Chapter 2

Methods to solve linear system

Gauss Method:

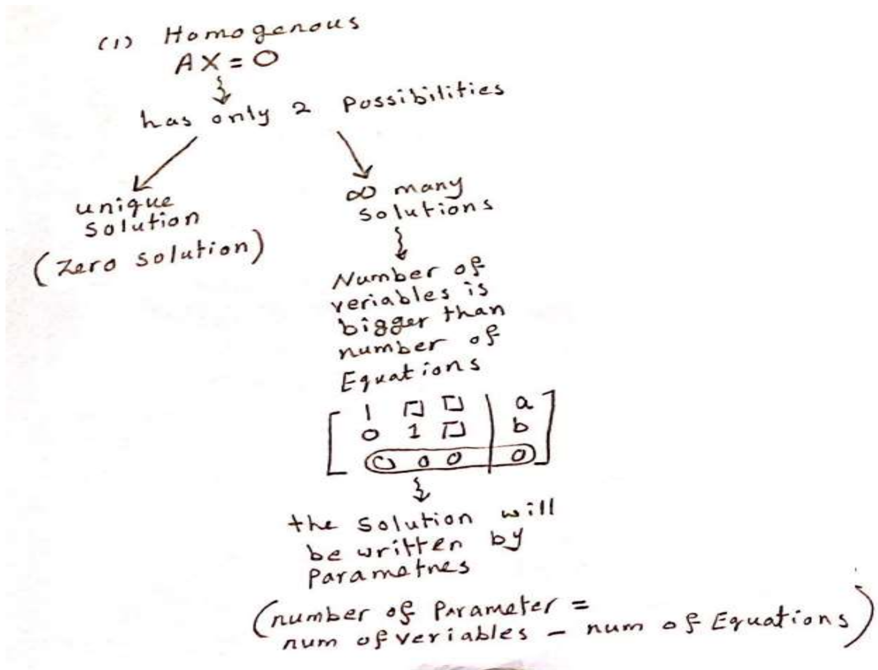
We will do the following steps:

- Orgnize the system
- Make the Augmented matrix of the system
- Eliminate the Augmented matrix which means (by using elementary row operation):
    1- Number (1) at the beginning of the first row
    2- Zeros below 1
    3- Make 1 the first non-zero number, and zeros below 1
    4- Continue as the manner even you complete all rows.

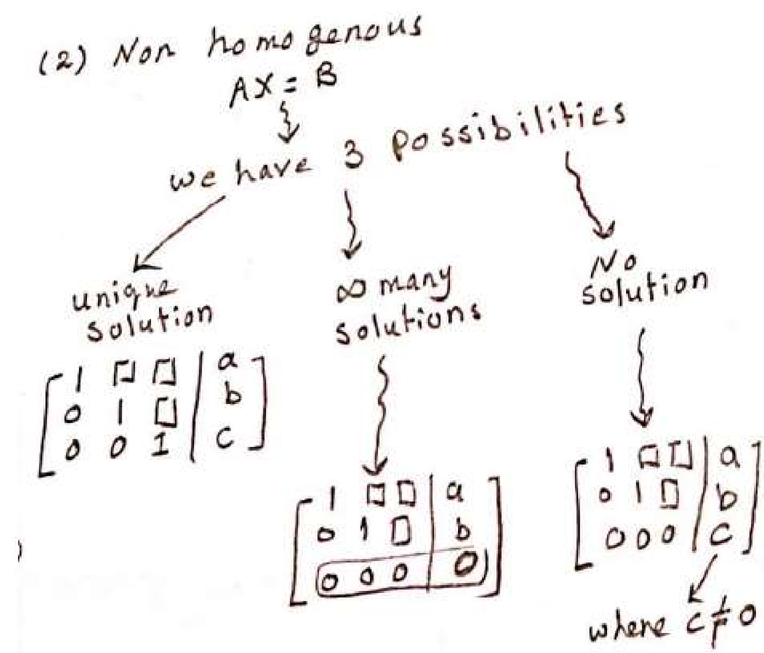
    Example of an eliminated matrix:

$$\left[\begin{array}{ccc|c} 1 & \square & \square & a \\ 0 & 1 & \square & b \\ 0 & 0 & 1 & c \end{array}\right]$$

A- Homogenous system AX=0 (always has at least solution which is zero solution)



B- Non-Homogenous System AX=B



<u>Definition</u>

If the linear system has solution, then it is called by Consistent system. Otherwise, It is called by inconsistent.

Remark: Every homogenous system is consistent.

e.g : After elimination we get the following :

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 5 \\ 0 & 6 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has ∞ many solutions (i.e. the solution will be written by Parameters)

Equations : $x + 2y + 3z - f = 5 \rightsquigarrow (1)$

$6y + 1z = 3 \rightsquigarrow (2)$

Number of Parameters = $\boxed{4} - \boxed{2} = 2$

Let $\boxed{z = t}$

$\Rightarrow y = \frac{3-t}{6}$ (By Equation (2))

Let $\boxed{x = s}$ (By Equation (1))

$$f = x + 2y + 3z - 5$$
$$= s + \frac{3-t}{3} + 3t - 5$$

$$\therefore S = \left\{ \begin{bmatrix} s \\ \frac{3-t}{6} \\ t \\ s + \frac{3-t}{3} + 3t - 5 \end{bmatrix} ; \ s, t \in \mathbb{R} \right\}$$

e.g : After elemination we get the following

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

" It is clear that it is Homogenous "

$$\Rightarrow \left.\begin{array}{l} z = 0 \\ y + 3z = 0 \\ x + 2y - z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} z = 0 \\ y = 0 \\ x = 0 \end{array} \Rightarrow S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

unique solution

e.g: After elomination we get the following:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

then, we have

(1) .... $y + z = 0$

(2) ... $x + 2y - f = 0$

$\Rightarrow$ we have ∞ many solutions (Nu of Para = Num of Ver - Num of Equations $= \boxed{4} - \boxed{2} = 2$)

Let $y = t \overset{\text{By (1)}}{\Rightarrow} z = -t$

$x = s \overset{\text{By (1)}}{\Rightarrow} f = x + 2y = s + 2t$

$$\therefore \; S = \left\{ \begin{bmatrix} s \\ t \\ -t \\ s+2t \end{bmatrix} ; \; s, t \in \mathbb{R} \right\}$$

e.g: Find value of a to make the system

① ~~unique~~ ∞ many solutions

② No solution

③ unique solutio

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & a-1 & 0 \end{array}\right]$$

Solution (1) If $a - 1 = 0$ then number of veriables > Number of Equations then there are ∞ many solutions $\Rightarrow$ if $\boxed{a = 1}$

(2) If $a - 1 \neq 0 \Rightarrow$ the system has unique solution

(3) It is (impossible ~~[illegible]~~. the case of no solution)

Therefore, The system is Consistent

Example: Solve the system of linear equations by Gaussion- elimination method

$$x-2y-z=3$$
$$3x-6y-5z=3$$
$$2x-y+z=0$$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

STEP 1. Creating 0 in the first below first entry by performing row operations $-3R_1+R_2 \Rightarrow R_2$, $-2R_1+R_3 \Rightarrow R_3$

$$\approx \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & 3 & 3 & -6 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & -2 & -6 \end{array}\right] R_2 \Leftrightarrow R_3$$

Creating 1 in second entry of the second row and in third entry of the third row by performing row operations $\frac{1}{3}R_2 \Rightarrow R_2, -\frac{1}{2}R_3 \Rightarrow R_3$

$$\approx \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Equivalent system of equations form is:

$$x-2y-z=3$$
$$y+z=-2$$
$$z=3$$

STEP 2. __Back Substitution__

$$z=3$$
$$y=-z-2=-3-2=-5$$
$$x=2y+z+3=-10+3+3=-4$$

Solution is

$$x=-4,\ y=-5,\ z=3$$

Example **. Find values of *x, y, and z* by solving the system of equations by Gauss Elimination method**

$$\frac{1}{x} + \frac{8}{y} + \frac{2}{z} = 7$$
$$\frac{2}{x} + \frac{4}{y} - \frac{4}{z} = 3$$
$$\frac{2}{x} + \frac{1}{y} + \frac{1}{z} = 2$$

Solution:

Step I *is to eliminate the values below the leading entries to zero of the Augmented matrix* [A:b]

$$[A:b] \cong \begin{bmatrix} 1 & 8 & 2 & 7 \\ 2 & 4 & -4 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2,-2R_1+R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & -12 & -8 & -11 \\ 0 & -15 & -3 & -12 \end{bmatrix} \xrightarrow{-\frac{1}{12}R_2} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & -15 & -3 & -12 \end{bmatrix}$$

$$\xrightarrow{15R_2+R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & 0 & 7 & \frac{7}{4} \end{bmatrix} \xrightarrow{\frac{1}{7}R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$$

$$\frac{1}{z} = \frac{1}{4} \quad \Rightarrow \quad z = 4$$
$$\frac{1}{y} = -\frac{2}{3z} + \frac{11}{12} = -\frac{1}{6} + \frac{11}{12} = \frac{3}{4} \quad \Rightarrow \quad y = \frac{4}{3}$$
$$\frac{1}{x} = -\frac{8}{y} - \frac{2}{z} + 7 = -8\left(\frac{3}{4}\right) - 2\left(\frac{1}{4}\right) + 7 = \frac{1}{2} \quad \Rightarrow \quad x = 2$$

Example Suppose that points (-2,-1), (-1,2), (1,2) lie on parabola

$$y = a + bx + cx^2,$$

(i) Determine a linear system of equations in three unknown a, b and c,
(ii) Find the equation of parabola by solving the system of linear equation.

Solution:

(i) The system of linear equations can be obtained by substituting these points in tl equation of parabola as these lie on the parabola.

Through point $(-2,-1)$ $a - 2b + 4c = -1$
through point $(-1,2)$ $a - b + c = 2$
through point $(1,2)$ $a + b + c = 2$

The system of linear equations is

$$\begin{aligned} a - 2b + 4c &= -1 \\ a - b + c &= 2 \\ a + b + c &= 2 \end{aligned}$$

(ii)

Matrix form of the system is:

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Example **. Find values of *x, y, and z* by solving the system of equations by Gauss Elimination method**

$$\frac{1}{x} + \frac{8}{y} + \frac{2}{z} = 7$$
$$\frac{2}{x} + \frac{4}{y} - \frac{4}{z} = 3$$
$$\frac{2}{x} + \frac{1}{y} + \frac{1}{z} = 2$$

Solution:

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$$\xrightarrow{15R_2+R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & 0 & 7 & \frac{7}{4} \end{bmatrix} \xrightarrow{\frac{1}{7}R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$$

$$\frac{1}{z} = \frac{1}{4} \quad \Rightarrow \quad z = 4$$
$$\frac{1}{y} = -\frac{2}{3z} + \frac{11}{12} = -\frac{1}{6} + \frac{11}{12} = \frac{3}{4} \quad \Rightarrow \quad y = \frac{4}{3}$$
$$\frac{1}{x} = -\frac{8}{y} - \frac{2}{z} + 7 = -8\left(\frac{3}{4}\right) - 2\left(\frac{1}{4}\right) + 7 = \frac{1}{2} \quad \Rightarrow \quad x = 2$$

Example Suppose that points (-2,-1), (-1,2), (1,2) lie on parabola

$$y = a + bx + cx^2,$$

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Solution:

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Through point $(-2,-1)$ $a - 2b + 4c = -1$
through point $(-1,2)$ $a - b + c = 2$
through point $(1,2)$ $a + b + c = 2$

The system of linear equations is

$$\begin{aligned} a - 2b + 4c &= -1 \\ a - b + c &= 2 \\ a + b + c &= 2 \end{aligned}$$

(ii)

Matrix form of the system is:

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Augmented matrix form is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

Creating 0 in the first below first entry by performing row operations
$-R_1+R_2$ **and** $-R_1+R_3$

$$\approx \left[\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 3 & -3 & 3 \end{array}\right]$$

Creating 0 in second entry of the third row by performing row operations
$-3R_2+\mathbf{R_3}$

$$\approx \left[\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 6 & -6 \end{array}\right]$$

We are using Gauss Elimination method , so we write the equation of the matrix

$$a \; -2b \; + \; 4c = -1$$
$$-b \; -3c \; = \; 3$$
$$6c \; = -6$$

Solving by backward substitution

$$6c \; = -6 \quad \Rightarrow c = -1$$
$$-b \; = 3c \; + \; 3 = -3 + 3 = 0 \qquad \Rightarrow b = 0$$
$$a \; = 2b \; - \; 4c - 1 = 0 + 4 - 1 = 3 \;\; \Rightarrow a = 3$$

Solution of the system *is* $a = 3,\ b = 0$ and $c = -1$

Equation of parabola is $y = 3 - x^2$

1.8 Gauss – Jorden Elimination Method

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{array}\right]$$

Example.4. Solve the system of linear equations by Gauss - Jorden elimination method

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \qquad \mathbf{R_1+R_2,\ -3R_1+R_3}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] \qquad \mathbf{-R_2,\ 10R_2+R_3}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad -R_3/52$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad -2R_3+R_1,\ 5R_3+R_2$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad -R_2+R_1$$

Equivalent system of equations form is:

$x_1 = 3$
$x_2 = 1$
$x_3 = 2$ is the solution of the system.

1.9 Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. Below the leading entry all values must be zero,
iv. A row containing all zero values must be in the bottom.

Examples:

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

1.10 Reduced Row Echelon Form

A form of a matrix, which satisfies following conditions, is row echelon form

i. '1' (leading entry) must be in the beginning of each row,
ii. '1' must be on the right of the above leading entry,
iii. All entries in the column containing leading entry must be zero,
iv. A row containing all zero values must be in the bottom.

Examples

(i) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Remark:

: 1. Gaussian Elimination method is reducing the given Augmented matrix to Row echelon form and backward substitution.
: 2. Gauss- Jordan Elimination method is reducing the given Augmented matrix to Reduced Row echelon form.

. Solve the system of linear equations

$$
\begin{aligned}
x - 2y + z - 4u &= 1 \\
x + 3y + 7z + 2u &= 2 \\
x - 12y - 11z - 16u &= 5
\end{aligned}
$$

Solution:

Augmented matrix is:

$$
\left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{array}\right]
$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$
\approx \left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right] \qquad R_2 - R_1,\ R_3 - R_1
$$

$$
\approx \left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right] \qquad -R_3 + 2R_2
$$

Last equation is

$$
0x + 0y + 0z + 0u = 6
$$

$$
\text{but} \qquad 0 \neq 6
$$

hence there is no solution for the given system of linear equations.

Example: For which values of 'a' will be following system

$$\begin{aligned} 2x + 3y + z &= -1 \\ x + 2y + z &= 0 \\ 3x + y + (a^2 - 6)z &= a - 3 \end{aligned}$$

(i) infinitely many solutions?
(ii) No solution?
(iii) Exactly one solution?

Solution:

Using Gaussian Elimination method: Reducing the Augmented matrix to row Echelon form

The augmented matrix:

$[A:b] \equiv$

$$\left[\begin{array}{cccc} 2 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & a^2-6 & a-3 \end{array}\right] R_1 \leftrightarrow R_2 \equiv \left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & -1 \\ 3 & 1 & a^2-6 & a-3 \end{array}\right] \begin{array}{l} R_2+(-2R_1) \\ R_3+(-3R_1) \end{array}$$

$$\equiv \left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -5 & a^2-9 & a-3 \end{array}\right] \begin{array}{l} -R_2 \\ R_3+(5R_2) \end{array} \equiv \left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

I. *infinitely many solutions:* $a = 2$, $a^2 - 4 = a - 2 \Leftrightarrow 0 = 0$, *as number of equations are reduced to two and number of variables are three.*

II. *no solution:* $a = -2$, $a^2 - 4 = a - 2 \Leftrightarrow 0 \neq -4$, It is never true statement

III. *one solution:* $a = R - \{2, -2\}$, *for every value of a in the given interval there will have only one solutions.*

Note: *System is inconsistent is case* $a = -2$, *otherwise the system is consistent.*

Example: **For what values of λ does the system of equations**

$$\begin{aligned} 3x \quad\quad + \lambda z &= 2 \\ 3x + 3y + 4z &= 4 \\ y + 2z &= 3 \end{aligned}$$

have (i) unique solution, (ii) infinitely many solutions and (iii) no solution.

Solution: (a) Augmented matrix is Form

$$[A|b] \equiv \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 3 & 3 & 4 & 4 \\ 0 & 1 & 2 & \lambda \end{bmatrix}$$

$$\xrightarrow{-R_1+R_2} \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 0 & 3 & 4-\lambda & 2 \\ 0 & 1 & 2 & \lambda \end{bmatrix}$$

$$\xrightarrow{3R_3-R_2} \begin{bmatrix} 3 & 0 & \lambda & 2 \\ 0 & 3 & 4-\lambda & 2 \\ 0 & 0 & \lambda+2 & 3\lambda-2 \end{bmatrix}$$

considering last row the Augmented matrix

$$0x + 0y + (\lambda+2)z = 3\lambda - 2$$

(i) If $\lambda = -2$, then $0 = -8$, but $0 \neq -8$ which is not possible, so there is no solution.

(ii) If $\lambda \neq -2$, then , we have three equations and three unkowns, so we have unique solution.

(iii) As both side of last row of the matrix will not have all zero value for any value of λ, so system will not have infinitly many solutions.

Example What conditions must a, and b satisfy in order for the system of equations

$$x-2y+3z = 4$$
$$2x-3y+az = 5$$
$$3x-4y+5z = b$$

to have (i)infinitely many solutions? (ii) No solution? (iii) Exactly one solution?

Solution: The augmented matrix is

$$[A:B]=\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix}$$

reducing it to reduced row echelon form

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{bmatrix} \quad R_2\text{-}2R_1,\ R_3\text{-}3R_1$$

$$\approx\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{bmatrix} \quad R_3\text{-}2R_2$$

(i) Infinitely many solutions?
If $a=4$ and $b=6$

(ii) No solution?
If $a=4$ and $b\neq 6$

(iii) Exactly one solution?
If $a\neq 4$ and $b\in R$

Example Solve the homogeneous system of linear equations

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w \quad\; - y - 3z &= 0 \\ 2w + 3x + \; y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Leftrightarrow} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \Leftrightarrow$$

$R_2/2,\ R_3 - 2R_1,\ R_4 + 2R_1$ $\qquad$ $R_3 - 3R_2,\ -R_2 + R_4$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \Leftrightarrow$$

$R_1 + 3R_3,\ R_2 - 2R_3,\ R_4 + 10R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

System form is;

$$\begin{aligned} w - y &= 0 \\ x + y &= 0 \\ z &= 0 \end{aligned}$$

leading entries are w, x, and z , free entry is y

let $y = t$

$$\begin{aligned} \mathrm{w} &= y = t \\ x &= -y = -t \\ z &= 0 \end{aligned}$$

solution is $w = t, x = -t, y = t, z = 0,$ where $t \in R, \quad t \neq 0.$

so there are inifitly many solutions.

Challenge Question:

For which value (s) of λ, the system of equations have non – trivial solutions,

$$(\lambda - 3)x + \quad y = 0$$
$$x + (\lambda - 3)y = 0$$

Inverse of Matrix

a square matrix A has an inverse B iff AB=BA=I. In this case, A is called inverible. Otherwise, A is called singular.

A^{-1} denotes the inverse of A (if it is existed)

for example: if $A^2 - A^3 = I$, then $A(A - A^2) = I$. Therefore, $A^{-1} = A - A^2$

1. $A^{-1}A = A A^{-1} = I$
2. If A and B are invertible matrices of the same size , then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$

2.5 Power of a matrix

1. $A^0 = I$
2. $A^n = A.A.A \ldots A$ (n-factors), where n>0.
3. $A^{-n} = (A^{-1})^n = A^{-1}.A^{-1}.A^{-1} \ldots A^{-1}$ (n- factors) , where n>0.
4. $A^r A^s = A^{r+s}$
5. $(A^r)^s = A^{rs}$
6. $(A^{-1})^{-1} = A$
7. $(A^n)^{-1} = (A^{-1})^n$, $n = 0,1,2,\ldots$
8. $(kA)^{-1} = \frac{1}{k}A^{-1}$, where k is a scalar.

Challenge question: if A is an ivertible matrix , prove A has a unique inverse.

Inverse of a 2x2 matrix

Consider a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:3. Find inverse of matrix $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$? $A^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$.

Example: Let A be an invertible matrix and suppose that inverse of 7A is $\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$, find matrix A

Solution: $(7A)^{-1} = \frac{1}{7}A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$

$$A^{-1} = 7\begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -14 & 49 \\ 7 & -21 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = -\frac{1}{49}\begin{bmatrix} -21 & -49 \\ -7 & -14 \end{bmatrix} = \frac{7}{49}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}.$$

Example: . Let A be a matrix $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ compute $A^3, A^{-3}, A^2 - 2A + I$.

Solution:

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$$

$$A^{-3} = (A^3)^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 0 \\ -28 & 8 \end{bmatrix}$$

$$A^2 - 2A + I = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

Elementary Matrix

An nxn matrix is called *elementary matrix*, if it can be obtained from nxn identity matrix by performing a single row operation.

Examples: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3x3 identity matrix.

Elementary matrices E_1,E_2 and E_3 can be obtained by single row operation.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \; -3R_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \; -2R_3+R_2$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \; R_1 \leftrightarrow R_3$$

Remark (1)

If you multiplied an elementary matrix with a matrix, we would get the same effect of the elementary row operation of elementary matrix on the given matrix

Example:

Let A be a 3x4 matrix, $A=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}$ and

E be 3x3 elementary matrix obtained by row operation $3R_1+R_3$ from an Identity matrix

$$E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}$$

$$EA=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 1 & 4 & 4 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 & 3\\ 2 & -1 & 3 & 6\\ 4 & 4 & 10 & 9\end{bmatrix}, 3R_1+R_3 .$$

Remark 2:

An elementary matrix is invertible, and its inverse is elementary matrix, too

- If E is obtained by switching rows i and j, then E^{-1} is also obtained by switching rows i and j.
- If E is obtained by multiplying row i by the scalar k, then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- If E is obtained by adding k times row i to row j, then E^{-1} is obtained by subtracting k times row i from row j.

$$E=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$

Here, E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence

$$E^{-1}=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}$$

Remark 3:

Let A be a matrix and B be a reduced row echelon form of A. Then B=UA where U is product of elementary matrices.

For example:

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. First, set up the matrix $[A|I_m]$ = $\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right]$

Now, row reduce this matrix until the left side equals the reduced row-echelon form of A.

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right]$$

B U

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}.$$

Remark 4:

A matrix A is invertible iff A is a product of elementary matrices.

Question: Let A be a matrix where its square power subtracts it is a unite matrix. Prove that A could be written as product of elementary matrices.

<u>Expansion matrix to find an inverse of a matrix (elementary matrix method)</u>

To find an inverse of matrix A, we perform a sequence of elementary row operations that reduce

$$[A \mid I] \text{ to } [I \mid A^{-1}]$$

Example:2. Find inverse of a matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ by using Elementary ma

method.

Solution:

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \quad -2R_1 + R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -R_2$$

$$\approx \left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad -4R_2 + R_1$$

$$= [I|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Example Use Elementary matrix method to find inverses of

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 0 & 3\\ 2 & 5 & -4\end{bmatrix} \quad \text{if A is invertible.}$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc}3 & 4 & -1 & 1 & 0 & 0\\ 1 & 0 & 3 & 0 & 1 & 0\\ 2 & 5 & -4 & 0 & 0 & 1\end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 3 & 4 & -1 & 1 & 0 & 0\\ 2 & 5 & -4 & 0 & 0 & 1\end{array}\right] R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 4 & -10 & 1 & -3 & 0\\ 0 & 5 & -10 & 0 & -2 & 1\end{array}\right] -3R_1 + R_2, -2R_1 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 4 & -10 & 1 & -3 & 0\\ 0 & 1 & 0 & -1 & -2 & 1\end{array}\right] -R_2 + R_3$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 0 & -1 & 1 & 1\\ 0 & 0 & -1 & \frac{1}{2} & \frac{-7}{10} & \frac{-2}{5}\end{array}\right] \quad R_2 \leftrightarrow R_3, \frac{(-4R_3 + R_2)}{10}$$

$$\approx \left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{2} & \frac{-11}{10} & \frac{-6}{5}\\ 0 & 1 & 0 & -1 & 1 & 1\\ 0 & 0 & 1 & \frac{-1}{2} & \frac{7}{10} & \frac{2}{5}\end{array}\right] \quad -3R_3 + R_1, -R_3$$

$$\approx [I|A^{-1}]$$

Remark: A square matrix A is invertible iff its reduced row echelon form is I.

Solving Linear system by Inverse Matrix

SUPPOSE $AX = B$ is non homogenouse system where A is invertible. Then

$$A^{-1}AX = A^{-1}B$$
$$I\,X = A^{-1}B$$
$$X = A^{-1}B \text{ is a solution.}$$

unique solution

if A is singular, we have to use Gauss method to dtermine the system has infinite many solutions or does NOT have solution at all.

SUPPOSE $AX = 0$ is homogenouse system where A is invertible. Then

Then we have a unique solution which is the trivial solution.

If A is sigular, then we have infinite many solutions. To determinate them exactly, we should use Gauss method.

Remark : To use inverse method, The linear system should be square.

Example:

Write the system of equations in a matrix form, find A^{-1}, use A^{-1} to solve the system

$$x_1 + 3x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 3$$

Solution: 1. Matrix Form is:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \quad \text{is in form of } AX = B$$

2. Find A^{-1} by using Elementary Matrix method

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

By elemntary row operations, we get the following:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right] = [\, I \,|A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$$

Solution set is $x_1 = -1, x_2 = 4, x_3 = -7.$

Question: Let AX=B and CX=D two n-square linear system where A and C are invertible. Determine whether the system ACX=0 has only trivial solution or more than?

Determinant of matrix

1- Square matrix of 3X3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of 3x3 matrix

$$B = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 5 & 9 \end{bmatrix}$$

$$\det(B) = 2\begin{vmatrix} 6 & 8 \\ 5 & 9 \end{vmatrix} - 4\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} + 5\begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix}$$

$$= 2(54 - 40) - 4(27 - 32) + 5(15 - 24)$$

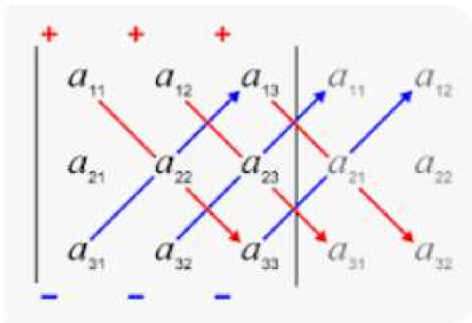
$$= 2(14) - 4(-5) + 5(-9)$$

$$= 28 + 20 - 45$$

$$= 48 - 45$$

$$= 3$$

Remark: **Determinant of 3X3 matrix could be calculated by Sarrus Method:**



Finding determinant by method of co-factors

Minor The minor of an element of a matrix a_{ij} of a matrix A, denoted by M_{ij}, is the determinant of the matrix obtained by deleting the row and column containing a_{ij}.

$$\text{Example: } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor M_{23} of the element a_{23} of matrix

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the determinant of 2x2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$. Thus

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}.$$

Cofactor of an element a_{ij} of a matrix A, denoted by C_{ij}, is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of the element } a_{ij}.$$

Remark: Method of co-factors can be used to nXn determinant

Example:3. Find determinant of matrix if $A = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 1 & 0 & 4 & 0 \end{bmatrix}$

Solution: Expanding from 4th row

$$\text{Det}(A) = -(1)\begin{vmatrix} 1 & 2 & 5 \\ -1 & 2 & 3 \\ 2 & 1 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 2 & 5 \\ 2 & 2 & 3 \\ 3 & 1 & 5 \end{vmatrix} - (4)\begin{vmatrix} 0 & 1 & 5 \\ 2 & -1 & 3 \\ 3 & 2 & 5 \end{vmatrix} + (0)\begin{vmatrix} 0 & 1 & 2 \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= -(1)(4) + (0)(?) - (4)(34) + (0)(?)$$

$$= -4 - 136 = -140.$$

Example : Find all values of λ for which $\det(A) = 0$ for matrix

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Solution: $\det(A) = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & 2 \\ 0 & \lambda - 1 \end{vmatrix} + (0)\begin{vmatrix} 0 & \lambda \\ 0 & 3 \end{vmatrix}$

$= (\lambda-4)\,[\lambda(\lambda-1) - 6]$

$= (\lambda-4)\,[\lambda^2 - \lambda - 6]$

$= (\lambda-4)(\lambda-3)(\lambda+2)$

$\det(A) = 0.$

$(\lambda-4)(\lambda-3)(\lambda+2) = 0.$

$\Rightarrow \lambda = 4, \ \lambda = 3, \ \lambda = -2.$

Evaluating Determinant by row operations (reduction methode)

1. If matrix A_1 is obtained from matrix A by the interchange of two rows , then **$\det(A_1) = -\det(A)$.**
2. If matrix A_2 is obtained from matrix A by the multiplication of a row of A by a constant k , then **$\det(A_2) = k\det(A)$.**
3. If matrix A_3 is obtained from the matrix A by addition of a multiple of one row to another row , then **$\det(A_3) = \det(A)$.**

Example:5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 8 \end{bmatrix}$, and $\det(A) = 2$. Find determinant of

(i) $A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 2 \end{bmatrix}$, (ii) $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, (iii) $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution: (i) A_1 is obtained from A by interchanging R_2 and R_3 of A,
$\det(A_1) = -\det(A) = -2$.

(ii) A_2 is obtained from A by multiplying R_3 of A by $\frac{1}{2}$,
$\det(A_2) = \frac{1}{2}\det(A) = \frac{1}{2}(2) = 1$.

(iii) A_3 is obtained by row operation $-2R_2+R_1$,
$\det(A_3) = \det(A) = 2$.

Some remarks:

1. If A is any square matrix that contains a row of zeros, then $\det(A) = 0$.
2. If a square matrix has two proportional rows, then $\det(A) = 0$.
3. In case of upper or lower triangular matrix, determinant is the product of the diagonal elements.

Example:6.

Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$, find (a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$, (b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$, (d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$.

Solution:

(a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\underset{R_1 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}} = (-1)(-1)\underset{R_2 \leftrightarrow R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-1)(-1)(6) = 6$

(b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(6) = -72$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \underset{R_1 - R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = 6$

(d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = (-3)\underset{4R_2 + R_3}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = (-3)(6) = -18$

example: Evaluate the determinant by row reduction

$$\text{Det A} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

Solution:

$$\det A = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \quad 2R_1 + R_2, -2R_3 + R_4$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \quad -R_4 + R_5$$

$$= (1)(-1)(1)(1)(2) = -2$$

Example: . Find the value(s) of x if det A = -12, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-3 & -3 \\ 1 & x-4 & 0 \end{bmatrix}$$

Solution: Performing row operations $-2R_1+R_2$, $-R_1+R_3$

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x-3 & -3 \\ 0 & x-4 & 0 \end{vmatrix} = (1)\begin{vmatrix} x-3 & -3 \\ x-4 & 0 \end{vmatrix} - (0) + (0)$$

$$= 3(x-4)$$

$$\det A = -12 \quad \Rightarrow -3x - 12 = -12$$

$$-3x = 0$$

$$x = 0.//$$

NOTE: Operations on columns are same as on rows.

Theorem:

For an nxn matrix A, following are equivalent:

1. $\det(A) \neq 0$,
2. A^{-1} exists, and
3. $AX = B$ has a unique solution for any B.
4. A is invertible

3.5 Properties of Determinantial Function

1. **If A is a nxn matric $\det(kA) = k^n \det(A)$,**
2. **$\det(A + B) \neq \det(A) + \det(B)$,**
3. **$\det(AB) = \det(A).\det(B)$,**
4. $\det(A^{-1}) = \dfrac{1}{\det A}$,
5. **A square matrix is invertible if and only if $\det(A) \neq 0$,and**
6. **$\det(A^t) = \det(A)$**
7. If $\det A = 0$, then matrix A is singular matrix
8. $AX = 0$, will have non-trivial solution iff $\det A = 0$

Example :9. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and det(A) = -7 find

(a) det (3A) , (b) $\det(2A)^{-1}$, (c) $\det(2A^{-1})$ and (d) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

Solution: a. $\det(3A) = 3^3 \det A = 27(-7) = -189$

b. $\det(2A)^{-1} = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3 \det(A)} = \dfrac{1}{8(-7)} = \dfrac{-1}{56}$

c. $\det(2A^{-1}) = 2^3 \det(A) = \dfrac{2^3}{\det(A)} = \dfrac{8}{-7} = \dfrac{-8}{7}$

d. $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}^t = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7) = 7$

Example: Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Solution.

det(A) = det (A^t)

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$R_2 - R_1, R_3 - R_1$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$R_3 - R_2$

$$= (b-c)(c-a)(c-b)$$

Example: Without directly evaluating by using properties of determinant show that

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Solution:

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} R_1 + R_2$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0.$$

evaluate the determinant by using row operations

$$\begin{vmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 4 & 2 & 1 & 1 \end{vmatrix}$$

Find values of λ the determinant of the matrix

$$\begin{bmatrix} \lambda^2 & 4 & 1 \\ -4 & -\lambda & 2 \\ 6 & 3 & \lambda^2 \end{bmatrix}, \text{ if the inverse of matrix } \begin{bmatrix} \lambda^2 & 1 \\ 1 & \lambda \end{bmatrix}$$

does not exist.

. **Inverse by method of Cofactors:**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{det\ A \neq 0.}$$

Step:1. Find Matrix of cofactors

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. Find Adjoint of matrix A , adj(A)

$$\mathbf{Adj(A)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A)≠ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

Example: Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$ by the method of cofactors.

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \; C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

Matrix of cofactors, $C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

Adjoint of matrix A, $\mathrm{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-12)+0(-4)+3(6) \\ &= -24 + 18 = -6 \neq 0 \end{aligned}$$

Inverse of matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

Cramer's Rule

If A is nxn marix with $\det(A) \neq 0$, then the linear system $AX = B$ has a unique solution $X = (x_j)$ given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad , j = 1, 2, \ldots, n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

NOTE: If A is 3x3 matrix , then the solution of the system $AX = B$ is

$$x = \frac{\det(A_1)}{\det(A)}, \quad y = \frac{\det(A_2)}{\det(A)}, \quad z = \frac{\det(A_{31})}{\det(A)}$$

Example: Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution: $A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$; $A_2 = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$\det(A) = -132$, $\det(A_1) = -36$, $\det(A_2) = -24$, $\det(A_3) = 12$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$

$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

NOTE: when $\det(A) = 0$, then there does not exist any solution of the system.

Summary

* Linear system $AX=B$
  - If A is rectengular
    1. Gauss
    2. Gauss - Jordan
  - If A is square
    1. Gauss
    2. Gauss - Jordon
    3. Inverse - method
    4. Cramer

* suppose $AX=B$ is Linear system where A is square matrix
  - If $B=0$ → system is Homogenous
    - $|A|\neq 0$ → A^{-1} is existed → The system has unique soluton (Zero - solution)
    - $|A|=0$ → A^{-1} is not existed → The system has ∞ many solutions → There are non-trivial solution (we can find them through Gauss)
  - If $B\neq 0$ → system is not-Homogenous
    - $|A|\neq 0$ → A^{-1} is existed → The system has unique solution $A^{-1}\cdot B$
    - $|A|=0$ → A^{-1} is not existed → we use Gauss method to decide
      - No Solution
      - ∞ many Solutions

① Let $A \in M_3(\mathbb{R})$ where A is invertible. If $B \in M_3(\mathbb{R})$ is singular, find $|2A^{-1}A^T + 3B\, adj(B)|$?

Solution As B is singular then $|B| = 0$

Now $adj(B) = |B|\, B^{-1} = 0$

So, $3B\, adj(B) = 0$

There fore, $2A^{-1}A^T + 3B\, adj(B) = 2A^{-1}A^T$

Now

$$|2A^{-1}A^T + 3B\, adj(B)| = |2A^{-1}A^T| = 2^3 \frac{|A|}{|A|} = 2^3 = 8$$

---

② Let $(x,y,z) = (1,-1,1)$ be a solution of the following system :

$$2x - y + z = r$$
$$x + 2y - z = s$$
$$3x + 4y + rz = t$$

find r, s and t.

Solution since $(1,-1,1)$ is a solution, by substitute in equations

(From E1) $2+1+1 = r \Rightarrow r = 4$

(From E2) $1-2-1 = s \Rightarrow s = -2$

$3-4+4 = t \Rightarrow t = 3$

③ write a relation of α, β and γ to make the following system is consistent :

$$x + 2y + 3z = \alpha$$
$$2x + 5y + 9z = \beta$$
$$x + 3y + 6z = \gamma$$

Solution we will write the augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \alpha \\ 2 & 5 & 9 & \beta \\ 1 & 3 & 6 & \gamma \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & \alpha \\ 0 & 1 & 3 & -2\alpha+\beta \\ 0 & 1 & 3 & -\alpha+\gamma \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & \alpha \\ 0 & 1 & 3 & -2\alpha+\beta \\ 0 & 0 & 0 & \alpha-\beta+\gamma \end{array}\right]$$

the system is consistent iff $\alpha - \beta + \gamma = 0$

(4) find the value of m where $\begin{bmatrix} 1 & 1 & 1 \\ 1 & m & 2 \\ 1 & 10 & m \end{bmatrix}$ is invertible ?

solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & m & 2 \\ 1 & 10 & m \end{vmatrix} \underset{-R_1+R_3}{\overset{-R_1+R_2}{=\!=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & m-1 & 1 \\ 0 & 9 & m-1 \end{vmatrix}$$

$$= (m-1)^2 - 9$$

Now A is invertible iff $(m-1)^2 - 9 \neq 0$

$\Leftrightarrow (m-1)^2 \neq 9$

$\Leftrightarrow (m-1) \neq \pm 3$

$m \neq -2$ or $m \neq 4$

So, $m \in \mathbb{R} - \{-2, 4\}$

(5) Let $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \alpha & 3 \\ 1 & \alpha & 3 & 2 \end{array}\right]$ be the augmented matrix of a linear system. Find the value of α where the system has unique solution. Find the solution of the system than.

Solution clearly, the system is not homogenous. So, it has a unique solution iff $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & \alpha \\ 1 & \alpha & 3 \end{bmatrix}$ is invertible iff $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & \alpha \\ 1 & \alpha & 3 \end{vmatrix} \neq 0$.

Now $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & \alpha \\ 1 & \alpha & 3 \end{vmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2+\alpha \\ 0 & \alpha-1 & 4 \end{vmatrix} =$

$$4-(\alpha-1)(2+\alpha) = 4-(2\alpha+\alpha^2-2-\alpha)$$
$$= -\alpha^2-\alpha+6$$

If $|A| \neq 0 \Rightarrow -\alpha^2-\alpha+6 \neq 0$
$\Rightarrow \alpha^2+\alpha-6 \neq 0$
$\Rightarrow (\alpha+3)(\alpha-2) \neq 0$
$\Rightarrow \alpha \neq -3$ or $\alpha \neq 2$
$\Rightarrow \alpha \in \mathbb{R} - \{-3, 2\}$.

In this case, the unique solution is $A^{-1} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

(6) Let $A \in M_n(\mathbb{R})$ such that $A^2 - 3A + I_n = 0$. Find A^{-1}?

Solution

$\because A^2 - 3A + I_n = 0 \Rightarrow 3A - A^2 = I_n$
$\Rightarrow A(3I - A) = I_n$

Notice that

$(3I - A)A = 3A - A^2 = I$ and $A(3I - A) = 3A - A^2 = I$

So,

$A^{-1} = 3I - A$

⑦ Let $A, B \in M_3(\mathbb{R})$ where $|A| = -3$ $|B| = 3$. Find

$|A^T B^3 \operatorname{adj}(A^2) B^{-1}|$.

Solution : Notice that $|A| = 3$, $|B| = -1$

also, $|\operatorname{adj}(A^2)| = |A^2| \cdot (A^2)^{-1}$

$= |A| \cdot |A| \cdot (A^{-1})^2 = |A|^2 \cdot (A^{-1})^2$

So, $|\operatorname{adj}(A^2)| = \left| |A|^2 \cdot (A^{-1})^2 \right|$

$= (|A|^2)^3 \cdot |A^{-1}| \cdot |A^{-1}| = \dfrac{|A|^6}{|A|^2} = |A|^4$ (

Now

$$|A^T B^3 \operatorname{adj}(A^2) B^{-1}| = |A^T| \cdot |B^3| \cdot |\operatorname{adj}(A^2)| \cdot |B^{-1}|$$

(by (*))

$$= |A| \cdot |B|^3 \cdot |A|^4 \cdot \frac{1}{|B|}$$

$$= (3)(-1)^3 (3)^4 (-1) = 3^5$$

⑥ Let $A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$. ① find $|-4I_3 - A|$, and decide whether the system $(-4I_3 - A)X = 0$ has unique solution or not ? Find all solution.

$$-4I_3 - A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & -5 \\ -1 & -5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$; $|-4I_3 - A| = 0 \Rightarrow (-4I_3 - A)X = 0$ has ∞ many solutions.

So, the Linear system $(-4I_3 - A)X = 0$ is can be presented by the following augmented matrix :

$$\left[\begin{array}{ccc|c} -5 & 0 & -5 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\overset{-1/5 R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & -5 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\overset{R_1+R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\overset{-1/5 R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\overset{R_1+R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (\infty \text{ many solutions})$$

The number of Equations = 2
The number of Veriables = 3 } ⇒ The solution will be written by parameter t

$$\Rightarrow \left.\begin{array}{c} x + z = 0 \\ y = 0 \end{array}\right\} \quad \text{put } x = t \Rightarrow z = -t$$

$$\text{so, } S = \left\{ \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}, t \in \mathbb{R} \right\}$$