INTRODUCTION

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Differential Equations

Classification of Differential Equations

- ODE and PDE
- Order and Degree of Differential Equations
- Linear Differential Equations

Applications of ODEs

- Population models
- Predator-prey models
- Electrical circuits
- Bessel's equation

Solution of Differential Equations

The Elimination of Arbitrary Constant

Families of Curves

This course is concerned with differential equations (DEs) and their solutions methods. Methods will be employed to obtain approximate solutions of the differential equations which consider the equation $y = y(x)$, which describes a relationship between two quantities. The **dependent** variable y, and the **independent** variable x.

Definition

A *differential equation* is an equation containing an unknown function and its derivatives.

Example (1)

- $\frac{dy}{dx} = 5x + 2$
- $4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = ay$
- $\left(\frac{d^{100}y}{dx^{100}}\right)^{99} + \left(\frac{dy}{dx}\right)^{101} = 3,$

we show that in this example, y is *depentent* variable and x is *independent* variable. These equations are called *ordinary differentail equations (ODE)*.

Example (2)

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
- $3\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial^2 u}{\partial t^2} = 0.$

In the previous example, u is *dependent* variable and x and y are *independent* variables. These equations are called *partial differential equations(PDE)*.

ODE and PDE

Definition

We say that an equation is ODE if the unknown function depends only on one variable.

Example

- $\frac{dy}{dx} = 5x + 2.$
- $4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = ay.$
- $\left(\frac{d^{100}y}{dx^{100}}\right)^{99} + \left(\frac{dy}{dx}\right)^{101} = 3.$

Definition

We say that an equation is PDE if the unknown function depends on more than one variable.

Example

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
- $3\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial^2 u}{\partial t^2} = 0.$

An equation, or systems of equations, may involve more than one dependent variables, but provided only a single independent variable appears then it is still termed ordinary. For example,

$$u' = 2u + 3v^2; v'' = u^2 + 4v;$$

where

$$u = u(x) \text{ and } v = v(x),$$

is a system of ODEs since there is only one independent variable x.

Table: Note

$\frac{dy}{dx} = y^{'}$	$\frac{d^2y}{dx^2} = y^{''}$	$\dots$	$\frac{d^ny}{dx^n} = y^n$
$\left(\frac{dy}{dx}\right)^2 = (y^{'})^2$	$\left(\frac{dy}{dx}\right)^3 = (y^{'})^3$	$\dots$	$\left(\frac{dy}{dx}\right)^n = (y^{'})^n$

Order and Degree of Differential Equations

Definition

1. The **order** of the differential equation is order of the highest derivative in the differential equation.
2. The **degree** of a differential equation is given by the degree of the power of the highest derivative used.

Table: Examples of the **order** and the **degree** of the differential equations

Differential Equation	Order	Degree
$y^{''} + 2y^{'} + y = 0$	2	1
$(y^{'})^2 + 3y = x + 2$	1	2
$y^{'''} + y = xe^8$	3	1
$(y^{''''})^8 + (y^{'})^{20} = 0$	4	8

Linear Differential Equations

Definition

An ordinary differential equation of order n is called linear if it might be on the form

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x), \tag{1}$$

or

$$f(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^{n-1}y}{dx^{n-1}},\frac{d^ny}{dx^n})=g(x),$$

or

$$f(x,y,y^{'},y^{''},\ldots,y^{n-1},y^{n})=g(x),$$

where the coefficient functions $a_n, a_{n-1}\ldots a_0$ in (1), and $g(x)$ depend only on x, and all of the terms involving the dependent variable (y or its derivatives) appear in simple algebraic form.

If the coefficient functions $a_n(x), a_{n-1}(x)$, etc, are constants, then the equation is called linear with constant coefficients. If the equation can't be written as (1), then **the ODE is non-linear**.

Exercises

Classify the following equations as either linear or nonlinear, state the order and identify the dependent and independent variables.

(a) $\frac{d^3y}{dx^3} - \frac{dy}{dx} = \sin x$ (b) $\frac{d^2u}{dt^2} - \frac{du}{dt} = e^u$

(c) $\left(\frac{d^3\theta}{dz^3}\right)^2 - z\frac{d\theta}{dz} = 0$ (d) $\frac{d^2y}{dt^2} - y^3 = 1 - t$

Answers

(a) ...., third order, dependent: y, independent: x.

(b) Nonlinear, second order, dependent: ..., independent: t.

(c) ...., ..., dependent: ...., independent: ......

(d)................., ..., dependent: ...., independent: .....

Applications of ODEs

There are countless applications that are modelled by ODEs. Pioneers in developing ODEs include Leibniz (1646–1716), Newton (1643–1727) and Euler (1707–1783).

Population models

For some population $P = P(t)$, we might assume that the rate of increase or decrease is proportional to the current population

$$\frac{dP}{dt} = \lambda P,$$

which is used to model populations of biological cells, species and radioactive atoms.

Another model is

$$\frac{dP}{dt} = rP - aP^2,$$

which is called the Logistic equation. In this model the population does not increase without bound forever, but is limited at large populations.

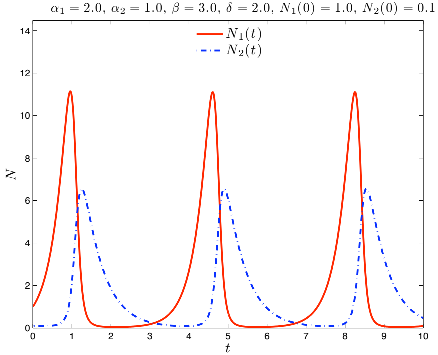
Predator-prey models

For some prey population $N_1 = N_1(t)$, and predator population $N_2 = N_2(t)$, we have the model

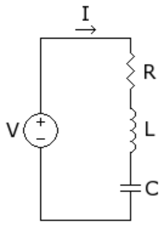
$$\frac{dN_1}{dt} = N_1(\alpha_1 - \beta N_2), \quad \frac{dN_2}{dt} = N_2(-\alpha_2 + \delta N_1),$$

called the Lotka-Volterra equations. This model was developed in the 1920s and was originally used to model populations of sharks and prey fish in the Adriatic Sea. Other examples of predator-prey populations include foxes and rabbits, ladybirds and aphids.

An interesting feature of the solution to this model is that the two populations can oscillate on a coupled periodic basis.


$\alpha_1 = 2.0,\ \alpha_2 = 1.0,\ \beta = 3.0,\ \delta = 2.0,\ N_1(0) = 1.0,\ N_2(0) = 0.1$
$N_1(t)$
$N_2(t)$
N
t
0
2
4
6
8
10
12
14
0
1
2
3
4
5
6
7
8
9
10

Electrical circuits



An RLC circuit comprises an resistor, inductor and capacitor in series with a cell supplying a voltage $V(t)$. The equation for the current $I(t)$ is given by

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dV}{dt}, \tag{2}$$

where R is the resistance, L is the inductance and C is the capacitance.

Equation (2) is a second order ODE with constant coefficients and forcing and it is the same equation that describes a mass-spring-damper system

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t),$$

where $x(t)$ is the position of the mass at time t, m is the mass, c is the damping constant, k is the spring constant and $F(t)$ is the applied force. Thus RLC circuits and mass-spring-damper systems exhibit similar phenomena, such as resonance and damping frequencies, which can be solved for in terms of the parameters of the system.

Bessel's equation

The second-order ODE

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-p^2)y=0,$$

is called Bessel's equation and it has many applications in heat-diffusion, electromagnetics and elasticity. The two linearly-independent solutions of this equations are

- $J_p(x)$ Bessel function of the first kind of order p
- $Y_p(x)$ Bessel function of the second kind of order p

Due to the importance of this ODE, many analytical and asymptotic properties of Bessel functions have been tabulated and recorded.

Solution of Differential Equations

we can say that the Solution of the Differential Equations (DEs) means finding an equation with no derivatives that satisfies the given DE.

Example

Prove that $y = e^{2x}$ is a solution of the equation

$$y'' + y' - 6y = 0$$

Solution

to prove that we substitute $y = e^{2x}$, $y' = 2e^{2x}$ and $y'' = 4e^{2x}$ in the equation, thus

$$y'' + y' - 6y = 4e^{2x} + 2e^{2x} - 6e^{2x} = 0 \quad \text{for all } x \in \mathbb{R}$$

Example

Verify that $y = x^3 e^x$ is a solution of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0; x > 0$$

Solution By substituting $y = x^3 e^x$, $y' = 3x^2 e^x + x^3 e^x$ and $y'' = 6xe^x + 6x^2 e^x + x^3 e^x$ in the equation thus,

$$\begin{aligned} xy'' & -2(x+1)y' + (x+2)y \\ = & \quad 6x^2e^x + 6x^3e^x + x^4e^x - 6x^3e^x - 2x^4e^x - 6x^2e^x - 2x^3e^x + x^4e^x + x^3e^x \\ = & \quad 0 \end{aligned}$$

Example

verify that $F(x,y) = x^2 + y^2 - 25 = 0$ satisfies an implicit solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, on the interval $-5 < x < 5$.

Note: we say a relation $F(x,y) = 0$ is an **implicit solution** of an ordinary differential equation on an interval I if the relation defines implicitly a function $y = \phi(x)$ satisfy the differential equation.

Solution

by using the implicit differentiation we obtain that $2x + 2yy' = 0$ or
$\frac{dy}{dx} = -\frac{x}{y}$; $y \neq 0$.
We write y explicitly as a function of x. Thus

$$y^2 + x^2 = 25 \;\Rightarrow\; y = \pm\sqrt{25 - x^2},$$

then the two explicit solutions defined on the interval $(-5, 5)$

The Elimination of Arbitrary Constant

Creating a differential equation by eliminate the arbitrary constants:

Example (1)

Eliminate the arbitrary constants c from the equation

$$y = c\sin x \quad (1)$$

Solution

Since one constant is to be eliminated, we obtain the one derivative

$$y' = c\cos x$$

$$\Rightarrow \ c = \frac{y'}{\cos x}$$

by substitution the value of c into (1) we have

$$y = \frac{y'}{\cos x}\sin x$$

$$\frac{y}{y'} = \tan x$$

Example (2)

Eliminate the arbitrary constants a from the equation
$$y^2 = 4ax.$$

Solution Since one constant is to be eliminated, we obtain the one derivative and by using the implicit differentiation we have

$$2yy' = 4a$$

$$a = \frac{yy'}{2} \quad (2)$$

by substitution from (2) into (1) we have the differential equation of the first order and the first degree

$$2xy' - y = 0$$

Example (3)

Eliminate the arbitrary constants c_1 and c_2 from the relation

$$y = c_1 e^{-2x} + c_2 e^{3x} \quad (1)$$

Solution

Since one constant is to be eliminated, we obtain the two derivative

$$y' = -2c_1 e^{-2x} + 3c_2 e^{3x} \quad (2)$$

$$y'' = 4c_1 e^{-2x} + 9c_2 e^{3x} \quad (3)$$

the elimination of c_1 from (2) and (3) yields

$$y'' + 2y' = 15c_2 e^{3x} \quad (4)$$

the elimination of c_1 from (1) and (2) yields

$$y' + 2y = 5c_2 e^{3x} \quad (5)$$

the elimination of c_2 from (4) and (5) yeilds

$$y'' - y' - 6y = 0$$

Example (4)

Eliminate the arbitrary constant a from the equation

$$(x-a)^2 + y^2 = a^2$$

Solution

$$2(x-a) + 2yy' = 0$$

$$\Rightarrow \ a = 2yy' + x$$

thus,

$$(yy')^2 + y^2 = (x + yy')^2$$

$$\Rightarrow \ 2xyy' + x^2 - y^2 = 0$$

Example (5)

Eliminate B and α from the relation

$$x = B\cos(\omega t + \alpha).$$

Solution

in this example we have B and α are arbitrary constants and ω is a fixed constant. First we obtain two derivatives of x with respect to t

$$x' = -\omega B\sin(\omega t + \alpha)$$

$$x'' = -\omega^2 B\cos(\omega t + \alpha)$$

by comparison of equation and shows at once that

$$x'' + \omega^2 x = 0$$

Exercise

Eliminate the arbitrary constant c from the family of curves

$$cxy + c^2 x + 4 = 0.$$

Families of Curves

An equation involving a parameter, also one both of the coordinates of a plane, might represent a family of curve corresponding to each value of the parameter. For example the equation

$$(x-c)^2 + (y-c)^2 = 2c^2 \tag{3}$$

represents a family of circles with their centers on $y = x$. If we assume that c in the equation (3) is arbitrary constant, then by using the elimination of arbitrary constant, then result equation is called differential equation of the family of curve (3).
We can rearrange equation (3) to be the following

$$x^2 + y^2 - 2c(x+y) = 0$$

$$\frac{x^2 + y^2}{x+y} = 2c, \quad y \neq -x.$$

Families of Curves

Example (1)

Find a differential equation satisfied by the family of parabolas having their vertices at the origin and their foci (focus) on the $y-axis$.

Solution An equation of this family of parabolas is

$$y = ax^2,$$

where $a \neq 0$ is a arbitrary constant Thus

$$y' = 2ax$$

$$\Rightarrow \; xy' - 2y = 0,$$

which is the differential equation of the family, and it is a first-order linear differential equation.

Families of Curves

Example (2)

Find a differential equation of the family of Find circles having their centers on the $y - axis$.

Solution we have to deal with the two parameters family because of a member of the family of circles of this examples have its center anywhere on the $y - axis$ and its radius of any magnitude, thus

$$x^2 + (y - b)^2 = c^2,$$

now we will eliminate the both b and c parameters and we will have a non-linear second-order differential equation as blew

$$xy'' - (y')^3 - y' = 0.$$