Multiple Compounding Periods in a Year

Example: credit card debt

Example 2.36

Rebecca Carlson purchased a car for $25,000 by borrowing the money at 8% per year compounded monthly. She paid off the loan with 60 equal monthly payments, the first of which was paid one month after receiving the car. How much was her monthly payment?

A = $25,000(A|P ⅔%,60) = $25,000(0.02028) = $507.00

A =PMT(0.08/12,60,-25000)

A = $506.91

Compounding and Timing

- To date, we have assumed the frequency with which interest was compounded matched the frequency of payments and deposits.
- Such is not always the case, e.g., monthly deposits and annual compounding, annual compounding and bi-annual deposits. The effective annual interest rate (i_{eff}) is used to deal with differences in the timing of cash flows and the compounding frequency.

$$i_{eff} = \left(1+\frac{r}{m}\right)^{m} - 1 \qquad = \qquad \left(F \mid P\ \frac{r\%}{m}, m\right) - 1$$

  - r = nominal annual interest rate
  - m = number of compound periods per year
  - i_{eff} =EFFECT(r,m)

Example 2.37

- What is the effective interest rate for 12% per annum compounded annually?

  r = 12%, m = 1, r/m = 12%

  i_{eff} = (F|P 12%,1)-1 = 1.1200-1.0 = 12.00%

  =EFFECT(12%,1) = 12%

- What is the effective interest rate for 12% per annum compounded semiannually?

  r = 12%, m = 2, r/m = 6%

  ieff = (F|P 6%,2)-1 = 1.1236-1.0 = 12.36%

  =EFFECT(12%,2) = 12.36%

Example 2.37

- What is the effective interest rate for 12% per annum compounded quarterly?

  r = 12%, m = 4, r/m = 3%

  ieff = (F|P 3%,4)-1 = 1.12551-1.0 = 12.551%

  =EFFECT(12%,4) = 12.551%

- What is the effective interest rate for 12% per annum compounded monthly?

  r = 12%, m = 12, r/m = 1%

  ieff = (F|P 1%,12)-1 = 1.12683-1.0 = 12.683%

  =EFFECT(12%,12) = 12.683%

Example 2.37

- What is the effective interest rate for 12% per annum compounded weekly?

  r = 12%   m = 52,

  ieff = (1 + 0.12/52)52 -1 = 1.1273409872 - 1.0

  = 12.73409872%

  =EFFECT(12%,52) = 12.73409872%

- What is the effective interest rate for 12% per annum compounded daily?

  r = 12%   m = 365

  ieff = (1 + 0.12/365)365 -1 = 1.1274746156 - 1.0

  = 12.74746156%

  =EFFECT(12%,365) = 12.74746156%

Example 2.37

- What is the effective interest rate for 12% per annum compounded hourly?

  r = 12% m = 8760,

  i_{eff} = $(1 + 0.12/8760)^{8760}$ -1 = 1.1274959248782 - 1.0

  = 12.74959248782%

  =EFFECT(12%,8760) = 12.7495924877**6**% (!!!)

- What is the effective interest rate for 12% per annum compounded every minute?

  r = 12% m = 525,600,

  i_{eff} = $(1 + 0.12/525,600)^{525,600}$ -1 = 1.127496836146 - 1.0

  = 12.7496836146%

  =EFFECT(12%,525600) = 12.7496836133% (!!!)

Example 2.37

- What is the effective interest rate for 12% per annum compounded every second?

  $r = 12\%$ $m = 31{,}536{,}000$

  $i_{eff} = (1 + 0.12/31{,}536{,}000)^{31{,}536{,}000} - 1 = 1.127496851431 - 1.0$

  $= 12.7496851431\%$

  =EFFECT(12%,31536000) = 12.7496852242% (!!!)

- What is the effective interest rate for 12% per annum compounded continuously?

  $r = 12\%$ $m =$ infinitely large

  From Appendix 2A, we'll find that

  $i_{eff} = e^{0.12} - 1 = 12.7496851579\%$

Shorthand Notation

- In some end-of-chapter problems in subsequent chapters, we use a shorthand notation when ***money is compounded multiple times per year*.**

- For example,

8% per annum compounded quarterly is denoted ***8%/year/quarter***

and

12% per annum compounded monthly is denoted ***12%/year/month***.

Example 2.38

Wenfeng Li borrowed $1,000 and paid off the loan in 4.5 years with a single lump sum payment of $1,500. Based on a 6-month interest period, what was the annual effective interest rate paid?

Example 2.38

Wenfeng Li borrowed $1,000 and paid off the loan in 4.5 years with a single lump sum payment of $1,500. Based on a 6-month interest period, what was the annual effective interest rate paid?

n = 9 (6-mo periods), P = $1,000, F = $1,500, i = ?

$1,500 = $1,000(F|P i%,9)

$(1 + i)^9 = 1.5$

i = 0.046 or 4.6%

$i_{eff} = (1 + 0.046)^2 - 1 = 0.0943$ or 9.43%

Example 2.38

Wenfeng Li borrowed $1,000 and paid off the loan in 4.5 years with a single lump sum payment of $1,500. Based on a 6-month interest period, what was the annual effective interest rate paid?

n = 9 (6-mo periods), P = $1,000, F = $1,500, i = ?

$1,500 = $1,000(F|P i%,9)

$(1 + i)^9 = 1.5$

i = 0.046 or 4.6%

$i_{eff} = (1 + 0.046)^2 - 1 = 0.0943$ or 9.43%

i_{eff} =RATE(4.5,,-1000,1500) = 9.429%

Example 2.39

Greg Wilhelm borrowed $100,000 to purchase a house. It will be repaid with 360 equal monthly payments at a nominal annual rate of 6% (6%/year/month). Determine the monthly payments.

A = $100,000(*A*|*P* 0.5%,360)

= $100,000(0.0059955)

= $599.55/month

=PMT(0.06/12,360,-100000)

= $599.55

Example 2.39

In addition to the usual interest charges, Mr. Wilhelm had to pay $2,000 in closing costs to the lending firm. If the closing costs are financed, what would be the size of the monthly payments?

A = $102,000(*A*|*P* 0.5%,360)

= $102,000(0.0059955)

= $611.54/month

=PMT(0.06/12,360,-102000)

= $611.54

When Compounding and Cash Flow Frequencies Differ

When compounding frequency and cash flow frequency differ, the following approach is taken.

$$i = (1 + r/m)^{m/k} - 1 \qquad (2.49)$$

r = the nominal annual interest rate for money
m = the number of compounding periods in a year
k = the number of cash flows in a year and;
i = the interest rate per cash flow period.

Equation 2.49 results from setting the effective annual interest rate for the stated compounding frequency of money equal to the effective annual interest rate for the cash flow frequency.

$$(1 + i)^k - 1 = (1 + r/m)^m - 1$$

and solving for i.

Example 2.40

What size monthly payments should occur when \$10,000 is borrowed at 8% compounded quarterly (8%/year/quarter) and the loan is repaid with 36 equal monthly payments?

From Equation 2.49:

r = 0.08, m = 4, and k = 12. Therefore,

$i = (1 + 0.08/4)^{4/12} - 1 = 0.006623$ or 0.6623%/month

Knowing the monthly interest rate, the monthly payment can be determined,

$$A = \$10{,}000(A|P\ 0.6623\%,36)$$

$$= \$10{,}000[(0.006623)(1.006623)^{36}]/[(1.006623)^{36} - 1]$$

$$= \$313.12$$

Using the Excel® PMT worksheet function,

***A* =PMT(1.02^(1/3)-1,36,-10000)**

= \$313.12

Summary of discrete compounding interest factors.

To Find	Given	Factor	Symbol	Name
P	F	$(1+i)^{-n}$	$(P\mid F\ i\%,n)$	Single sum, present worth factor
F	P	$(1+i)^{n}$	$(F\mid P\ i\%,n)$	Single sum, compound amount factor
P	A	$\frac{(1+i)^n - 1}{i(1+i)^n}$	$(P\mid A\ i\%,n)$	Uniform series, present worth factor
A	P	$\frac{i(1+i)^n}{(1+i)^n - 1}$	$(A\mid P\ i\%,n)$	Uniform series, capital recovery factor
F	A	$\frac{(1+i)^n - 1}{i}$	$(F\mid A\ i\%,n)$	Uniform series, compound amount factor
A	F	$\frac{i}{(1+i)^n - 1}$	$(A\mid F\ i\%,n)$	Uniform series, sinking fund factor
P	G	$\frac{[1-(1+ni)(1+i)^{-n}]}{i^2}$	$(P\mid G\ i\%,n)$	Gradient series, present worth factor
A	G	$\frac{(1+i)^n - (1+ni)}{i[(1+i)^n - 1]}$	$(A\mid G\ i\%,n)$	Gradient series, uniform series factor
P	A_1,j	$\frac{1-(1+j)^n(1+i)^{-n}}{i-j}$ for $i \neq j$	$(P\mid A_1\ i\%,j\%,n)$	Geometric series, present worth factor
F	A_1,j	$\frac{(1+i)^n - (1+j)^n}{i-j}$ for $i \neq j$	$(F\mid A_1\ i\%,j\%,n)$	Geometric series, future worth factor

Pit Stop #2 — Hang On!

1. **True of False: If money is worth 5% compounded annually to you, then you should prefer to receive $2,750 today than to receive $3,500 five years from today.**

2. **True or False: If money is worth 7% compounded annually to you, then you should prefer to receive $1,000,000 thirty years from now than to receive $200,000 today.**

3. **True of False: If money is worth 6% compounded annually to you, then you should prefer to receive $1,750 per year for 5 years than to receive $1,000 per year for 10 years, assuming the first receipt occurs one year from today in both cases.**

4. **True or False: If money is worth 7% compounded annually to you, then you would prefer to receive $2,200 each year for 5 years than to receive $4,000 the first year, $3,000 the second year, $2,000 the third year, $1,000 the fourth year, and $0 the fifth year.**

5. **True of False: If money has a time value of 8% compounded annually, you should prefer to receive a uniform series of ten $1,000 cash flows over the interval [1,10] to receiving a uniform series of ten $1,260 cash flows over the interval [4,13].**

Pit Stop #2 — Hang On!

1. **True of False: If money is worth 5% compounded annually to you, then you should prefer to receive $2,750 today than to receive $3,500 five years from today. True PV(5%,5,,-3500) = $2,742.34 < $2,750.00**

2. **True or False: If money is worth 7% compounded annually to you, then you should prefer to receive $1,000,000 thirty years from now than to receive $200,000 today. False $1,000,000.00 < FV(7%,30,,-200000) = $1,522,451.01**

3. **True of False: If money is worth 6% compounded annually to you, then you should prefer to receive $1750 per year for 5 years than to receive $1,000 per year for 10 years, assuming the first receipt occurs one year from today in both cases. True PV(6%,5,-1750) = $7,371.64 > PV(6%,10,-1000) = $7,360.09**

4. **True or False: If money is worth 7% compounded annually to you, then you would prefer to receive $2,200 each year for 5 years than to receive $4,000 the first year, $3,000 the second year, $2,000 the third year, $1,000 the fourth year, and $0 the fifth year. True PV(7%,5,-2200) = $9,020.43 > NPV(7%,4000,3000,2000,1000) = $8,754.12**

5. **True of False: If money has a time value of 8% compounded annually, you should prefer to receive a uniform series of ten $1,000 cash flows over the interval [1,10] to receiving a uniform series of ten $1,260 cash flows over the interval [4,13]. False PV(8%,10,-1000) = $6,710.08 < PV(8%,3,,PV(8%,10,1260)) = $6,711.62**

Appendix 2A
Continuous Compounding

- In businesses and governments, transactions occur every year, every month, every day, every hour, every minute, every second!
- In multi-national corporations, for example, money is "put to work" immediately.
- Via electronic transfers, money is invested around the world continuously.
- Explicit consideration of "around the clock" and "around the world" money management motivates the use of continuous compounding.

Continuous Compounding

Let m denote *the number of compounding periods in a year*, r denote *the nominal annual interest rate*, and n denote *the number of years*.

The single payment compound amount factor is given by

$$(1+r/m)^{mn}$$

Letting the number of compounding periods in a year become infinitely large, the single payment compound amount factor reduces to

$$\lim_{m\to\infty}(1+r/m)^{mn} = e^{rn}$$

Hence,

$F = P e^{rn}$

$F = P\ (F|P\ r\%, n)_{\infty}$

$P = F e^{-rn}$ and

$P = F(P|F\ r\%, n)_{\infty}$

$by - tables$

Continuous Compounding

Recall, in discussing effective interest rates we claimed the effective interest rate for 12% compounded continuously was 12.7496851579%.

The effective interest rate under continuous compounding is given by:

$$i_{eff} = e^{r} - 1 \qquad \text{or} \qquad i_{eff} = (F|P\ r\%,1)_{\infty} - 1$$

Therefore, the effective interest rate for 12% compounded continuously is equal to $e^{0.12}-1$, or 12.7496851579%.

Continuous Compounding

For the case of discrete cash flows, the continuous compounding equivalents for the discounted cash flow formulas can be obtained by substituting (e^r-1) for i.

$(P|F\ r\%,n)_\infty$ **= e^{-rn}**

$(F|P\ r\%,n)_\infty$ **= e^{rn}**

$(F|A\ r\%,n)_\infty$ **= $(e^{rn}-1)/(e^r-1)$**

$(A|F\ r\%,n)_\infty$ **= $(e^r-1)/(e^{rn}-1)$**

$(P|A\ r\%,n)_\infty$ **= $(e^{rn}-1)/[e^{rn}(e^r-1)]$**

$(A|P\ r\%,n)_\infty$ **= $e^{rn}(e^r-1)/(e^{rn}-1)$**

See Table 2.A.1

Summary of continuous compounding interest factors for discrete flows.

To Find	Given	Factor	Symbol
P	F	e^{-rn}	$(P\|F\ r,n)_\infty$
F	P	e^{rn}	$(F\|P\ r,n)_\infty$
F	A	$\dfrac{e^{rn}-1}{e^r-1}$	$(F\|A\ r,n)_\infty$
A	F	$\dfrac{e^r-1}{e^{rn}-1}$	$(A\|F\ r,n)_\infty$
P	A	$\dfrac{e^{rn}-1}{e^{rn}(e^r-1)}$	$(P\|A\ r,n)_\infty$
A	P	$\dfrac{e^{rn}(e^r-1)}{e^{rn}-1}$	$(A\|P\ r,n)_\infty$
P	G	$\dfrac{e^{rn}-1-n(e^r-1)}{e^{rn}(e^r-1)^2}$	$(P\|G\ r,n)_\infty$
A	G	$\dfrac{1}{e^r-1}-\dfrac{n}{e^{rn}-1}$	$(A\|G\ r,n)_\infty$
P	A_1, c	$\dfrac{1-e^{(c-r)n}}{e^r-e^c}$	$(P\|A_1\ r,c,n)_\infty^*$
F	A_1, c	$\dfrac{e^{rn}-e^{cn}}{e^r-e^c}$	$(F\|A_1\ r,c,n)_\infty^*$

* $r \neq c$.

Example 2.A.1

If \$2,000 is invested in a fund that pays interest at a rate of 12% compounded continuously, after 5 years how much will be in the fund?

$$F = P(F|P\ 12\%,5)_{\infty} = \$2,000(1.82212) = \$3,644.24$$

Example 2.A.2

If \$1,000 is deposited annually in an account that pays interest at a rate of 12% compounded continuously, after the 10th deposit how much will be in the fund?

$F = \$1,000(F|A\ 12\%,10)_{\infty}$

$F = \$1,000(18.19744)$

$F = \$18,197.44$

F =FV(exp(0.12)-1,10,-1000)

F = \$18,197.44

Example 2.A.2

$1,000 is deposited annually in an account that pays interest at a rate of 12% compounded continuously. What is the present worth of the 10-year investment?

P = $1,000(P|A 12%,10)$_{\infty}$

P = $1,000(5.48097)

P = $5,480.97

P =PV(exp(0.12)-1,10,-1000)

P = $5,480.97

Example 2.A.3

Annual bonuses are deposited in a savings account that pays 8 percent compounded continuously. The size of the bonus increases at a rate of 10% compounded continuously; the initial bonus was $500. How much will be in the account immediately after the 10th deposit?

A_1 = $500, r = 8%, c = 10%, n = 10, F = ?

F = $500($F|A_1$ 8%,10%,10)$_\infty$

F = $500(22.51619)

F = $11,258.09

Continuous Compounding, Continuous Flow

Not only does compounding of money occur "continuously," but also expenditures occur by the hour, minute, and second! Instead of cash flows for labor, material, energy, etc. occurring at the end of the year, they occur throughout the year, even daily or hourly. Hence, funds also flow continuously.

In the text, we show that the annual discrete cash flow (A) equivalent to an annual continuous cash flow ($\bar{A}$), based on an annual nominal interest rate of r, is given by

$$A = \bar{A}(e^r - 1)/r$$

Summary of continuous compounding interest factors for **continuous flows**.

Find	Given	Factor	Symbol
P	Ā	$(e^{rn}-1)/(re^{rn})$	(P|Ā r%, n)
Ā	P	$re^{rn}/(e^{rn}-1)$	(Ā|P r%, n)
F	Ā	$(e^{rn}-1)/r$	(F|Ā r%, n)
Ā	F	$r/(e^{rn}-1)$	(Ā|F r%, n)

Example 2.A.4

Determine the present worth equivalent of a uniform series of continuous cash flows totaling $10,000/yr for 10 years when the interest rate is 20% compounded continuously.

Example 2.A.4

Determine the present worth equivalent of a uniform series of continuous cash flows totaling $10,000/yr for 10 years when the interest rate is 20% compounded continuously.

P = $10,000(P|Ā 20%,10)

P = $10,000(4.32332)

P = $43,233.20

Example 2.A.4

Determine the present worth equivalent of a uniform series of continuous cash flows totaling $10,000/yr for 10 years when the interest rate is 20% compounded continuously.

P = $10,000(P|Ā 20%,10)

P = $10,000(4.32332)

P = $43,233.20

P =PV(exp(0.2)-1,10,-10000*(exp(0.2)-1)/0.2)

P = $43,233.24

Example 2.A.4

Determine the future worth equivalent of a uniform series of continuous cash flows totaling $10,000/yr for 10 years when the interest rate is 20% per year compounded continuously.

Example 2.A.4

Determine the future worth equivalent of a uniform series of continuous cash flows totaling $10,000/yr for 10 years when the interest rate is 20% per year compounded continuously.

F = $10,000(F|Ā 20%,10)

F = $10,000(31.94528)

F = $319,452.80

Example 2.A.4

Determine the future worth equivalent of a uniform series of continuous cash flows totaling $10,000/yr for 10 years when the interest rate is 20% per year compounded continuously.

F = $10,000(F|Ā 20%,10)

F = $10,000(31.94528)

F = $319,452.80

F =FV(exp(0.2)-1,10,-10000*(exp(0.2)-1)/0.2)

F = $319,452.80