Interpolation & Polynomial Approximation
Divided Differences

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A new algebraic representation for $P_n(x)$

- Suppose that $P_n(x)$ is the $n$th Lagrange polynomial that agrees with the function $f$ at the distinct numbers $x_0, x_1, \ldots, x_n$.
- Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations.
- The divided differences of $f$ with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0) \cdots (x-x_{n-1})$$

for appropriate constants $a_0, a_1, \ldots, a_n$. 
\[ P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1}) \]

- To determine the first of these constants, \( a_0 \), note that if \( P_n(x) \) is written in the form of the above equation, then evaluating \( P_n(x) \) at \( x_0 \) leaves only the constant term \( a_0 \); that is,

\[ a_0 = P_n(x_0) = f(x_0) \]

- Similarly, when \( P(x) \) is evaluated at \( x_1 \), the only nonzero terms in the evaluation of \( P_n(x_1) \) are the constant and linear terms,

\[ f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1) \]

\[ \Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]
The Divided Difference Notation

- We now introduce the divided-difference notation, which is related to Aitken's $\Delta^2$ notation.

- The zeroth divided difference of the function $f$ with respect to $x_i$, denoted $f[x_i]$, is simply the value of $f$ at $x_i$:

\[ f[x_i] = f(x_i) \]

- The remaining divided differences are defined recursively.

**Forward Difference Operator $\Delta$**

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the forward difference $\Delta p_n$ (read “delta $p_n$”) is defined by

\[ \Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0. \]

Higher powers of the operator $\Delta$ are defined recursively by

\[ \Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2. \]
The first divided difference of $f$ with respect to $x_i$ and $x_{i+1}$ is denoted $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

The second divided difference, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$
Similarly, after the \((k - 1)\)st divided differences,

\[
f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}] \quad \text{and} \quad f[x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}, x_{i+k}]
\]

have been determined, the \(k\)th divided difference relative to \(x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k}\) is

\[
f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}
\]

The process ends with the single \(n\)th divided difference,

\[
f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}
\]
Generating the Divided Difference Table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>First divided differences</th>
<th>Second divided differences</th>
<th>Third divided differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$f[x_0]$</td>
<td>$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$</td>
<td>$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$</td>
<td>$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$f[x_1]$</td>
<td>$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$</td>
<td></td>
<td>$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$f[x_2]$</td>
<td>$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$</td>
<td></td>
<td>$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$f[x_3]$</td>
<td>$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$</td>
<td></td>
<td>$f[x_2, x_3, x_4, x_5] = \frac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$f[x_4]$</td>
<td>$f[x_3, x_4, x_5] = \frac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_5$</td>
<td>$f[x_5]$</td>
<td>$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Newton’s Divided Difference Interpolating Polynomial

\[ P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \]

Using the Divided Difference Notation

- Returning to the interpolating polynomial, we can now use the divided difference notation to write:

\[
 a_0 = f(x_0) = f[x_0] \\
 a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]
\]

- Hence, the interpolating polynomial is

\[
P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})
\]
Newton’s Divided Difference Interpolating Polynomial

\[ P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}). \]

- As might be expected from the evaluation of \( a_0 \) and \( a_1 \), the required constants are

\[ a_k = f[x_0, x_1, x_2, \ldots, x_k], \]

for each \( k = 0, 1, \ldots, n \).

- So \( P_n(x) \) can be rewritten in a form called Newton’s Divided-Difference:

\[ P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}) \]
Example 1 Complete the divided difference table for the data in Table below, and construct the interpolating polynomial that uses all this data.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.7651977</td>
</tr>
<tr>
<td>1.3</td>
<td>0.6200860</td>
</tr>
<tr>
<td>1.6</td>
<td>0.4554022</td>
</tr>
<tr>
<td>1.9</td>
<td>0.2818186</td>
</tr>
<tr>
<td>2.2</td>
<td>0.1103623</td>
</tr>
</tbody>
</table>

**Solution**  
The first divided difference involving $x_0$ and $x_1$ is

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{0.6200860 - 0.7651977}{1.3 - 1.0} = -0.4837057.$$  

The second divided difference involving $x_0$, $x_1$, and $x_2$ is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-0.5489460 - (-0.4837057)}{1.6 - 1.0} = -0.1087339.$$
The third divided difference involving \( x_0, x_1, x_2, \) and \( x_3 \) and the fourth divided difference involving all the data points are, respectively,

\[
f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-0.0494433 - (-0.1087339)}{1.9 - 1.0} = 0.0658784,
\]

and

\[
f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0} = \frac{0.0680685 - 0.0658784}{2.2 - 1.0} = 0.0018251.
\]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f[x_i] )</th>
<th>( f[x_{i-1}, x_i] )</th>
<th>( f[x_{i-2}, x_{i-1}, x_i] )</th>
<th>( f[x_{i-3}, \ldots, x_i] )</th>
<th>( f[x_{i-4}, \ldots, x_i] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.7651977</td>
<td>0.7651977</td>
<td>0.7651977</td>
<td>0.7651977</td>
<td>0.7651977</td>
</tr>
<tr>
<td>1</td>
<td>1.3</td>
<td>0.6200860</td>
<td>0.6200860</td>
<td>0.6200860</td>
<td>0.6200860</td>
<td>0.6200860</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>0.4554022</td>
<td>0.4554022</td>
<td>0.4554022</td>
<td>0.4554022</td>
<td>0.4554022</td>
</tr>
<tr>
<td>3</td>
<td>1.9</td>
<td>0.2818186</td>
<td>0.2818186</td>
<td>0.2818186</td>
<td>0.2818186</td>
<td>0.2818186</td>
</tr>
<tr>
<td>4</td>
<td>2.2</td>
<td>0.1103623</td>
<td>0.1103623</td>
<td>0.1103623</td>
<td>0.1103623</td>
<td>0.1103623</td>
</tr>
<tr>
<td>$i$</td>
<td>$x_i$</td>
<td>$f[x_i]$</td>
<td>$f[x_{i-1}, x_i]$</td>
<td>$f[x_{i-2}, x_{i-1}, x_i]$</td>
<td>$f[x_{i-3}, \ldots, x_i]$</td>
<td>$f[x_{i-4}, \ldots, x_i]$</td>
</tr>
<tr>
<td>-----</td>
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<td>-----------------------------</td>
<td>-----------------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.7651977</td>
<td></td>
<td>-0.4837057</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.3</td>
<td>0.6200860</td>
<td></td>
<td>-0.5489460</td>
<td>-0.1087339</td>
<td>0.0658784</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>0.4554022</td>
<td></td>
<td>-0.5786120</td>
<td>-0.0494433</td>
<td>0.0680685</td>
</tr>
<tr>
<td>3</td>
<td>1.9</td>
<td>0.2818186</td>
<td></td>
<td></td>
<td>0.0118183</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.2</td>
<td>0.1103623</td>
<td></td>
<td></td>
<td>-0.5715210</td>
<td></td>
</tr>
</tbody>
</table>

The coefficients of the Newton forward divided-difference form of the interpolating polynomial are along the diagonal in the table. This polynomial is

$$P_4(x) = 0.7651977 - 0.4837057(x - 1.0) - 0.1087339(x - 1.0)(x - 1.3)$$

$$+ 0.0658784(x - 1.0)(x - 1.3)(x - 1.6)$$

$$+ 0.0018251(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9).$$
Knowledge

- Forward Difference
- Backward Difference
Newton’s divided-difference formula can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing. In this case, we introduce the notation \( h = x_{i+1} - x_i \), for each \( i = 0, 1, \ldots, n - 1 \) and let \( x = x_0 + sh \). Then the difference \( x - x_i \) is \( x - x_i = (s - i)h \). So Eq. (3.10) becomes

\[
P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}). \tag{3.10}
\]

\[
P_n(x) = P_n(x_0 + sh) = f[x_0] + sh f[x_0, x_1] + s(s - 1)h^2 f[x_0, x_1, x_2] + \cdots + s(s - 1) \cdots (s - n + 1)h^n f[x_0, x_1, \ldots, x_n]
\]

\[
= f[x_0] + \sum_{k=1}^{n} s(s - 1) \cdots (s - k + 1)h^k f[x_0, x_1, \ldots, x_k].
\]

Using binomial-coefficient notation,

\[
\binom{s}{k} = \frac{s(s - 1) \cdots (s - k + 1)}{k!},
\]

we can express \( P_n(x) \) compactly as

\[
P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} h^k f[x_0, x_i, \ldots, x_k]. \tag{3.11}
\]
The Newton forward-difference formula, is constructed by making use of the forward difference notation $\Delta$ introduced in Aitken’s $\Delta^2$ method. With this notation,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}(f(x_1) - f(x_0)) = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{1}{2h} \left[ \frac{\Delta f(x_1) - \Delta f(x_0)}{h} \right] = \frac{1}{2h^2}\Delta^2 f(x_0),$$

and, in general,

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Since $f[x_0] = f(x_0)$, Eq. (3.11) has the following form.

**Newton Forward-Difference Formula**

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{S}{k} \Delta^k f(x_0)$$
If the interpolating nodes are reordered from last to first as \( x_n, x_{n-1}, \ldots, x_0 \), we can write the interpolatory formula as

\[
P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1}) + \cdots + f[x_n, \ldots, x_0](x - x_n)(x - x_{n-1}) \cdots (x - x_1).
\]

If, in addition, the nodes are equally spaced with \( x = x_n + sh \) and \( x = x_i + (s + n - i)h \), then

\[
P_n(x) = P_n(x_n + sh)
\]

\[
= f[x_n] + sh f[x_n, x_{n-1}] + s(s + 1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \cdots + s(s + 1) \cdots (s + n - 1)h^n f[x_n, \ldots, x_0].
\]

**Newton Backward–Difference Formula**

\[
P_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n)
\]