244 math

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2022\2023

Syllabus:

Matrices and their operations. Types of matrices. Elementary transformations. Linear systems of equations. Determinants, elementary properties. Inverse of a matrix. Vector spaces, linear independence, finite dimensional spaces, linear subspaces. Inner product spaces. Linear transformations, kernel and image of a linear transformation. Eigen values and Eigen vectors of a matrix and of a linear operator.

Textbook:

Elementary Linear Algebra, with Supplemental Applications, 11th Edition by Howard Anton, Chris Rorres

Grading:

- **30 marks- Midterm**
- **10 marks- Quiz 1**
- **10 marks- Quiz 2**
- **10 marks-Tutorial**
- **40 marks- Final Exam**

Targeted skills:

- Development of computational kills
- Development of Logical thinking skills
- Development of research skills
- Development of programming skills.

Chapter 1: Linear systems

define a ***linear equation*** in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (1)$$

where $a_1, a_2, \ldots, a_n$ and b are constants, and the a's are not all zero.

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \qquad (4)$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \ldots, x_n$.

The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$
$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$
$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

A finite set of linear equations is called a ***system of linear equations***

For example:

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \qquad (5\text{–}6)$$

Solution of the system is a sequence of numbers that satisfy every equation in the system.

We can write solution of a system as **explicit form** or as ordered **n tuples form** the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$ **explicite form**

solutions can be written more succinctly as

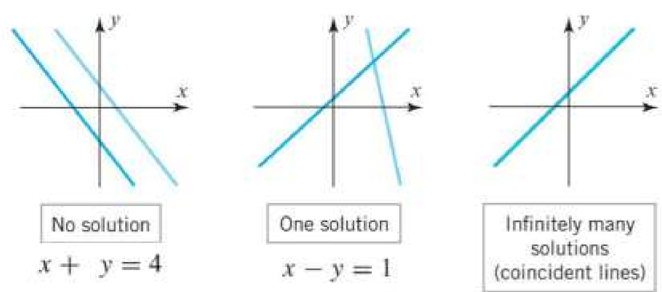
$(1, 2, -1)$ **ordered 3-tuples**

POSSIBILITIES OF SOLUTION OF A LINEAR SYSTEM

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy-plane.



No solution:
$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$
WHY?

One solution:
$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$
WHY?

Infinitely many solutions (coincident lines):
$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$
WHY?

Result:

> *Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

Remark: if the system has infinite many solutions, then we can write them in terms of parameters.

For example: 2x+y-z=3 is a linear system in 3 variables and 1 equation which has infinite many solutions:
S= {(t, s, 2t+s-3): t and s any real numbers}

How to represent a linear system?

We have 3 manners to write a linear system.

1- **Explicit method**

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

2- **Matrix form** AX=B where A is a matrix of coefficients , X is column of variables and B is a column of constants

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

3- **Augmented matrix:**

$$\overset{A}{} \qquad \overset{B}{}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Remark:

You should learn how induce the explicit form from the Augmented matrix form.

Ex) Suppose that

$$\left[\begin{array}{rrrr} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{array}\right],$$ is the augmented matrix of a linear system. Write the system as explicit form.

Answer:

elementary row operations

1. Multiply an equation through by a nonzero constant. **3Ri**
2. Interchange two equations. **Rij**
3. Add a constant times one equation to another. **aRi+Rj**

For example:

To make them zeros ⇨

$$\begin{bmatrix} 1 & 8 & 2 & 7 \\ 2 & 4 & -4 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2,\,-2R_1+R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & -12 & -8 & -11 \\ 0 & -15 & -3 & -12 \end{bmatrix} \xrightarrow{-\frac{1}{12}R_2} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & -15 & -3 & -12 \end{bmatrix}$$

⇦ To make it zero

$$\xrightarrow{15R_2+R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & 0 & 7 & \frac{7}{4} \end{bmatrix} \xrightarrow{\frac{1}{7}R_3} \begin{bmatrix} 1 & 8 & 2 & 7 \\ 0 & 1 & \frac{2}{3} & \frac{11}{12} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$$

Definition:

Row equivalent matrix to A is a matrix which is resulting from one or more row elementary operations

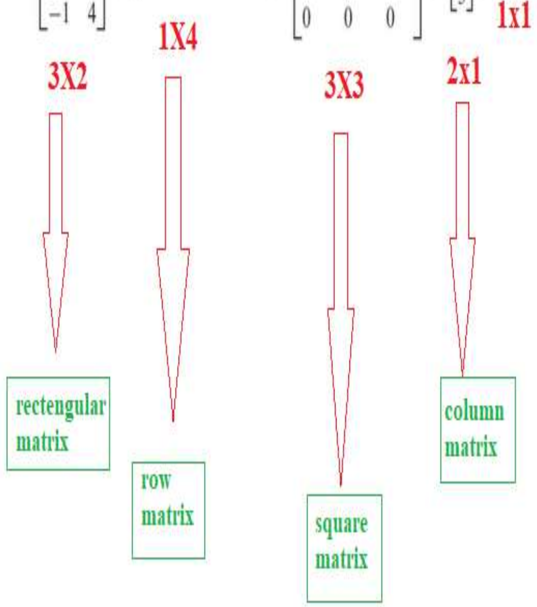
Matrices and Matrix Operations

DEFINITION 1 A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains.

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$



Remark:

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Equality of Matrices

DEFINITION 2 Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

EX) If A=B, finf values of x, y and z?

$$A = \begin{bmatrix} x-2 & y-3 \\ x+y & z+3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3+z \\ z & y \end{bmatrix}$$

Answer:

Addition and Subtraction

DEFINITION 3 If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

scalar multiple of A

DEFINITION 4 If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

Example: Find the value of x and y in the following matrix equation

$$\begin{bmatrix} 5 & x \\ 3y & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \end{bmatrix}$$

Answer:

Multiplying Matrices

Determining Whether a Product Is Defined

$$\underset{m \times r}{A} \quad \underset{r \times n}{B} = \underset{m \times n}{AB} \tag{3}$$

Inside: r and r; Outside: m and n

EXAMPLE 6

Suppose that A, B, and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

Then by (3), AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix; and CA is defined and is a 7×4 matrix. The products AC, CB, and BA are all undefined. ◀

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

The computations for the remaining entries are

$$\begin{aligned} (1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\ (1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\ (1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\ (2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\ (2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\ (2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12 \end{aligned} \qquad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$ ◀

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **[Commutative law for matrix addition]**

(b) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(c) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(d) $A(B + C) = AB + AC$ **[Left distributive law]**

(e) $(B + C)A = BA + CA$ **[Right distributive law]**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Remark: We have to noticed some properties which are **Not** satisfied in the class of matrices.

1- Product of matrices is not commutative.

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$. ◀

Notation : Mnxm(R) is the set of all nxm matrices.
Mn(R) is the set of all nxn matrices.

(Challenge): Would you bring an example on infinite subset of M2(R) which is the product of its matrices is commutative?

Answer:

2- Failure of the Cancellation Law

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

But B doesn't equal to C
In other words, we can not cancell A

3- A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Definition (power of Matrix) A^n

$A^n = A.A.A \ldots . A$ **(n-times)**

Question: Let A and B be two square matrices in the same size.

1- write formula to $(A+B)^2$

2- Is $(A^2 - B^2) = (A-B)(A+B)$ **?**

Remark:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Transpose of a Matrix

> **DEFINITION 7** If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix},$$

3x2 $\quad a_{ij} = a_{ji}$ $\quad$ 2x3

Properties of the Transpose of a matrix

1. $(A^t)^t = A$
2. $(AB)^t = B^t A^t$
3. $(kA)^t = kA^t$, where k is a scalar.
4. $(A+B)^t = A^t + B^t$

> **DEFINITION 8** If A is a square matrix, then the ***trace of A***, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

nd their traces.

$$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Working with Proofs

35. Prove: If A and B are $n \times n$ matrices, then

$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$

36. (a) Prove: If AB and BA are both defined, then AB and BA are square matrices.

(b) Prove: If A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

Zero Matrix: A zero matrix is a matrix of nay order whose all entries are zero.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$ is a **zero matrix**.

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(*a*) $A + 0 = 0 + A = A$

(*b*) $A - 0 = A$

(*c*) $A - A = A + (-A) = 0$

(*d*) $0A = 0$

(*e*) *If* $cA = 0$, *then* $c = 0$ *or* $A = 0$.

Remember that if AB=O where A, B are matrices, then we can NOT deduce that A=O or B=O

Diagonal Matrix: A square matrix with all its non- diagonal entries are zero.

Examples.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

Unit Matrix: A diagonal matrix with all diagonal entries are one '1'

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rule : I A = A where IA is defined
A I = A where AI is defined

Example:

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and multiplying on the left by the 2×2 identity matrix yields

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Symmetric Matrix:

A square matrix is symmetric if $A^t = A$. **In other words,** $a_{ij} = a_{ji}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad A^t = A$$

Skew – symmetric Matrix :

A square matrix is skew symmetric if $A^t = -A$. **In other words, $a_{ij} = -a_{ji}$** **which means all elements in the main diognal are zeros**

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, \quad A^t = -A.$$

Question) If A is skew-symmetric matrix, find trace(A)?

Challenge question**: Give an example on Symmetric matrix and skew-symmetric matrix in the same time?**

Give an example on diagonal symmetric matrix?

Upper triangular matrix:

It is a square matric such that all elements under the main diagonal are zeros.

Give an example:

Lower triangular matrix:

It is a square matric such that all elements above the main diagonal are zeros.

Give an example:

Some problems with solutions:

(1) Let A, B, C and D be matrices where

$$(A+BA)^t + C^tA^t = D$$

if $size(D) = 10\times5$, find $size(A)$, $size(B)$ and $size(C)$?

Solution

we have $(A+BA)^t + C^tA^t = D$.

$D \to 10\times5$

So, $size\,(A+BA)^t = 10\times5$

$\Rightarrow size\,(A+BA) = 5\times10$

$size(A) = 5\times10$

$size(BA) = 5\times10$

$BA \to 5\times10$

$\Rightarrow size(B) = 5\times5$

$size\,(C^tA^t) = 10\times5$

$A^t \to 10\times5$

$\Rightarrow size\ C^t = 10\times10$

$\Rightarrow$ $size(C) = 10\times10$

(2) Let A, B and I be matrices of $M_3(\mathbb{R})$. Find

$$tr\left(AB + (B^tA^t + I)^t\right)$$

Solution

$$AB - (B^tA^t + I)^t = AB - (B^tA^t)^t + I_3^t$$
$$= AB - AB + I_3$$
$$= I_3$$

$$\therefore \; tr\left(AB - (B^tA^t + I)^t\right) = tr(I_3)$$
$$= 1 + 1 + 1 = 3$$

(3) If A is a square matrix such that $A^2 = A$ then $(I + A)^2 - 3A = I$, Prove that ?

Solution $\quad L.H.S = (I + A)^2 - 3A$

$$= I^2 + IA + AI + A^2 - 3A$$
$$= I^2 + A + A + A^2 - 3A$$
$$= I = R.H.S$$

Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $f(x) = 1 + x + x^2 + x^4 + x^8 + x^{16}$

find $f(A)$?

Solution $f(A) = I + A + A^2 + A^4 + A^8 + A^{16}$

$A^2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$A^4 = A^2 A^2 = 0$

$A^8 = 0$

$A^{16} = 0$

So, $f(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$