Ch 3 - Fourier Series Representation

Jean Baptiste Joseph **Fourier**

- Born in Auxerre, France
- Mathematician and physicist
- Developed Fourier series, Fourier transforms and their applications on heat and vibration
- Life span: 21 March 1768 – 16 May 1830
- Also known as an Egyptologist.

Ch 3 - Fourier Series Representation

3.2 The response of LTI systems to complex exponentials

The response of an LTI system to a complex exponential input is the same complex exponential with only a change of amplitude.

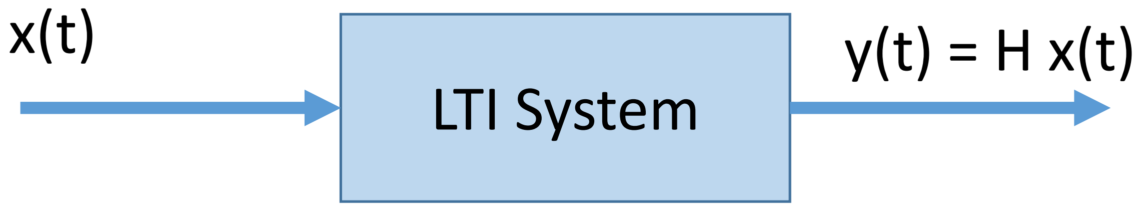
$$\text{Continuous time: } e^{st} \rightarrow H(s)e^{st}$$

$$\text{Discrete time: } z^{n} \rightarrow H(z)z^{n}$$

H(s) and *H(z)* are the complex amplitude factor.

A signal for which the system output is a (possibly complex) constant times the input is referred to as an ‘eigen function’ of the system, and the amplitude factor is referred to as the system’s ‘eigenvalue’.

Continuous time case



$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- The complex exponentials are ‘eigenfunctions’ of LTI systems.
- The constant H(s) for a specific value of s is the ‘eigenvalue’ associated with the eigenfunction e^{st}.

If $x(t) = e^{st}$ (a complex exponential)

then $y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$

$$e^{s(t-\tau)} = e^{st}.e^{-s\tau}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{st}.e^{-s\tau}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$$y(t) = H(s)e^{st}$$

where, $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

Discrete time case

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If $x[n] = z^n$

then $y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

$$y[n] = H(z)z^n$$

where $H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$

- The complex exponentials are 'eigenfunctions' of LTI systems.
- The constant H(z) for a specific value of z is the 'eigenvalue' associated with the eigenfunction z^n.

3.3 Fourier Series Representation of Continuous-Time Periodic Signals

3.3.1 Linear combination of harmonically related complex exponentials

A signal is periodic, if, for some positive value of *T*,

$$x(t) = x(t+T), \quad \text{for all } t$$

The fundamental period of x(t) is the minimum, positive, nonzero value of *T* for which the above equation is satisfied.

$$\text{Fundamental frequency: } \omega_0 = \frac{2\pi}{T}$$

Basic periodic signals:

Sinusoidal: $x(t) = \cos(\omega_0 t)$

Complex exponential: $x(t) = e^{j\omega_0 t}$

Harmonically related signals with the complex exponential:

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Linear combination of harmonically related complex exponentials

x(t) with n mutually orthogonal functions is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$ is periodic with period T.

- The term for $k = 0$ is a constant.
- The terms for $k = \pm 1$ are the ‘first harmonic components’ or ‘fundamental components’.
- The terms for $k = \pm 2$ are the ‘second harmonic components’.
- The terms for $k = \pm N$ are the ‘N-th harmonic components’.

→ Fourier Series Representation

Example 3.2

Consider a periodic signal x(t) with fundamental frequency 2π, expressed as:

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

where,

$$a_0 = 1$$
$$a_1 = a_{-1} = 1/4$$
$$a_2 = a_{-2} = 1/2$$
$$a_3 = a_{-3} = 1/3$$

With these values, Eq. (3.26) can be re-written as:

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

$$2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$$

3.3.2 Determination of the Fourier Series Representation of Continuous-Time Signals

Given a periodic continuous signal x(t) with a fundamental period *T* and the fundamental frequency $\omega_0 = \frac{2\pi}{T}$

We can express it as a Fourier series as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

How to find a_k?

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}$$

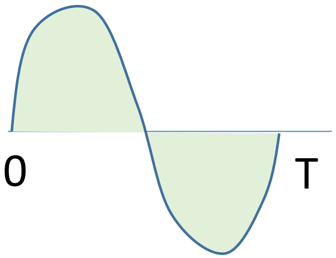
$$\int_0^T x(t)e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

Fourier Series Representation : Continued

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(k-n)\omega_0 t dt + j\int_0^T \sin(k-n)\omega_0 t dt$$

For $k \neq n$,

$$\int_0^T \cos(k-n)\omega_0 t dt = 0, \quad \int_0^T \sin(k-n)\omega_0 t dt = 0$$



Area = 0

For $k = n$,

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(0)dt + j\int_0^T \sin(0)dt = \int_0^T dt = T = T\,\delta_{kn}$$

Kronecker Delta: $\delta_{kn} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$

Fourier Series Representation : Continued

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k T\,\delta_{kn} = a_n T$$

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

$$\Rightarrow a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

a_k are called **Fourier series coefficients**, or spectral coefficients

Synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

a_0: dc component

Analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-jk(2\pi/T)t}dt$$

Fourier Series Representation : Example 3.3

A CT signal with fundamental frequency w_0: $x(t) = \sin(\omega_0 t)$

Using Euler's formula:

$$\left.\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned}\right| \Rightarrow \sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing with Fourier synthesis equation:

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t}$$

We get,

$$a_0 = 0;$$

$$a_1 = (1/2j) = \frac{1}{2j} \times \frac{j}{j} = -\frac{1}{2} j$$

$$a_{-1} = (-1/2j) = \frac{-1}{2j} \times \frac{j}{j} = \frac{1}{2} j$$

$$a_k = 0, \quad \text{for } |k| > 1$$

Fourier Series Representation : Example 3.4

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \pi/4)$$

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \frac{2}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

Comparing with Fourier series expansion,

$$x(t) = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$
$$+ a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t}$$

$$a_0 = 1$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j; \quad a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$$

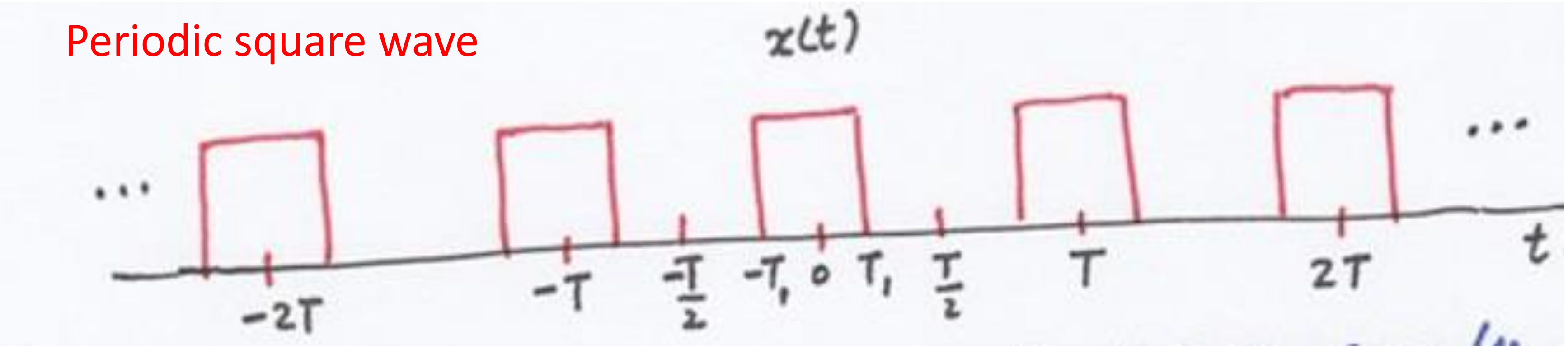
$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = \frac{1}{2\sqrt{2}}(1 + j)$$

$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) = \frac{1}{2\sqrt{2}}(1 - j)$$

$$a_k = 0, \quad \text{for} \quad |k| > 2$$

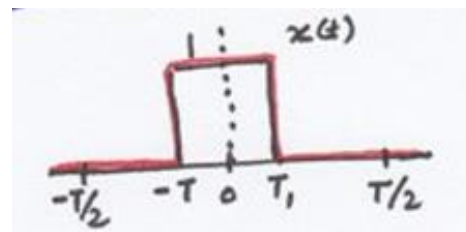
Fourier Series Representation : Example 3.5

Periodic square wave



One period:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



Analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T}\int_{-T_1/2}^{T_1/2} dt = \frac{2T_1}{T}$$

Fourier Series Representation : Example 3.5

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk(2\pi/T)t}dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk(2\pi/T)t}dt = \frac{1}{T}\cdot\frac{-1}{jk(2\pi/T)}e^{-jk(2\pi/T)t}\Big|_{-T_1}^{T_1}$$

$$= \frac{1}{k\pi}\left(\frac{-1}{2j}\right)\left[e^{-jk(2\pi/T)T_1} - e^{jk(2\pi/T)T_1}\right]$$

$$= \left(\frac{1}{k\pi}\right)\left[\frac{e^{jk(2\pi/T)T_1} - e^{-jk(2\pi/T)T_1}}{2j}\right]$$

$$\Rightarrow a_k = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi} = \frac{\sin\left(k\omega_0 T_1\right)}{k\pi}, \quad \text{for } k \neq 0$$

Problem No. 3.1

A continuous-time periodic signal x(t) is real valued and has a fundamental period T = 8. The non-zero Fourier series coefficients for x(t) are $a_1 = a_{-1} = 2$, $a_3 = a_{-3}^* = 4j$. Express x(t) in the form $x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \varphi_k)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$= a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t}$$
$$= 2e^{j\omega_0 t} + 2e^{-j\omega_0 t} + 4je^{j3\omega_0 t} - 4je^{-j3\omega_0 t}$$
$$= 4\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] + 8(-1)\left[\frac{e^{j3\omega_0 t} - e^{-j3\omega_0 t}}{2j}\right]$$
$$= 4\cos(\omega_0 t) - 8\sin(3\omega_0 t)$$
$$= 4\cos(\omega_0 t + 0) - 8\cos(3\omega_0 t + \pi/2)$$

$$A_1 = 4;\ \omega_1 = \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4};\ \varphi_1 = 0$$
$$A_3 = -8;\ \omega_3 = 3\omega_0 = 3\frac{2\pi}{8} = \frac{3\pi}{4};\ \varphi_1 = \frac{\pi}{2}$$
$$\text{All other } A_k, \omega_k, \varphi_k = 0$$

Problem No. 3.3

For the continuous-time periodic signal, $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$

Determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(t) = 2 + (1/2)\left[e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}\right] + (4/2j)\left[e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t}\right]$$

$$= 2 + (1/2)e^{j\frac{2\pi}{3}t} + (1/2)e^{-j\frac{2\pi}{3}t} + (-2j)e^{j\frac{5\pi}{3}t} + (2j)e^{-j\frac{5\pi}{3}t}$$

$$\begin{aligned} x(t) = a_0 &+ a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} \\ &+ a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} \\ &+ a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t} \\ &+ a_4 e^{j4\omega_0 t} + a_{-4} e^{-j4\omega_0 t} \\ &+ a_5 e^{j5\omega_0 t} + a_{-5} e^{-j5\omega_0 t} + \cdots \end{aligned}$$

$\Rightarrow$

$$\omega_0 = \frac{\pi}{3}, a_0 = 2$$

$$a_2 = 1/2, \quad a_{-2} = 1/2$$

$$a_5 = -2j, \quad a_{-5} = 2j$$

$$\text{for all other } k, a_k = 0$$

3.5 Properties of continuous-time Fourier series

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$	**Synthesis Equation**
$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$ $= \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$	**Analysis Equation**

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ periodic with perio fundamental frequency	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jkw_0 t_0}$ $= a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jMw_0 t} x(t)$ $= e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha$ > 0 (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau) y(t-\tau) d\tau$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$

Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$a_k = a_{-k}^*$ $\mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\}$ $\mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\}$ $\lvert a_k\rvert = \lvert a_{-k}\rvert$ $\sphericalangle a_k = -\sphericalangle a_{-k}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of real Signals	$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\}, & x(t) \text{ real} \\ x_o(t) = \mathcal{Od}\{x(t)\}, & x(t) \text{ real} \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\,\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Problem No. 3.6

Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk(2\pi/50)t}$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk(2\pi/50)t}$$

$$x_3(t) = \sum_{k=-100}^{100} j\sin(k\pi/2) e^{jk(2\pi/50)t}$$

Use Fourier series properties to help answer the following questions:

(a) Which of the three signals is/are real valued?

(b) Which of the three signals is/are even?

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

For $x_1(t)$:

$$\omega_0 = \frac{2\pi}{50}$$

$$a_k = \left(\frac{1}{2}\right)^k, \quad \text{for } k = 0, 1, 2, \ldots, 100$$

$$a_k = 0, \quad \text{for } k > 100, k < 0$$

For real $x_1(t)$, $a_k = a_{-k}^*$

However, here $a_{10} = (1/2)^{10}$

$a_{-10} = 0$; $a_{10} \neq a_{-10}^*$

$x_1(t)$ is not real.

Problem No. 3.6

For $x_1(t)$ to be even: $x_1(t) = x_1(-t)$

$$x_1(-t) = \sum_{k=0}^{100}\left(\frac{1}{2}\right)^k e^{-jk(2\pi/50)t} = \sum_{k=-100}^{0}\left(\frac{1}{2}\right)^{-k} e^{jk(2\pi/50)t} \neq \sum_{k=0}^{100}\left(\frac{1}{2}\right)^k e^{jk(2\pi/50)t} = x_1(t)$$

$x_1(t)$ is not even.

For $x_2(t)$:

$$\omega_0 = \frac{2\pi}{50}$$

$$a_k = \cos(k\pi), \quad \text{for } -100 \le k \le 100$$
$$a_k = 0, \quad \textit{otherwise}$$

$$a_{-k}{}^* = (\cos(-k\pi))^* = \cos(k\pi) = a_k$$
$$\mathrm{Re}\{a_k\} = \cos(k\pi), \quad \mathrm{Re}\{a_{-k}\} = \cos(-k\pi) = \cos(k\pi)$$
$$\Rightarrow \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\}$$
$$\mathrm{Im}\{a_k\} = 0 = \mathrm{Im}\{a_{-k}\}$$
$$|a_k| = |a_{-k}|, \quad \measuredangle a_k = 0 = -\measuredangle a_{-k}$$

$x_2(t)$ is real.

$$a_k = \cos(k\pi)$$
$$a_{-k} = \cos(-k\pi) = \cos(k\pi)$$
$$\Rightarrow a_k = a_{-k}$$

$x_2(t)$ is even.

Problem No. 3.6

For $x_3(t)$:

$$x_3(t) = \sum_{k=-100}^{100} j\sin(k\pi/2)e^{jk(2\pi/50)t}$$

$$\omega_0 = \frac{2\pi}{50}$$

$$a_k = j\sin(k\pi/2), \quad \text{for } -100 \le k \le 100$$

$$a_k = 0, \quad otherwise$$

$$a_{-k} = j\sin(-k\pi/2) = -j\sin(k\pi/2)$$

$$a_{-k}{}^* = j\sin(k\pi/2) = a_k$$

$$\text{Re}\{a_k\} = 0 = \text{Re}\{a_{-k}\}$$

$$\text{Im}\{a_k\} = \sin(k\pi/2); \quad \text{Im}\{a_{-k}\} = -\sin(k\pi/2)$$

$$\Rightarrow \text{Im}\{a_k\} = -\text{Im}\{a_{-k}\}$$

$$|a_k| = |\sin(k\pi/2)|, \quad |a_{-k}| = |\sin(-k\pi/2)| = |\sin(k\pi/2)|$$

$$\Rightarrow |a_k| = |a_{-k}|$$

$$\measuredangle a_k = \tan^{-1}\left(\frac{\sin(k\pi/2)}{0}\right) = \tan^{-1}(\infty) = \pi/2$$

$$\measuredangle a_{-k} = \tan^{-1}\left(\frac{\sin(-k\pi/2)}{0}\right) = \tan^{-1}(-\infty) = -\pi/2$$

$$\Rightarrow \measuredangle a_k = -\measuredangle a_{-k}$$

$x_3(t)$ is real.

$$a_{-k} = j\sin\left(\frac{-k\pi}{2}\right) = -j\sin\left(\frac{k\pi}{2}\right)$$

$$\Rightarrow a_k \ne a_{-k}$$

$x_3(t)$ is not even.

Problem No. 3.8

Suppose we are given the following information about a signal x(t):

(1) x(t) is real and odd
(2) x(t) is periodic with period T = 2, and has Fourier coefficients a_k.
(3) $a_k = 0$ for $|k| > 1$
(4) $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$

Specify a signal that satisfies these conditions.

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

From (2): $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

From (3): $x(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$

From (1): $a_0 = 0$, because x(t) is odd.

From (1): a_k is odd $\Rightarrow a_1 = -a_{-1}$

$$x(t) = a_1 e^{j\omega_0 t} - a_1 e^{-j\omega_0 t} = a_1\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$\Rightarrow x^*(t) = a_1^*\left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right)$$

$$|x(t)|^2 = |x(t)x^*(t)| = |a_1 a_1^*||1 - e^{j\omega_0 t} - e^{-j\omega_0 t} + 1|$$

$$= |a_1 a_1^*||2 - 2\cos(2\omega_0 t)| = 2|a_1 a_1^*||1 - \cos(2\omega_0 t)|$$

Problem No. 3.8

From (4): $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$

$$\frac{1}{2}\int_0^2 2|a_1 a_1^*||1-\cos(2\omega_0 t)|dt = 1$$

$$\Rightarrow |a_1 a_1^*|\int_0^2 |1-\cos(2\omega_0 t)|dt = 1$$

$$\Rightarrow |a_1 a_1^*|\left[t - \frac{\sin(2\omega_0 t)}{2\omega_0}\right]_0^2 = 1 \Rightarrow |a_1 a_1^*|[2-0-0+0] = 1 \Rightarrow a_1 a_1^* = 1/2$$

As a_1 is complex $\{\mathrm{Re}(a_1)\}^2 + \{\mathrm{Im}(a_1)\}^2 = 1/2$

As a_1 is purely imaginary $\{0\}^2 + \{\mathrm{Im}(a_1)\}^2 = 1/2 \Rightarrow \mathrm{Im}(a_1) = \pm 1/\sqrt{2}$

$$a_1 = \pm j1/\sqrt{2}$$

$$a_{-1} = -a_1 = \mp j1/\sqrt{2}$$

Therefore, the signals are

$$x_1(t) = \frac{1}{\sqrt{2}} je^{j\pi t} - \frac{1}{\sqrt{2}} je^{-j\pi t} = -\sqrt{2}\sin(\pi t)$$

$$x_2(t) = -\frac{1}{\sqrt{2}} je^{j\pi t} + \frac{1}{\sqrt{2}} je^{-j\pi t} = \sqrt{2}\sin(\pi t)$$