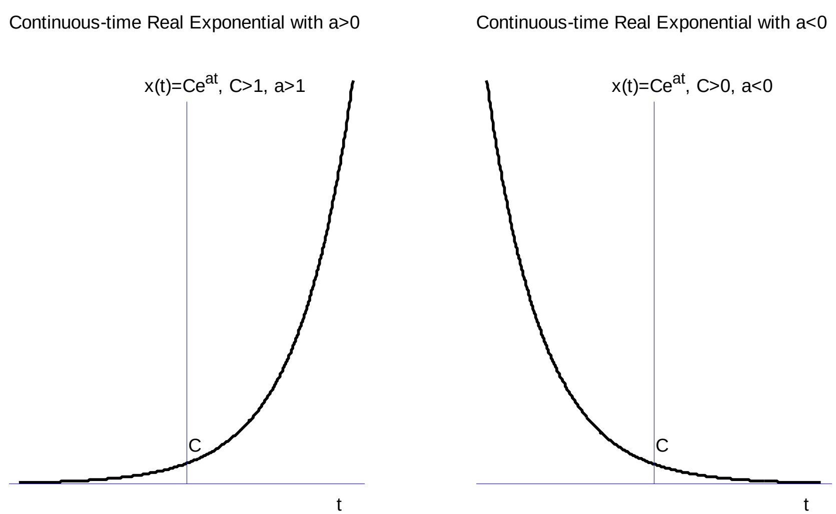
A continuous-time complex signal $x(t)$ can be written as

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers.

Real Exponential Signals

In this case both C and a are real numbers, and $x(t)$ is called a real exponential.



Periodic Complex Exponential and Sinusoidal Signals

Now we consider the case of complex exponentials where a is purely imaginary. More, specifically, we consider:

$$x(t) = e^{j\omega_0 t}$$

An important property of this signal is that it is periodic.

$$x(t) = x(t+T) \Rightarrow e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \Rightarrow e^{j\omega_0 T} = 1$$

This equation can be true,

1. If, $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T.

2. If, $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$, i.e. the smallest value of T for which the above equation holds, is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Periodic Signals

Replacing the value of T with this T_0, and using Euler's formula, that is,

$$e^{j\omega_0 T} = \cos(\omega_0 T) + j\sin(\omega_0 T)$$

We get

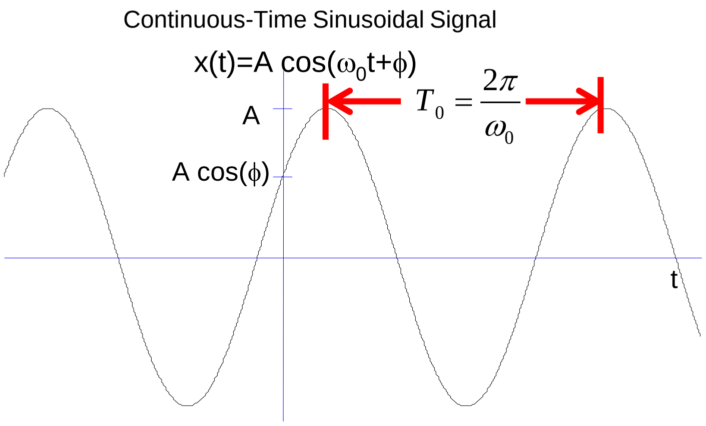
$$e^{j\omega_0 T} = \cos(2\pi) + j\sin(2\pi) = 1 + j0 = 1$$

Therefore, the signal $x(t)$ is a periodic signal.

Similarly, the signal $x(t) = e^{-j\omega_0 t}$ has the same fundamental period.

Sinusoidal Signal:

$$x(t) = A\cos(\omega_0 t + \phi)$$



Sinusoid Signals

$$A\cos(\omega_0 t+\phi)=A\left(\frac{e^{j(\omega_0 t+\phi)}+e^{-j(\omega_0 t+\phi)}}{2}\right)=\frac{A}{2}e^{j\phi}e^{j\omega_0 t}+\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

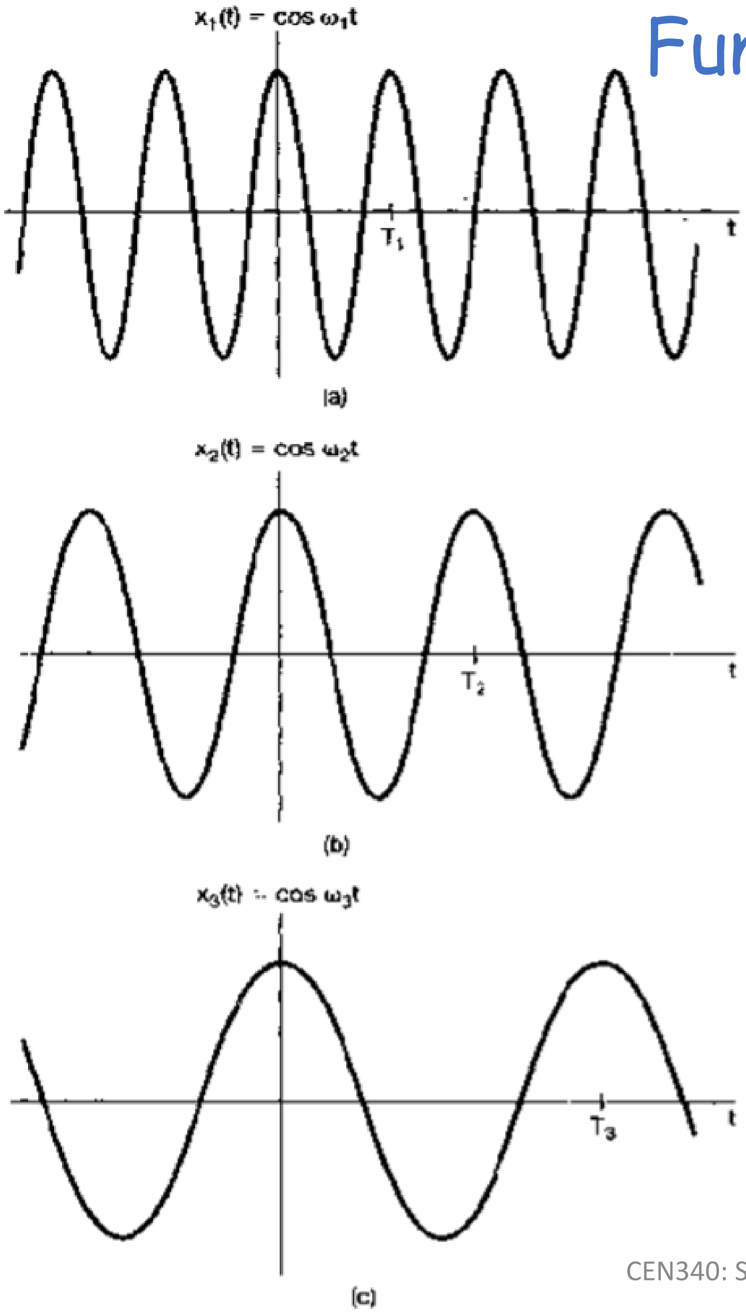
$$A\cos(\omega_0 t+\phi)=A\,\Re e\left\{e^{j(\omega_0 t+\phi)}\right\}$$

$$A\sin(\omega_0 t+\phi)=A\,\Im m\left\{e^{j(\omega_0 t+\phi)}\right\}$$

The fundamental period T_0 of a continuous-time sinusoidal or a periodic complex exponential signal, is inversely proportional to the $|\omega_0|$, which is called the *fundamental frequency*.

$$T_0=\frac{2\pi}{|\omega_0|}$$

Fundamental Period and Frequency



If we decrease the value of the magnitude of ω_0, we slow down the rate of oscillations and hence increase the period T_0. Alternatively, if we increase the value of the magnitude of ω_0, we increase the rate of oscillations and hence decrease the period T_0.

$$\omega_1 > \omega_2 > \omega_3$$
$$T_1 < T_2 < T_3$$

Energy & Power of Sinusoid / Complex Exp Signals

Over the one fundamental period T_0 of a continuous-time sinusoidal or a periodic complex exponential signal, the signal energy and power can be determined as:

$$E_{period} = \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt = \int_0^{T_0} 1dt = T_0$$

$$P_{period} = \frac{1}{T_0}\int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt = \frac{1}{T_0}\int_0^{T_0} 1dt = \frac{T_0}{T_0} = 1$$

As there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. The total average power is however remains 1, as by definition,

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt = \lim_{T\to\infty} \frac{1}{2T} 2T = 1$$

Harmonics of a Periodic Complex Exponential

We have noted that, $\boxed{e^{j\omega T_0} = 1}$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k \quad \text{where } \; k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

This shows that ω must be an integer multiple of ω_0, i.e., the fundamental frequency. We can therefore, write

$$\boxed{\phi_k(t) = e^{jk\omega_0 t}} \quad \text{where } \; k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

This is called the k-harmonic of the complex exponential signal.

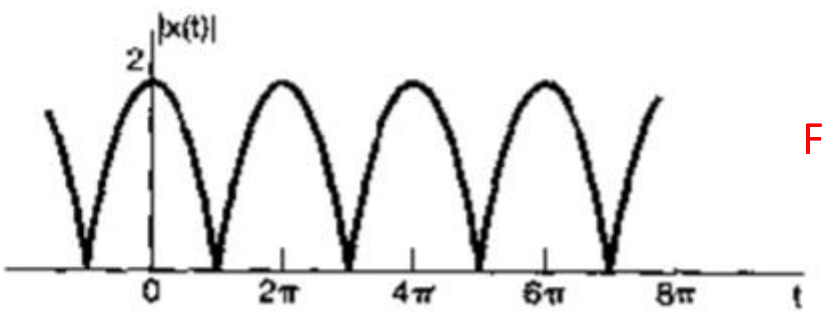
Expressing Two Complex Exponentials into a Product of One Complex Exp. & One Sinusoidal

$$x(t) = e^{j2t} + e^{j3t}$$

$$x(t) = e^{j2.5t}\left(e^{-j0.5t} + e^{j0.5t}\right) = 2e^{j2.5t}\cos(0.5t)$$

The magnitude of x(t) is:

$$|x(t)| = 2\,|\cos(0.5t)|$$



Full-wave rectified sinusoid.

General Complex Exponential Signals

The general complex exponential signals are of the form

$$x(t) = Ce^{at}$$

Where both C and a are complex numbers. Let us represent them as

$$\underbrace{C = |C|e^{j\theta}}_{\text{Polar form}} \qquad \underbrace{a = r + j\omega_0}_{\text{Cartesian form}}$$

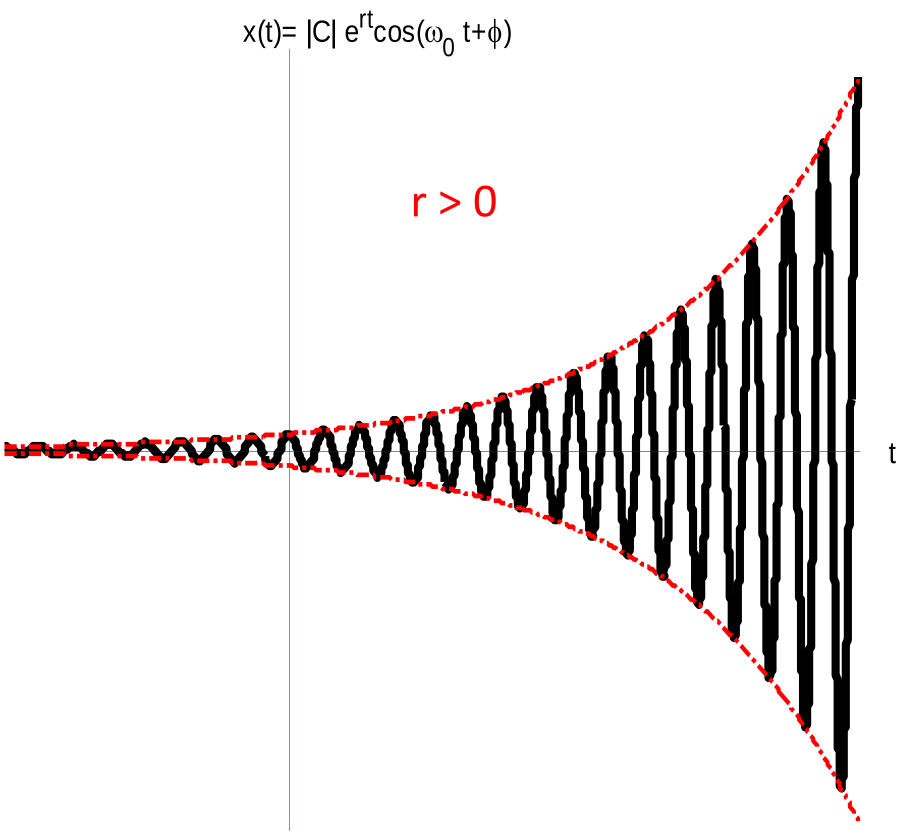
$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

$$x(t) = Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

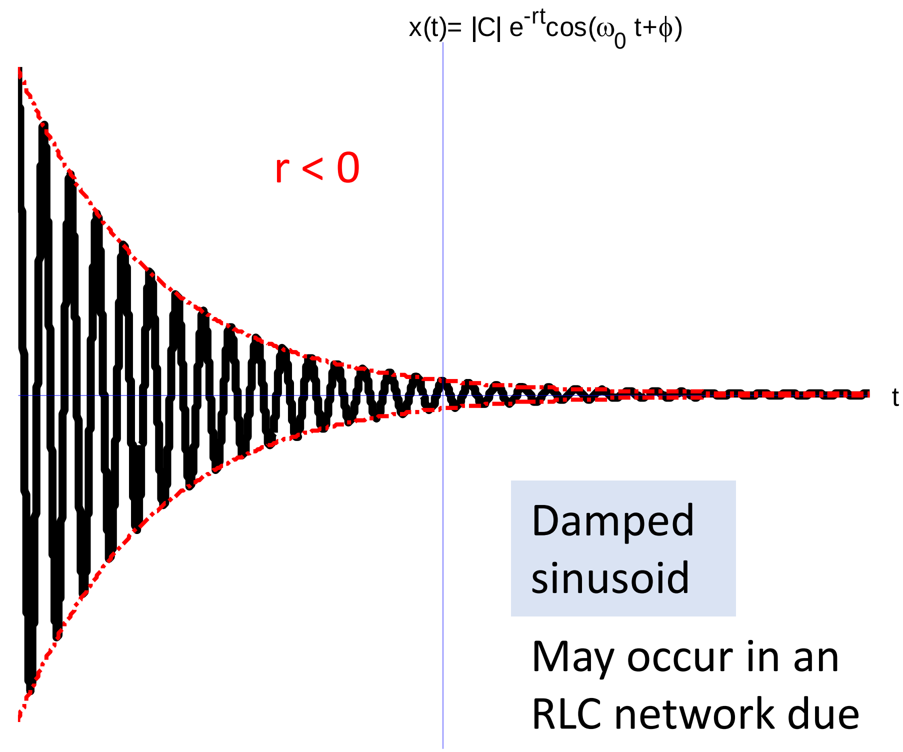
1. For $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal.
2. For $r > 0$, they correspond to sinusoidal signals multiplied with growing exponential.
3. For $r < 0$, they correspond to sinusoidal signals multiplied with decreasing exponentials.

Example: General Complex Exponential Signals

Sinusoid with growing exponential



Sinusoid with decaying exponential



Damped sinusoid

May occur in an RLC network due to resistors

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

A discrete-time complex exponential signal or sequence $x[n]$ can be written as

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. This could also be written as

$$x[n] = Ce^{\beta n}$$

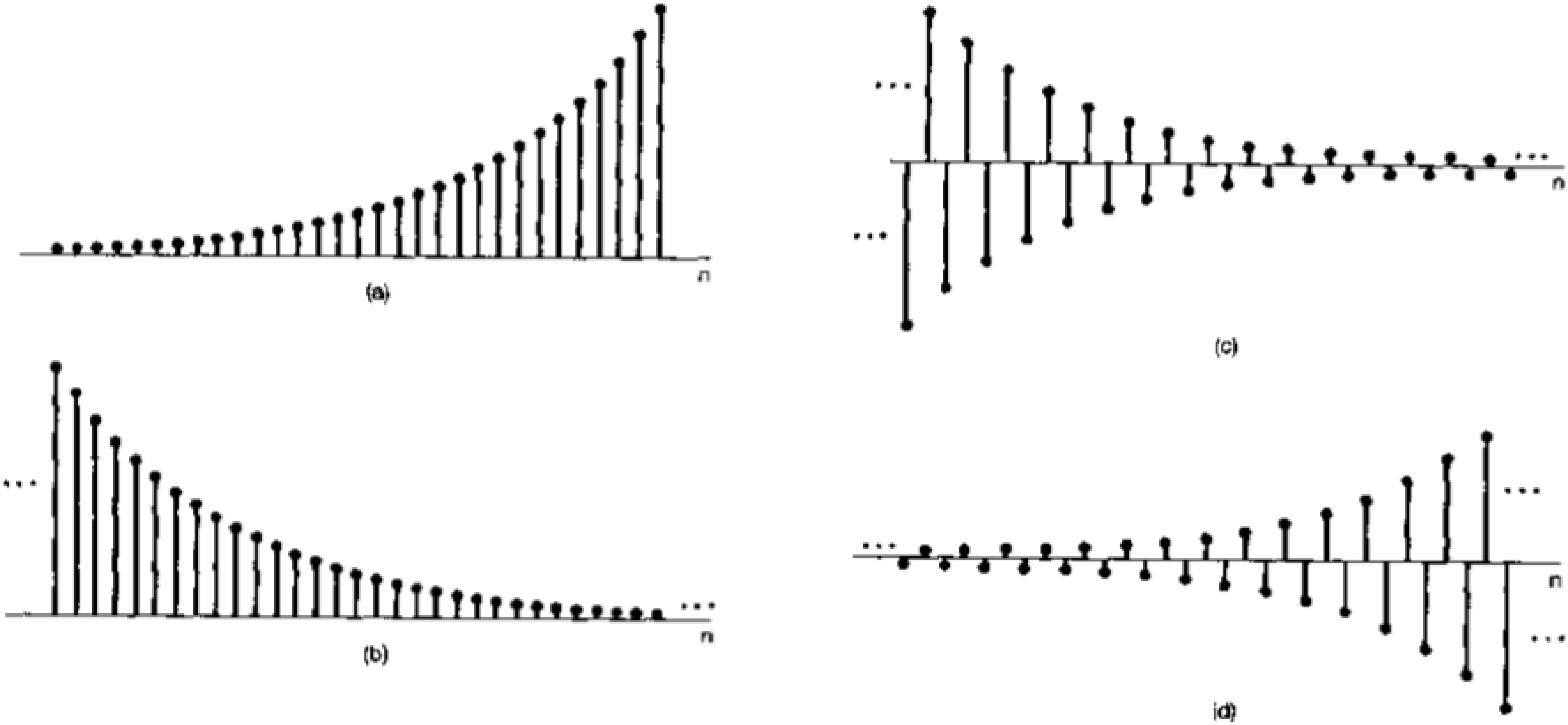
where $\alpha = e^{\beta}$

Real Exponential Signals

In this case both C and α are real numbers, and $x[n]$ is called a real exponential.

USAGE: Real-valued discrete-time exponentials are often used to describe population growth as a function of generation, and total return on investment as a function of day, month, a quarter.

Example: Real Exponential Signals



$$x[n] = C\alpha^n$$

(a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$

What will happen if (i) $\alpha = 1$, and (ii) $\alpha = -1$?

Discrete-Time Sinusoid Signals

$$x[n] = e^{j\omega_0 n} = \cos(\omega_0 n) + j\,\sin(\omega_0 n)$$

Therefore, a discrete-time sinusoid signal can be written as:

$$A\cos(\omega_0 n + \phi) = A\left(\frac{e^{j(\omega_0 n+\phi)} + e^{-j(\omega_0 n+\phi)}}{2}\right) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

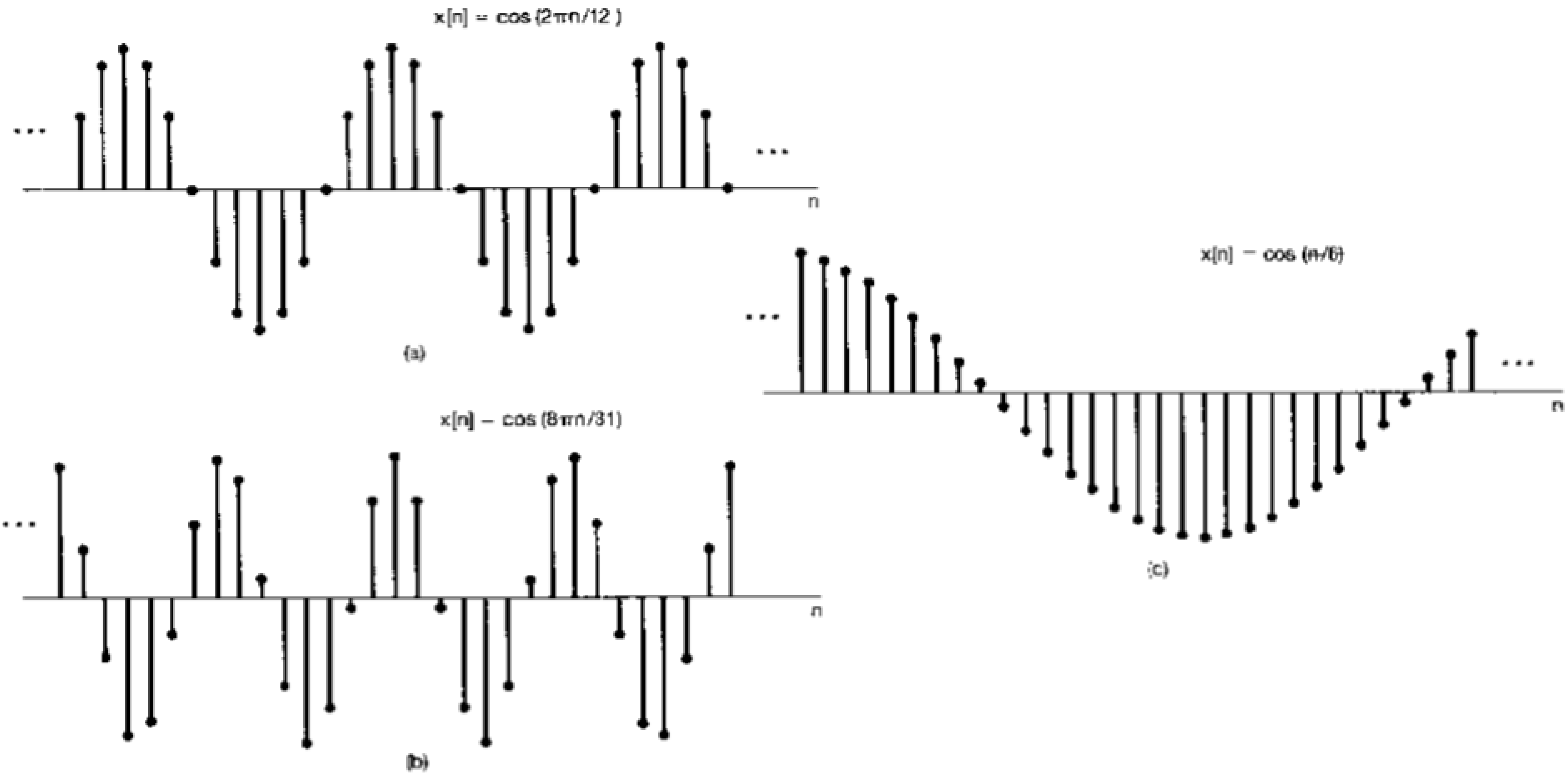
Using real and imaginary parts, we find:

$$A\cos(\omega_0 n + \phi) = A\,\Re e\left\{e^{j(\omega_0 n+\phi)}\right\}$$

$$A\sin(\omega_0 n + \phi) = A\,\Im m\left\{e^{j(\omega_0 n+\phi)}\right\}$$

Both the shaded signals have infinite total energy, but finite average power. For example, for every sample, $|e^{j\omega_0 n}|^2 = 1$, so it contributes to the total energy, making it infinite; however, per point time, the average power is 1.

Example: Discrete-Time Sinusoid Signals



Discrete-Time Complex Exponential Signals

The general discrete-time complex exponential signals are of the form

$$x[n] = C\alpha^n$$

where both C and α are complex numbers. Let us represent them as

$$\left.\begin{aligned} C &= |C|e^{j\theta} \\ \alpha &= |\alpha|e^{j\omega_0} \end{aligned}\right\} \quad \text{Polar form}$$

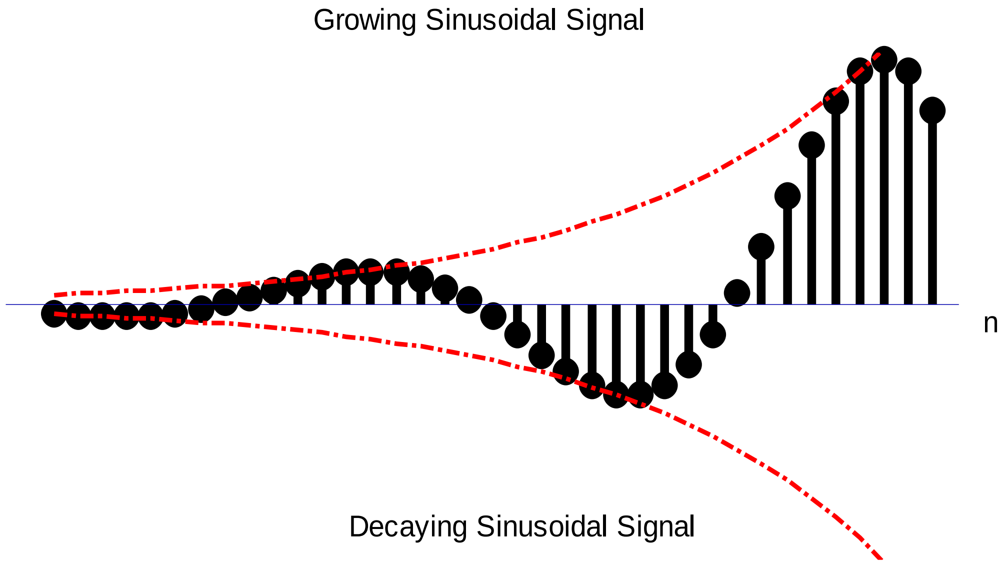
$$x[n] = C\alpha^n = |C|e^{j\theta}|\alpha|^n e^{j\omega_0 n} = |C||\alpha|^n e^{j(\omega_0 n+\theta)}$$

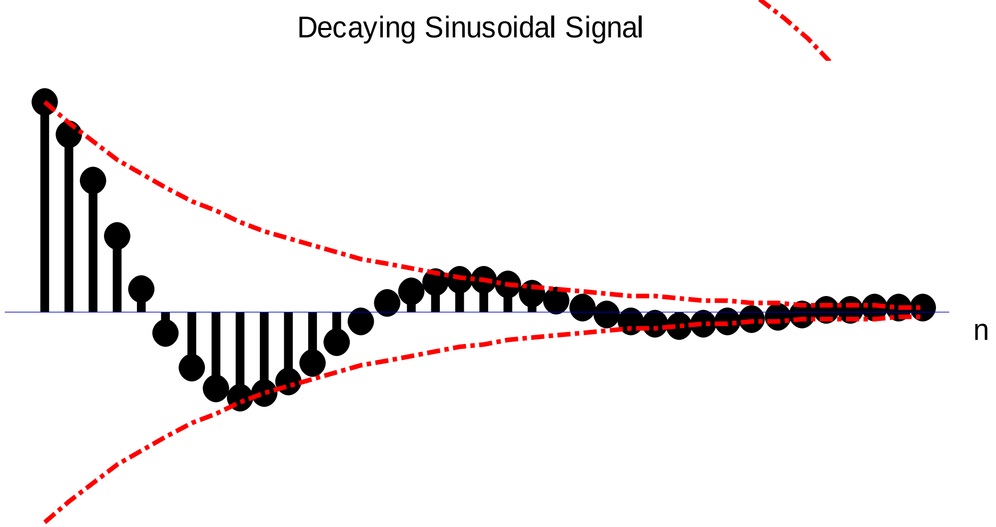
Using Euler's formula, it can be written as

$$x[n] = C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

1. For $|\alpha| = 1$, the real and imaginary parts of a complex exponential are sinusoidal.
2. For $|\alpha| > 1$, they correspond to sinusoidal signals / sequences multiplied with growing exponential.
3. For $|\alpha| < 1$, they correspond to sinusoidal signals / sequences multiplied with decreasing exponentials.

Discrete-Time Complex Exponential Signals





Discrete-Time Complex Exponential Signals

There are many similarities between continuous-time and discrete-time signals. But also there are many important differences. One of them is related with the the discrete-time exponential signal $e^{j\omega_0 n}$

The following properties were found with regard to the continuous-time exponential signal $e^{j\omega_0 t}$:

1. The larger the magnitude of ω_0, the higher is the rate of oscillations in the signal;
2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.

To see the difference for the first property, consider the discrete-time complex exponential:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}$$

This shows that the exponential at $\omega_0 + 2\pi$ is the same as that at frequency ω_0

Discrete-Time Complex Exponential Signals

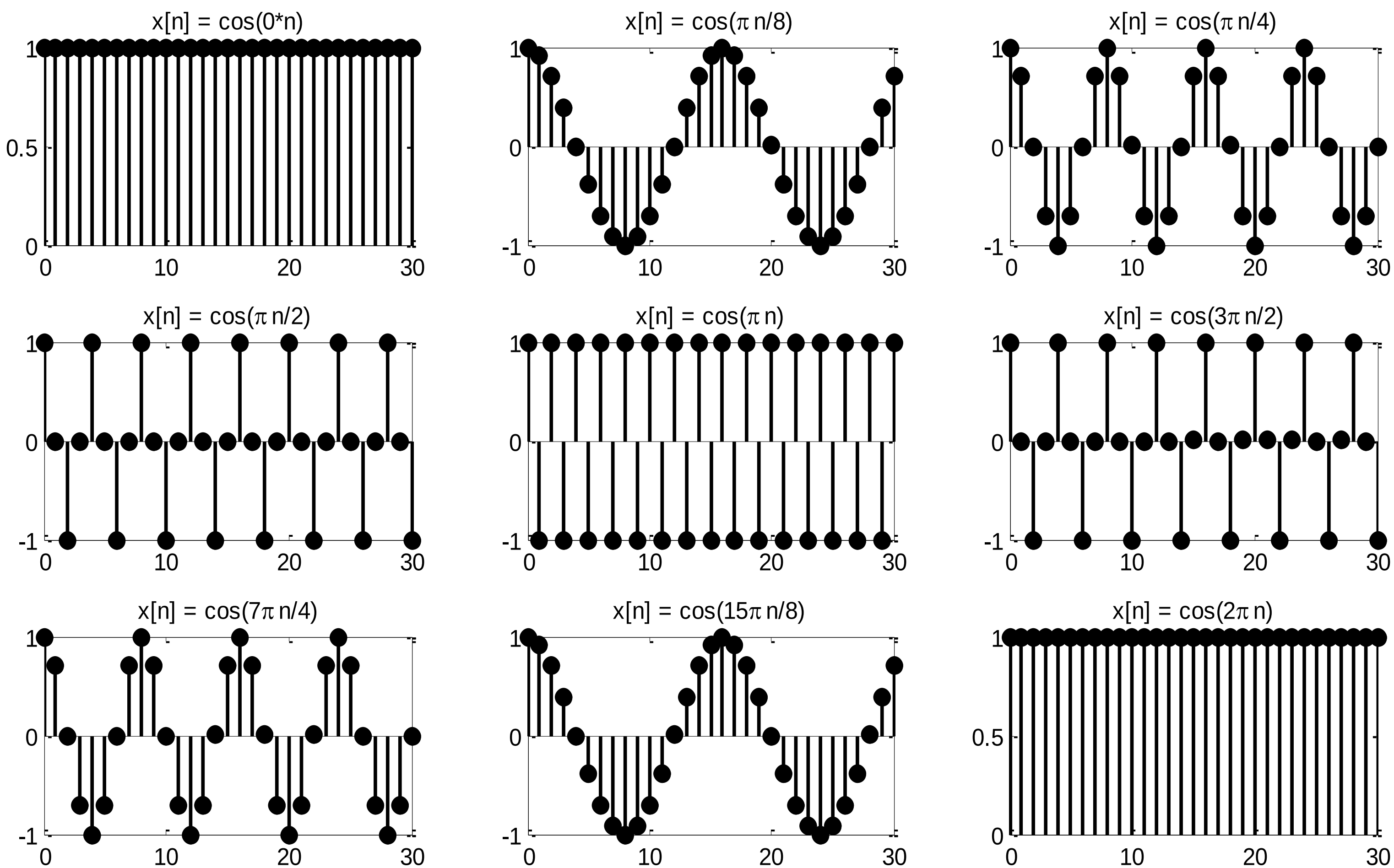
In case of continuous-time exponential, the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0.

- In discrete-time, these signals are not distinct. In fact, the signal with frequency ω_0 is identical to signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$ and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of size 2π. The most commonly used 2π intervals are $0 \leq \omega_0 \leq 2\pi$ or the interval $-\pi \leq \omega_0 \leq \pi$.
- As ω_0 is gradually increased, the rate of oscillations in the discrete-time signal does not keep on increasing. If ω_0 is increased from 0 to 2π, the rate of oscillations first increase and then decreases.
- Note in particular that for $\omega_0 = \pi$ or for any odd multiple of π,

$$e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$$

so that the signal oscillates rapidly, changing sign at each point in time.

Discrete-Time Complex Exponential Signals



Start decreasing from here

Periodicity of Discrete-Time Complex Exponential Signals

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \quad \rightarrow \quad e^{j\omega_0 N} = 1$$

It is true if $\omega_0 N$ is a multiple of 2π

$$\omega_0 N = 2\pi m \rightarrow \boxed{\frac{\omega_0}{2\pi} = \frac{m}{N}}$$

It means that the discrete-time signal $e^{j\omega_0 n}$ is periodic only when $\frac{\omega_0}{2\pi}$ is a rational number.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0.	Identical signals for values of ω_0 separated by multiples of 2π.
Periodic for any choice of ω_0.	Periodic only if $\omega_0 = 2\pi m/N$ for some integer $N > 0$ and m.
Fundamental frequency ω_0.	Fundamental frequency ω_0/m.
Fundamental period **$\omega_0 = 0$: undefined** **$\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$**	Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

Periodicity; Workout – (1)

Find fundamental period of the signal: $x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$

The first term

$$\omega_0 = 2\pi / 3$$

$$N = m\left(\frac{2\pi}{\omega_0}\right) = m\left(\frac{2\pi}{2\pi / 3}\right)$$

$$\frac{N}{m} = \frac{3}{1} \Rightarrow N = 3$$

The second term

$$\omega_0 = 3\pi / 4$$

$$N = m\left(\frac{2\pi}{\omega_0}\right) = m\left(\frac{2\pi}{3\pi / 4}\right)$$

$$\frac{N}{m} = \frac{8}{3} \Rightarrow N = 8$$

LCM(3, 8) = 24

Therefore, the fundamental period = 24

Cartesian to Polar & Vice Versa; Workout - (2)

1.1

$$\frac{1}{2}e^{j\pi}$$
$$= (1/2)\left(\cos\pi + j\sin\pi\right)$$
$$= (1/2)(-1 + j\,0)$$
$$= -(1/2) + j\,(0)$$

$$\frac{1}{2}e^{-j\pi}$$
$$= (1/2)\left(\cos(-\pi) + j\sin(-\pi)\right)$$
$$= (1/2)(-1 + j\,0)$$
$$= -(1/2) + j\,(0)$$

1.2

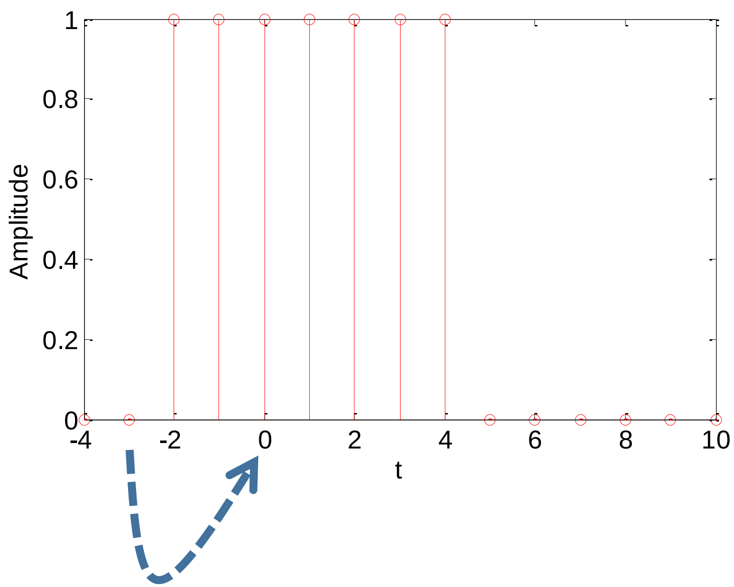
$$-3j$$
$$Ce^{-j\theta} = C\cos\theta - Cj\sin\theta = 0 - 3j\,(1)$$
$$\therefore C = 3$$
$$\sin\theta = 1 \Rightarrow \theta = \pi/2$$
$$-3j = 3e^{-j\pi/2}$$

1.4 (a)

Workout - (3)

Let, x[n] be a signal with x[n] = 0 for n < -2 and n > 4. For a signal x[n-3], determine the value of *n* for which it is guaranteed to be zero.

x[n-3] means shifting the signal towards right by 3 samples.



n < -2 → n+3 < -2+3 (=1)

n > 4 → n+ 3 > 4+3 (=7)

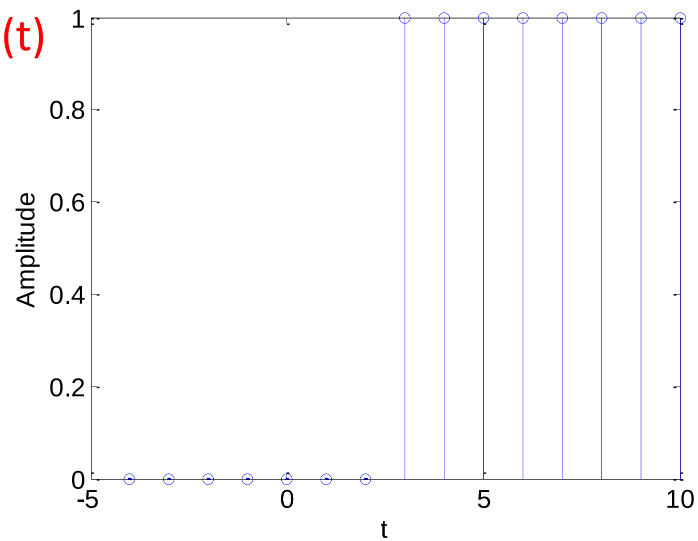
The shifted signal will be zero for n < 1 and n > 7.

1.5 (a)

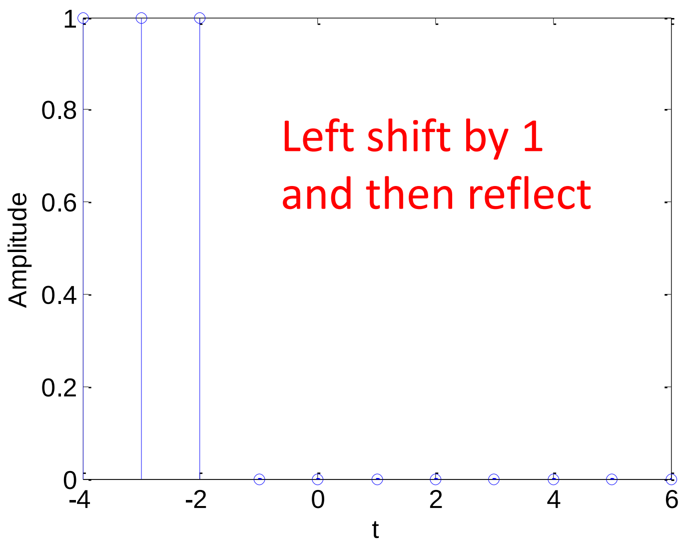
Workout - (4)

Let x(t) be a signal with x(t) = 0 for t < 3. For the signal x(1 – t), determine the value of t for which it is guaranteed to be zero.

x(t)



x(1-t)



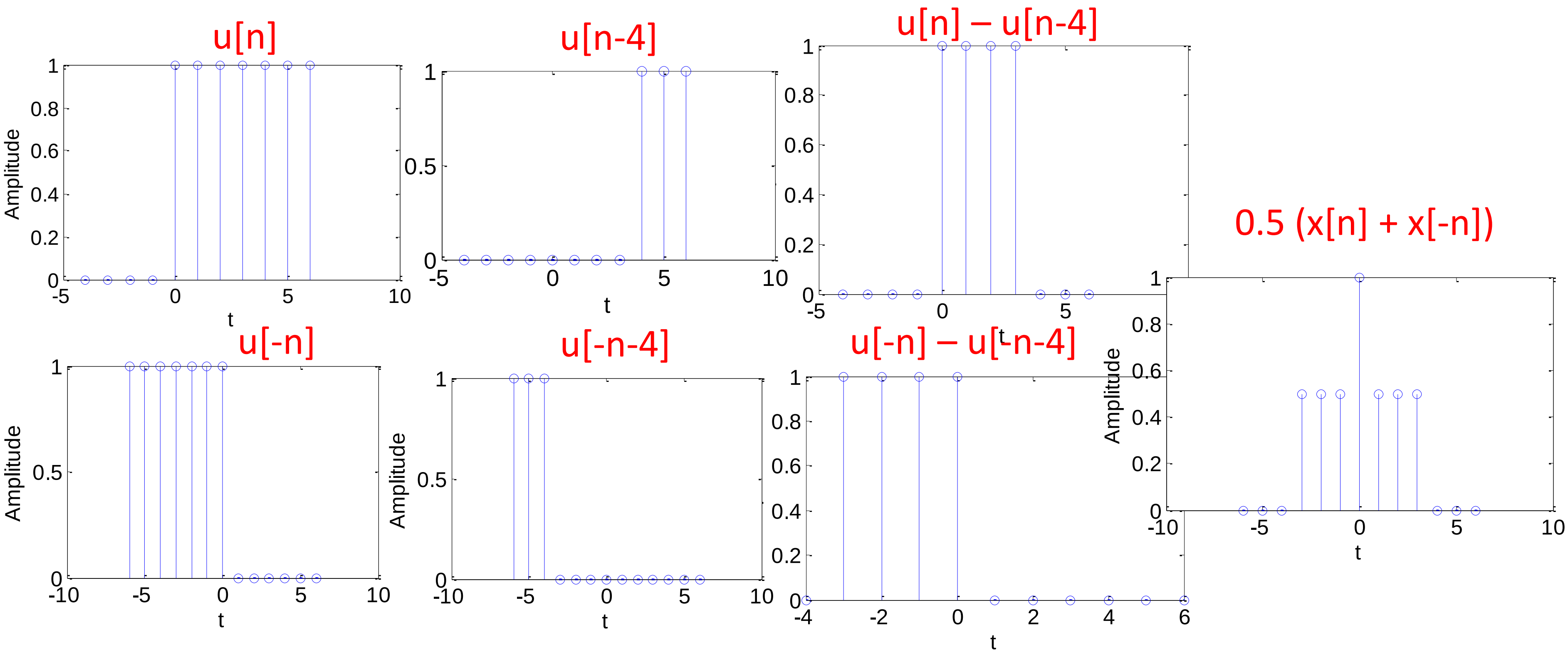
For t > -2, the signal is zero.

1.7 (a)

Workout - (5)

For a signal x[n] = u[n] – u[n-4], determine the values of the independent variable at which the even part of the signal is guaranteed to be zero.

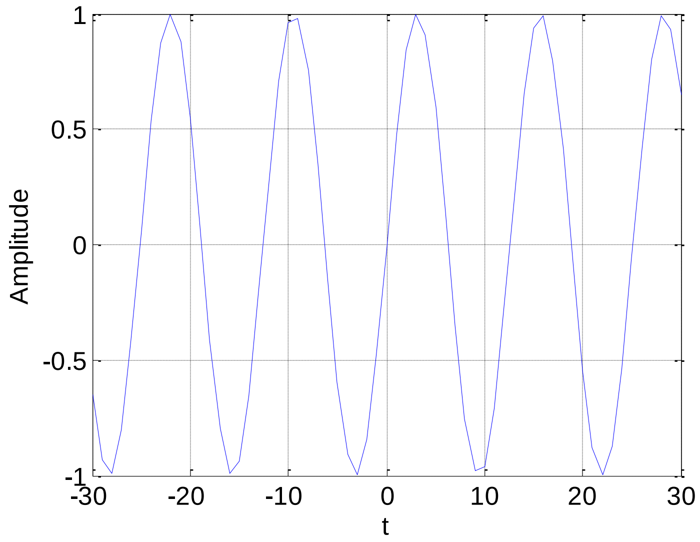
EVEN{ x[n]} = 0.5 (x[n] + x[-n]) = 0.5 (u[n] – u[n-4] + u[-n] – u[-n-4])



Zero for n > 3 and n <-3

Workout - (6)

For a signal x(t) = sin(0.5t), determine the values of the independent variable at which the even part of the signal is guaranteed to be zero.



It is always an odd signal, so the even part is zero for all values of t.

1.8 (a)

Workout - (7)

Express the real part of the signal, $x(t) = -2$, in the form $Ae^{-at}\cos(\omega t+\phi)$, where A, a, ω, and ϕ are real numbers with $A > 0$, and $-\pi < \phi < \pi$.

$$x(t) = A\,e^{-at}\cos(\omega t + \phi) = -2 = 2 \times 1 \times (-1) = 2\,e^{-0t}\cos(0t + \pi)$$

$A = 2$, $a = 0$, $\omega = 0$, and $\phi = \pi$

The above problem when the signal is $x(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi)$

$$x(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi) = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)\cos(3t + 2\pi)$$

$$\text{Real part} = \sqrt{2}\cos\frac{\pi}{4}\cos(3t + 2\pi) = \sqrt{2} \times \frac{1}{\sqrt{2}} \times \cos 3t = \cos 3t$$

$$= 1 \times e^{0t} \times \cos(3t + 0)$$

$A = 1$, $a = 0$, $\omega = 3$, and $\phi = 0$

1.9 (a)

Workout - (8)

If the signal x(t) is periodic, find the fundamental period.

$$x(t) = je^{j10t}$$

$$= j(\cos 10t + j\sin 10t) = j\cos 10t - \sin 10t$$

$$= j\sin\left(10t + \pi/2\right) + \cos\left(10t + \pi/2\right)$$

$$= e^{\left(10t + \pi/2\right)}$$

Fundamental period: $$T_0 = \frac{2\pi}{|\omega_0|} = \frac{2\pi}{10} = \frac{\pi}{5}$$

1.10

Workout - (9)

If the signal x(t) is periodic, find the fundamental period.

$$x(t) = 2\cos(10t + 1) - \sin(4t - 1)$$

$$T_0 = \frac{2\pi}{10} = \frac{\pi}{5} \qquad T_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

Fundamental period:

LCM ($\pi/5$, $\pi/2$) = LCM (π, π) / HCF (5, 2) = $\pi/1$ = π

$$LCM\left(\frac{a}{b}, \frac{c}{d}\right) = LCM(a, c) / HCF(b, d)$$

1.11

Workout - (10)

Determine the fundamental period of the following signal x[n].

$$x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$$

$$N_0 = m\left(\frac{2\pi}{\omega_0}\right)$$

$$N_0(\text{first part}) = 1$$

$$N_0(\text{second part}) = m\left(\frac{2\pi}{4\pi/7}\right) = m(7/2) = 7$$

$$N_0(\text{third part}) = m\left(\frac{2\pi}{2\pi/5}\right) = m(5/2) = 5$$

$$N_0 = LCM(1,7,5) = \boxed{35}$$

Acknowledgement

The slides are prepared based on the following textbook:

- Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab, *Signals & Systems*, 2nd Edition, Prentice-Hall, Inc., 1997.

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- *Prof. Anwar M. Mirza*, former faculty member, College of Computer and Information Sciences, King Saud University
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