2
Survival models

2.1 Summary

In this chapter we represent the future lifetime of an individual as a random variable, and show how probabilities of death or survival can be calculated under this framework. We then define an important quantity known as the force of mortality, introduce some actuarial notation, and discuss some properties of the distribution of future lifetime. We introduce the curtate future lifetime random variable. This is a function of the future lifetime random variable which represents the number of complete years of future life. We explain why this function is useful and derive its probability function.

2.2 The future lifetime random variable

In Chapter 1 we saw that many insurance policies provide a benefit on the death of the policyholder. When an insurance company issues such a policy, the policyholder's date of death is unknown, so the insurer does not know exactly when the death benefit will be payable. In order to estimate the time at which a death benefit is payable, the insurer needs a model of human mortality, from which probabilities of death at particular ages can be calculated, and this is the topic of this chapter.

We start with some notation. Let (x) denote a life aged x, where $x \geq 0$. The death of (x) can occur at any age greater than x, and we model the future lifetime of (x) by a continuous random variable which we denote by T_x. This means that $x + T_x$ represents the age-at-death random variable for (x).

Let F_x be the distribution function of T_x, so that

$$F_x(t) = \Pr[T_x \leq t].$$

Then $F_x(t)$ represents the probability that (x) does not survive beyond age $x + t$, and we refer to F_x as the **lifetime distribution** from age x. In many life insurance problems we are interested in the probability of survival rather than death, and so we define S_x as

$$S_x(t) = 1 - F_x(t) = \Pr[T_x > t].$$

Thus, $S_x(t)$ represents the probability that (x) survives for at least t years, and S_x is known as the **survival function.**

Given our interpretation of the collection of random variables $\{T_x\}_{x\geq 0}$ as the future lifetimes of individuals, we need a connection between any pair of them. To see this, consider T_0 and T_x for an individual who is now aged x. The random variable T_0 represented the future lifetime at birth for this individual, so that, at birth, the individual's age at death would have been represented by T_0. This individual could have died before reaching age x – the probability of this was $\Pr[T_0 < x]$ – but has survived. Now that the individual has survived to age x, so that we know that $T_0 > x$, her future lifetime is represented by T_x and her age at death is now $x + T_x$. If she dies within t years from now, then $T_x \leq t$ and $T_0 \leq x + t$. Loosely speaking, we require the events $[T_x \leq t]$ and $[T_0 \leq x + t]$ to be equivalent, given that the individual survives to age x. We achieve this by making the following assumption for all $x \geq 0$ and for all $t > 0$

$$\boxed{\Pr[T_x \leq t] = \Pr[T_0 \leq x + t | T_0 > x].} \tag{2.1}$$

This is an important relationship.

Now, recall from probability theory that for two events A and B

$$\Pr[A|B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]},$$

so, interpreting $[T_0 \leq x + t]$ as event A, and $[T_0 > x]$ as event B, we can rearrange the right-hand side of (2.1) to give

$$\Pr[T_x \leq t] = \frac{\Pr[x < T_0 \leq x + t]}{\Pr[T_0 > x]},$$

that is,

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)}. \tag{2.2}$$

Also, using $S_x(t) = 1 - F_x(t)$,

$$\boxed{S_x(t) = \frac{S_0(x+t)}{S_0(x)},} \tag{2.3}$$

which can be written as

$$\boxed{S_0(x+t) = S_0(x)\, S_x(t).} \tag{2.4}$$

This is a very important result. It shows that we can interpret the probability of survival from birth to age $x+t$ as the product of

(1) the probability of survival to age x from birth, and
(2) the probability, having survived to age x, of further surviving to age $x+t$.

Note that $S_x(t)$ can be thought of as the probability that (0) survives to at least age $x+t$ given that (0) survives to age x, so this result can be derived from the standard probability relationship

$$\Pr[A \text{ and } B] = \Pr[A|B]\Pr[B]$$

where the events here are $A=[T_0 > x+t]$ and $B=[T_0>x]$, so that

$$\Pr[A|B] = \Pr[T_0 > x+t | T_0 > x],$$

which we know from (2.1) is equal to $\Pr[T_x > t]$.

Similarly, any survival probability for (x), for, say, $t+u$ years can be split into the probability of surviving the first t years, and then, given survival to age $x+t$, subsequently surviving another u years. That is,

$$\begin{aligned} S_x(t+u) &= \frac{S_0(x+t+u)}{S_0(x)} \\ \Rightarrow S_x(t+u) &= \frac{S_0(x+t)}{S_0(x)}\frac{S_0(x+t+u)}{S_0(x+t)} \\ \Rightarrow S_x(t+u) &= S_x(t)S_{x+t}(u). \end{aligned} \tag{2.5}$$

We have already seen that if we know survival probabilities from birth, then, using formula (2.4), we also know survival probabilities for our individual from any future age x. Formula (2.5) takes this a stage further. It shows that if we know survival probabilities from any age x (≥ 0), then we also know survival probabilities from any future age $x+t\,(\geq x)$.

Any survival function for a lifetime distribution must satisfy the following conditions to be valid.

Condition 1 $S_x(0)=1$; that is, the probability that a life currently aged x survives 0 years is 1.

Condition 2 $\lim_{t\to\infty} S_x(t)=0$; that is, all lives eventually die.

Condition 3 The survival function must be a non-increasing function of t; it cannot be more likely that (x) survives, say 10.5 years than 10 years, because in order to survive 10.5 years, (x) must first survive 10 years.

These conditions are both necessary and sufficient, so that any function S_x which satisfies these three conditions as a function of t (≥ 0), for a fixed

x (≥ 0), defines a lifetime distribution from age x, and, using formula (2.5), for all ages greater than x.

For all the distributions used in this book, we make three additional assumptions:

Assumption 2.1 *$S_x(t)$ is differentiable for all $t > 0$. Note that together with Condition 3 above, this means that $\frac{d}{dt}S_x(t) \leq 0$ for all $t > 0$.*

Assumption 2.2 $\lim_{t\to\infty} t\, S_x(t) = 0$.

Assumption 2.3 $\lim_{t\to\infty} t^2\, S_x(t) = 0$.

These last two assumptions ensure that the mean and variance of the distribution of T_x exist. These are not particularly restrictive constraints – we do not need to worry about distributions with infinite mean or variance in the context of individuals' future lifetimes. These three extra assumptions are valid for all distributions that are feasible for human lifetime modelling.

Example 2.1 Let

$$F_0(t) = \begin{cases} 1-(1-t/120)^{1/6} & \text{for } 0 \leq t \leq 120 \\ 1 & \text{for } t > 120 \end{cases}.$$

Calculate the probability that

(a) a newborn life survives beyond age 30,
(b) a life aged 30 dies before age 50, and
(c) a life aged 40 survives beyond age 65.

Solution 2.1 (a) The required probability is

$$S_0(30) = 1 - F_0(30) = (1-30/120)^{1/6} = 0.9532.$$

(b) From formula (2.2), the required probability is

$$F_{30}(20) = \frac{F_0(50) - F_0(30)}{1 - F_0(30)} = 0.0410.$$

(c) From formula (2.3), the required probability is

$$S_{40}(25) = \frac{S_0(65)}{S_0(40)} = 0.9395.$$

□

We remark that in the above example, $S_0(120) = 0$, which means that under this model, survival beyond age 120 is not possible. In this case we refer to 120 as the limiting age of the model. In general, if there is a limiting age, we use the Greek letter ω to denote it. In models where there is no limiting age, it is often practical to introduce a limiting age in calculations, as we will see later in this chapter.

2.3 The force of mortality

The force of mortality is an important and fundamental concept in modelling future lifetime. We denote the force of mortality at age x by μ_x and define it as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_0 \le x + dx \mid T_0 > x]. \tag{2.6}$$

From equation (2.1) we see that an equivalent way of defining μ_x is

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_x \le dx],$$

which can be written in terms of the survival function S_x as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \left(1 - S_x(dx)\right). \tag{2.7}$$

Note that the force of mortality depends, numerically, on the unit of time; if we are measuring time in years, then μ_x is measured per year.

The force of mortality is best understood by noting that for very small dx, formula (2.6) gives the approximation

$$\mu_x \, dx \approx \Pr[T_0 \le x + dx \mid T_0 > x]. \tag{2.8}$$

Thus, for very small dx, we can interpret $\mu_x \, dx$ as the probability that a life who has attained age x dies before attaining age $x + dx$. For example, suppose we have a life aged exactly 50, and that the force of mortality at age 50 is 0.0044 per year. A small value of dx might be a single day, or 0.00274 years. Then the approximate probability that the life dies on his 50th birthday is $0.0044 \times 0.00274 = 1.2 \times 10^{-5}$.

We can relate the force of mortality to the survival function from birth, S_0. As

$$S_x(dx) = \frac{S_0(x + dx)}{S_0(x)},$$

formula (2.7) gives

$$\begin{aligned}
\mu_x &= \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x + dx)}{dx} \\
&= \frac{1}{S_0(x)} \left(-\frac{d}{dx} S_0(x) \right).
\end{aligned}$$

Thus,

$$\boxed{\mu_x = \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x).} \tag{2.9}$$

From standard results in probability theory, we know that the probability density function for the random variable T_x, which we denote f_x, is related to the distribution function F_x and the survival function S_x by

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t).$$

So, it follows from equation (2.9) that

$$\boxed{\mu_x = \frac{f_0(x)}{S_0(x)}.}$$

We can also relate the force of mortality function at any age $x+t$, $t>0$, to the lifetime distribution of T_x. Assume x is fixed and t is variable. Then $d(x+t) = dt$ and so

$$\begin{aligned}
\mu_{x+t} &= -\frac{1}{S_0(x+t)}\frac{d}{d(x+t)}S_0(x+t) \\
&= -\frac{1}{S_0(x+t)}\frac{d}{dt}S_0(x+t) \\
&= -\frac{1}{S_0(x+t)}\frac{d}{dt}(S_0(x)S_x(t)) \\
&= -\frac{S_0(x)}{S_0(x+t)}\frac{d}{dt}S_x(t) \\
&= \frac{-1}{S_x(t)}\frac{d}{dt}S_x(t).
\end{aligned}$$

Hence

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)}. \tag{2.10}$$

This relationship gives a way of finding μ_{x+t} given $S_x(t)$. We can also use equation (2.9) to develop a formula for $S_x(t)$ in terms of the force of mortality function. We use the fact that for a function h whose derivative exists,

$$\frac{d}{dx}\log h(x) = \frac{1}{h(x)}\frac{d}{dx}h(x),$$

so from equation (2.9) we have

$$\mu_x = -\frac{d}{dx}\log S_0(x),$$

and integrating this identity over $(0, y)$ yields

$$\int_0^y \mu_x dx = -\left(\log S_0(y) - \log S_0(0)\right).$$

As $\log S_0(0) = \log \Pr[T_0 > 0] = \log 1 = 0$, we obtain

$$S_0(y) = \exp\left\{-\int_0^y \mu_x dx\right\},$$

from which it follows that

$$\boxed{S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \exp\left\{-\int_x^{x+t} \mu_r dr\right\} = \exp\left\{-\int_0^t \mu_{x+s} ds\right\}.} \tag{2.11}$$

This means that if we know μ_x for all $x \geq 0$, then we can calculate all the survival probabilities $S_x(t)$, for any x and t. In other words, the force of mortality function fully describes the lifetime distribution, just as the function S_0 does. In fact, it is often more convenient to describe the lifetime distribution using the force of mortality function than the survival function.

Example 2.2 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \leq x \leq 120$. Derive an expression for μ_x.

Solution 2.2 As $S_0(x) = (1 - x/120)^{1/6}$, it follows that

$$\frac{d}{dx} S_0(x) = \tfrac{1}{6}(1 - x/120)^{-5/6}\left(-\tfrac{1}{120}\right),$$

and so

$$\mu_x = \frac{-1}{S_0(x)}\frac{d}{dx} S_0(x) = \tfrac{1}{720}(1 - x/120)^{-1} = \frac{1}{720 - 6x}.$$

As an alternative, we could use the relationship

$$\begin{aligned}\mu_x &= -\frac{d}{dx}\log S_0(x) = -\frac{d}{dx}\left(\frac{1}{6}\log(1 - x/120)\right) = \frac{1}{720(1 - x/120)} \\ &= \frac{1}{720 - 6x}.\end{aligned}$$

□

Example 2.3 Let $\mu_x = Bc^x$, $x > 0$, where B and c are constants such that $0 < B < 1$ and $c > 1$. This model is called **Gompertz' law of mortality**. Derive an expression for $S_x(t)$.

Solution 2.3 From equation (2.11),

$$S_x(t) = \exp\left\{-\int_x^{x+t} Bc^r dr\right\}.$$

Writing c^r as $\exp\{r \log c\}$,

$$\int_x^{x+t} Bc^r\,dr = B\int_x^{x+t} \exp\{r\log c\}dr$$

$$= \frac{B}{\log c}\exp\{r\log c\}\Big|_x^{x+t}$$

$$= \frac{B}{\log c}\left(c^{x+t} - c^x\right),$$

giving

$$S_x(t) = \exp\left\{\frac{-B}{\log c}c^x(c^t - 1)\right\}.$$

□

The force of mortality under Gompertz' law increases exponentially with age. At first sight this seems reasonable, but as we will see in the next chapter, the force of mortality for most populations is not an increasing function of age over the entire age range. Nevertheless, the Gompertz model does provide a fairly good fit to mortality data over some age ranges, particularly from middle age to early old age.

Example 2.4 Calculate the survival function and probability density function for T_x using Gompertz' law of mortality, with $B = 0.0003$ and $c = 1.07$, for $x = 20$, $x = 50$ and $x = 80$. Plot the results and comment on the features of the graphs.

Solution 2.4 For $x = 20$, the force of mortality is $\mu_{20+t} = Bc^{20+t}$ and the survival function is

$$S_{20}(t) = \exp\left\{\frac{-B}{\log c}c^{20}(c^t - 1)\right\}.$$

The probability density function is found from (2.10):

$$\mu_{20+t} = \frac{f_{20}(t)}{S_{20}(t)} \Rightarrow f_{20}(t) = \mu_{20+t}\,S_{20}(t) = Bc^{20+t}\exp\left\{\frac{-B}{\log c}c^{20}(c^t - 1)\right\}.$$

Figure 2.1 shows the survival functions for ages 20, 50 and 80, and Figure 2.2 shows the corresponding probability density functions. These figures illustrate some general points about lifetime distributions.

First, we see an effective limiting age, even though, in principle, there is no age to which the survival probability is exactly zero. Looking at Figure 2.1, we see that although $S_x(t) > 0$ for all combinations of x and t, survival beyond age 120 is very unlikely.

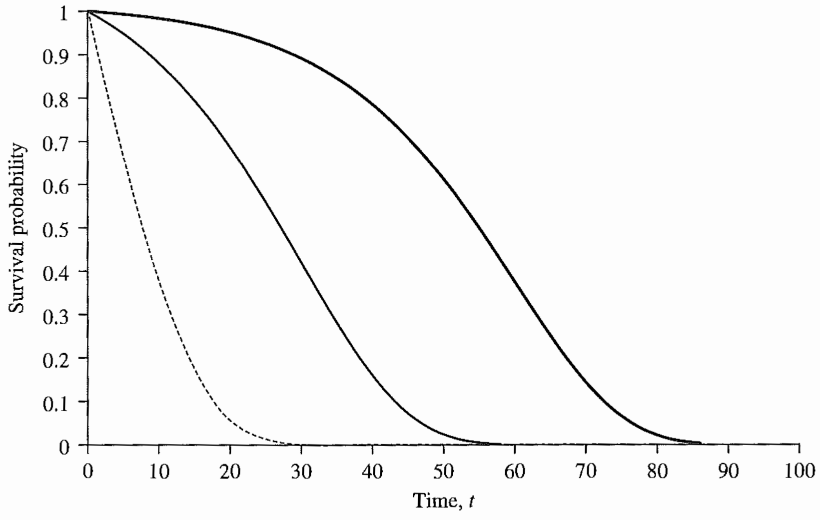

Figure 2.1 $S_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

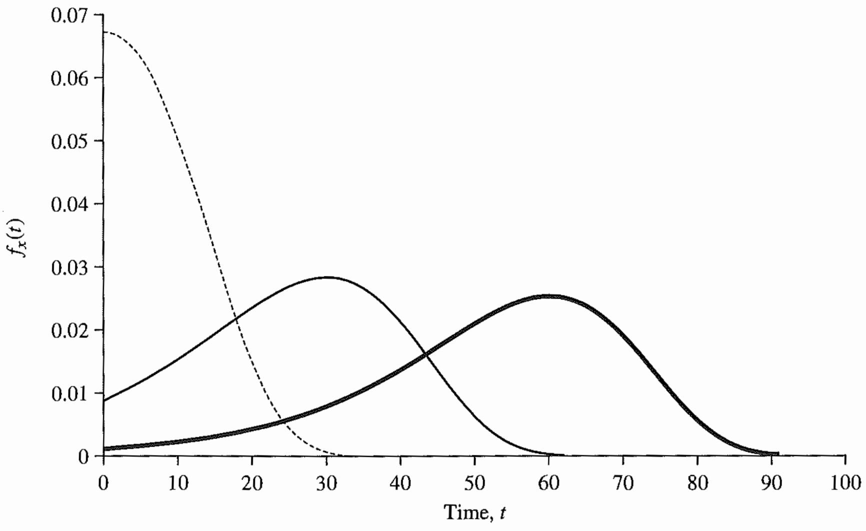

Figure 2.2 $f_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

Second, we note that the survival functions are ordered according to age, with the probability of survival for any given value of t being highest for age 20 and lowest for age 80. For survival functions that give a more realistic representation of human mortality, this ordering can be violated, but it usually holds at ages of interest to insurers. An example of the violation of this ordering is that $S_0(1)$ may be smaller than $S_x(1)$ for $x \geq 1$, as a result of perinatal mortality.

Looking at Figure 2.2, we see that the densities for ages 20 and 50 have similar shapes, but the density for age 80 has a quite different shape. For ages 20 and 50, the densities have their respective maximums at (approximately) $t = 60$ and $t = 30$, indicating that death is most likely to occur around age 80. The decreasing form of the density for age 80 also indicates that death is more likely to occur at age 80 than at any other age for a life now aged 80. A further point to note about these density functions is that although each density function is defined on $(0, \infty)$, the spread of values of $f_x(t)$ is much greater for $x = 20$ than for $x = 50$, which, as we will see in Table 2.1, results in a greater variance of future lifetime for $x = 20$ than for $x = 50$. □

2.4 Actuarial notation

The notation used in the previous sections, $S_x(t)$, $F_x(t)$ and $f_x(t)$, is standard in statistics. Actuarial science has developed its own notation, **International Actuarial Notation**, that encapsulates the probabilities and functions of greatest interest and usefulness to actuaries. The force of mortality notation, μ_x, comes from International Actuarial Notation. We summarize the relevant actuarial notation in this section, and rewrite the important results developed so far in this chapter in terms of actuarial functions. The actuarial notation for survival and mortality probabilities is

$$\boxed{{}_tp_x = \Pr[T_x > t] = S_x(t),} \tag{2.12}$$

$$\boxed{{}_tq_x = \Pr[T_x \leq t] = 1 - S_x(t) = F_x(t),} \tag{2.13}$$

$$\boxed{{}_{u|t}q_x = \Pr[u < T_x \leq u + t] = S_x(u) - S_x(u + t).} \tag{2.14}$$

That is

- ${}_t\boldsymbol{p}_x$ is the probability that (x) survives to at least age $x + t$,
- ${}_t\boldsymbol{q}_x$ is the probability that (x) dies before age $x + t$,
- ${}_{u|t}\boldsymbol{q}_x$ is the probability that (x) survives u years, and then dies in the subsequent t years, that is, between ages $x + u$ and $x + u + t$.

We may drop the subscript t if its value is 1, so that p_x represents the probability that (x) survives to at least age $x + 1$. Similarly, q_x is the probability that (x) dies before age $x + 1$. In actuarial terminology q_x is called the **mortality rate** at age x. We call ${}_{u|t}q_x$ a **deferred mortality probability**, because it is the probability that death occurs in the interval of t years, following a deferred period of u years.

The relationships below follow immediately from the definitions above and the previous results in this chapter:

$$
\begin{aligned}
{}_tp_x + {}_tq_x &= 1, \\
{}_{u|t}q_x &= {}_up_x - {}_{u+t}p_x, \\
{}_{t+u}p_x &= {}_tp_x\, {}_up_{x+t} \quad \text{from (2.5)}, && (2.15) \\
\mu_x &= -\frac{1}{{}_xp_0}\frac{d}{dx}{}_xp_0 \quad \text{from (2.9)}. && (2.16)
\end{aligned}
$$

Similarly,

$$\mu_{x+t} = -\frac{1}{{}_tp_x}\frac{d}{dt}{}_tp_x \Rightarrow \frac{d}{dt}\,{}_tp_x = -{}_tp_x\,\mu_{x+t}, \tag{2.17}$$

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \Rightarrow f_x(t) = {}_tp_x\,\mu_{x+t} \quad \text{from (2.10)}, \tag{2.18}$$

$${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}ds\right\} \quad \text{from (2.11)}. \tag{2.19}$$

As F_x is a distribution function and f_x is its density function, it follows that

$$F_x(t) = \int_0^t f_x(s)ds,$$

which can be written in actuarial notation as

$${}_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}ds. \tag{2.20}$$

This is an important formula, which can be interpreted as follows. Consider time s, where $0 \le s < t$. The probability that (x) is alive at time s is ${}_sp_x$, and the probability that (x) dies between ages $x+s$ and $x+s+ds$, having survived to age $x+s$, is (loosely) $\mu_{x+s}ds$, provided that ds is very small. Thus ${}_sp_x\,\mu_{x+s}ds$ can be interpreted as the probability that (x) dies between ages $x+s$ and $x+s+ds$. Now, we can sum over all the possible death intervals s to $s+ds$ – which requires integrating because these are infinitesimal intervals – to obtain the probability of death before age $x+t$.

We can illustrate this event sequence using the time-line diagram shown in Figure 2.3.

This type of interpretation is important as it can be applied to more complicated situations, and we will employ the time-line again in later chapters.

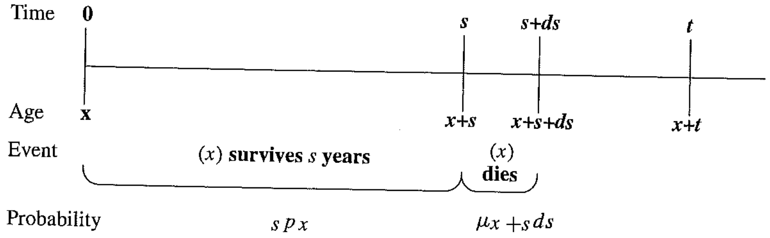

Figure 2.3 Time-line diagram for $_tq_x$.

In the special case when $t = 1$, formula (2.20) becomes

$$q_x = \int_0^1 {}_sp_x\ \mu_{x+s} ds.$$

When q_x is small, it follows that p_x is close to 1, and hence $_sp_x$ is close to 1 for $0 \leq s < 1$. Thus

$$q_x \approx \int_0^1 \mu_{x+s} ds \approx \mu_{x+1/2},$$

where the second relationship follows by the mid-point rule for numerical integration.

Example 2.5 As in Examples 2.1 and 2.2, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \leq x \leq 120$. Calculate both q_x and $\mu_{x+1/2}$ for $x = 20$ and for $x = 110$, and comment on these values.

Solution 2.5 We have

$$p_x = \frac{S_0(x+1)}{S_0(x)} = \left(1 - \frac{1}{120 - x}\right)^{1/6},$$

giving $q_{20} = 0.00167$ and $q_{110} = 0.01741$, and from the solution to Example 2.2, $\mu_{20\frac{1}{2}} = 0.00168$ and $\mu_{110\frac{1}{2}} = 0.01754$. We see that $\mu_{x+1/2}$ is a good approximation to q_x when the mortality rate is small, but is not such a good approximation, at least in absolute terms, when the mortality rate is not close to 0. □

2.5 Mean and standard deviation of T_x

Next, we consider the expected future lifetime of (x), $\mathrm{E}[T_x]$, denoted in actuarial notation by $\mathring{e}_x$. We call this the **complete expectation of life**. In order to evaluate $\overset{\circ}{e}_x$, we note from formulae (2.17) and (2.18) that

$$f_x(t) = {}_tp_x\,\mu_{x+t} = -\frac{d}{dt}\,{}_tp_x. \tag{2.21}$$

From the definition of an expected value, we have

$$\begin{aligned}\overset{\circ}{e}_x &= \int_0^\infty t\, f_x(t)dt \\ &= \int_0^\infty t\,{}_tp_x\,\mu_{x+t}dt.\end{aligned}$$

We can now use (2.21) to evaluate this integral using integration by parts as

$$\begin{aligned}\overset{\circ}{e}_x &= -\int_0^\infty t\left(\frac{d}{dt}\,{}_tp_x\right)dt \\ &= -\left(t\,{}_tp_x\Big|_0^\infty - \int_0^\infty {}_tp_x dt\right).\end{aligned}$$

In Section 2.2 we stated the assumption that $\lim_{t\to\infty} t\,{}_tp_x = 0$, which gives

$$\boxed{\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt.} \tag{2.22}$$

Similarly, for $\mathrm{E}[T_x^2]$, we have

$$\begin{aligned}\mathrm{E}[T_x^2] &= \int_0^\infty t^2\,{}_tp_x\,\mu_{x+t}dt \\ &= -\int_0^\infty t^2\left(\frac{d}{dt}\,{}_tp_x\right)dt \\ &= -\left(t^2\,{}_tp_x\Big|_0^\infty - \int_0^\infty {}_tp_x\, 2t\, dt\right) \\ &= 2\int_0^\infty t\,{}_tp_x\, dt. \qquad (2.23)\end{aligned}$$

So we have integral expressions for $\mathrm{E}[T_x]$ and $\mathrm{E}[T_x^2]$. For some lifetime distributions we are able to integrate directly. In other cases we have to use numerical integration techniques to evaluate the integrals in (2.22) and (2.23). The variance of T_x can then be calculated as

$$\mathrm{V}[T_x] = \mathrm{E}\left[T_x^2\right] - \left(\overset{\circ}{e}_x\right)^2.$$

Example 2.6 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \le x \le 120$. Calculate $\overset{\circ}{e}_x$ and $\mathrm{V}[T_x]$ for (a) $x = 30$ and (b) $x = 80$.

Solution 2.6 As $S_0(x) = (1 - x/120)^{1/6}$, we have

$$_tp_x = \frac{S_0(x+t)}{S_0(x)} = \left(1 - \frac{t}{120-x}\right)^{1/6}.$$

Now recall that this formula is valid for $0 \le t \le (120-x)$, because, under this model, survival beyond age 120 is impossible. Technically, we have

$$_tp_x = \begin{cases} \left(1 - \frac{t}{120-x}\right)^{1/6} & \text{for } x+t \le 120, \\ 0 & \text{for } x+t > 120. \end{cases}$$

So the upper limit of integration in equation (2.22) is $120 - x$, and

$$\mathring{e}_x = \int_0^{120-x} \left(1 - \frac{t}{120-x}\right)^{1/6} dt.$$

We make the substitution $y = 1 - t/(120-x)$, so that $t = (120-x)(1-y)$, giving

$$\mathring{e}_x = (120-x)\int_0^1 y^{1/6} dy$$
$$= \tfrac{6}{7}(120-x).$$

Then $\mathring{e}_{30} = 77.143$ and $\mathring{e}_{80} = 34.286$.

Under this model the expectation of life at any age x is 6/7 of the time to age 120.

For the variance we require $E[T_x^2]$. Using equation (2.23) we have

$$E\left[T_x^2\right] = 2\int_0^{120-x} t\ _tp_x dt$$
$$= 2\int_0^{120-x} t\left(1 - \frac{t}{120-x}\right)^{1/6} dt.$$

Again, we substitute $y = 1 - t/(120-x)$ giving

$$E\left[T_x^2\right] = 2(120-x)^2 \int_0^1 (y^{1/6} - y^{7/6})\, dy$$
$$= 2(120-x)^2 \left(\frac{6}{7} - \frac{6}{13}\right).$$

Then

$$V[T_x] = E[T_x^2] - \left(\mathring{e}_x\right)^2 = (120-x)^2\left(2(6/7 - 6/13) - (6/7)^2\right)$$
$$= (120-x)^2\,(0.056515) = ((120-x)\,(0.23773))^2.$$

So $V[T_{30}] = 21.396^2$ and $V[T_{80}] = 9.509^2$.

Table 2.1 *Values of* $\overset{\circ}{e}_x$, $SD[T_x]$ *and expected age at death for the Gompertz model with* $B = 0.0003$ *and* $c = 1.07$.

x	$\overset{\circ}{e}_x$	$SD[T_x]$	$x + \overset{\circ}{e}_x$
0	71.938	18.074	71.938
10	62.223	17.579	72.223
20	52.703	16.857	72.703
30	43.492	15.841	73.492
40	34.752	14.477	74.752
50	26.691	12.746	76.691
60	19.550	10.693	79.550
70	13.555	8.449	83.555
80	8.848	6.224	88.848
90	5.433	4.246	95.433
100	3.152	2.682	103.152

Since we know under this model that all lives will die before age 120, it makes sense that the uncertainty in the future lifetime should be greater for younger lives than for older lives. □

A feature of the model used in Example 2.6 is that we can obtain formulae for quantities of interest such as $\overset{\circ}{e}_x$, but for many models this is not possible. For example, when we model mortality using Gompertz' law, there is no explicit formula for $\overset{\circ}{e}_x$ and we must use numerical integration to calculate moments of T_x. In Appendix B we describe in detail how to do this.

Table 2.1 shows values of $\overset{\circ}{e}_x$ and the standard deviation of T_x (denoted $SD[T_x]$) for a range of values of x using Gompertz' law, $\mu_x = Bc^x$, where $B = 0.0003$ and $c = 1.07$. For this survival model, ${}_{130}p_0 = 1.9 \times 10^{-13}$, so that using 130 as the maximum attainable age in our numerical integration is accurate enough for practical purposes.

We see that $\overset{\circ}{e}_x$ is a decreasing function of x, as it was in Example 2.6. In that example $\overset{\circ}{e}_x$ was a linear function of x, but we see that this is not true in Table 2.1.

We are sometimes interested in the future lifetime random variable subject to a cap of n years, which is represented by the random variable $\min(T_x, n)$. For example, suppose that (x) is entitled to a benefit payable continuously for a maximum of n years, conditional on survival. Then $\min(T_x, n)$ would represent the payment period for the benefit. We derive the mean and variance of this random variable, using a similar approach to the derivation of the mean and variance of T_x. The expected value of $\min(T_x, n)$ is denoted $\mathring{e}_{x:\overline{n}|}$, and is called the **term expectation of life.**

$$\begin{aligned}
\mathrm{E}[\min(T_x,\ n)] = \mathring{e}_{x:\overline{n}|} &= \int_0^n t\ {}_tp_x\ \mu_{x+t}\,dt + \int_n^\infty n\ {}_tp_x\ \mu_{x+t}\,dt \\
&= \int_0^n t\left(-\frac{d}{dt}\,{}_tp_x\right)dt + n\ {}_np_x \\
&= -\left(t\ {}_tp_x\Big|_0^n - \int_0^n {}_tp_x\,dt\right) + n\ {}_np_x \\
\Rightarrow \mathring{e}_{x:\overline{n}|} &= \int_0^n {}_tp_x\,dt\,.
\end{aligned}$$

The $\overline{n}|$ notation is used to denote a period of n years (just as in annuity-certain notation), and is used extensively in later chapters.

2.6 Curtate future lifetime

2.6.1 K_x and e_x

In many insurance applications we are interested not only in the future lifetime of an individual, but also in what is known as the individual's curtate future lifetime. The **curtate future lifetime** random variable is defined as the integer part of future lifetime, and is denoted by K_x for a life aged x. If we let $\lfloor\ \rfloor$ denote the floor function, we have

$$K_x = \lfloor T_x \rfloor.$$

We can think of the curtate future lifetime as the number of whole years lived in the future by an individual. As an illustration of the importance of curtate future lifetime, consider the situation where a life aged x at time 0 is entitled to payments of 1 at times $1, 2, 3, \ldots$ provided that (x) is alive at these times. Then the number of payments made equals the number of complete years lived after time 0 by (x). This is the curtate future lifetime.

We can find the probability function of K_x by noting that for $k = 0, 1, 2, \ldots$, $K_x = k$ if and only if (x) dies between the ages of $x + k$ and $x + k + 1$. Thus for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
\Pr[K_x = k] &= \Pr[k \le T_x < k + 1] \\
&= {}_{k|}q_x \\
&= {}_kp_x - {}_{k+1}p_x \\
&= {}_kp_x - {}_kp_x\ p_{x+k} \\
&= {}_kp_x\ q_{x+k}.
\end{aligned}$$

The expected value of K_x is denoted by e_x, so that $e_x = \mathrm{E}[K_x]$, and is referred to as the **curtate expectation of life** (even though it represents the expected curtate lifetime). So

$$\begin{aligned}
\mathrm{E}[K_x] &= e_x \\
&= \sum_{k=0}^{\infty} k \Pr[K_x = k] \\
&= \sum_{k=0}^{\infty} k \left({}_k p_x - {}_{k+1} p_x \right) \\
&= ({}_1 p_x - {}_2 p_x) + 2({}_2 p_x - {}_3 p_x) + 3({}_3 p_x - {}_4 p_x) + \cdots \\
&= \sum_{k=1}^{\infty} {}_k p_x. \qquad (2.24)
\end{aligned}$$

Note that the lower limit of summation is $k = 1$.

Similarly,

$$\begin{aligned}
\mathrm{E}[K_x^2] &= \sum_{k=0}^{\infty} k^2 \left({}_k p_x - {}_{k+1} p_x \right) \\
&= ({}_1 p_x - {}_2 p_x) + 4({}_2 p_x - {}_3 p_x) + 9({}_3 p_x - {}_4 p_x) + 16({}_4 p_x - {}_5 p_x) + \cdots \\
&= 2 \sum_{k=1}^{\infty} k \, {}_k p_x - \sum_{k=1}^{\infty} {}_k p_x \\
&= 2 \sum_{k=1}^{\infty} k \, {}_k p_x - e_x.
\end{aligned}$$

As with the complete expectation of life, there are a few lifetime distributions that allow $\mathrm{E}[K_x]$ and $\mathrm{E}[K_x^2]$ to be calculated analytically. For more realistic models, such as Gompertz', we can calculate the values easily using Excel or other suitable software. Although in principle we have to evaluate an infinite sum, at some age the survival probability will be sufficiently small that we can treat it as an effective limiting age.

Analogous to the random variable $\min(T_x, n)$ we have the random variable $\min(K_x, n)$. For example, if a life aged x is entitled to payments of 1 at times $1, 2, 3, \ldots, n$, where n is an integer, then $\min(K_x, n)$ represents the number of payments made. An important difference between these two random variables is that $\min(T_x, n)$ is a mixed random variable (with a density over $(0, n)$ and a mass of probability at n), whereas $\min(K_x, n)$ is a discrete random variable since K_x is a discrete random variable. The expected value of $\min(K_x, n)$ is denoted $e_{x:\overline{n}|}$, and when n is an integer is given by

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x \, .$$

The proof of this result is set as Exercise 2.16.

2.6.2 *The complete and curtate expected future lifetimes, $\mathring{e}_x$ and e_x*

As the curtate future lifetime is the integer part of future lifetime, it is natural to ask if there is a simple relationship between $\mathring{e}_x$ and e_x. We can obtain an approximate relationship by writing

$$\mathring{e}_x = \int_0^\infty {}_t p_x \, dt = \sum_{j=0}^{\infty} \int_j^{j+1} {}_t p_x \, dt.$$

If we approximate each integral using the trapezium rule for numerical integration (see Appendix B), we obtain

$$\int_j^{j+1} {}_t p_x \, dt \approx \tfrac{1}{2} \left({}_j p_x + {}_{j+1} p_x \right),$$

and hence

$$\mathring{e}_x \approx \sum_{j=0}^{\infty} \tfrac{1}{2} \left({}_j p_x + {}_{j+1} p_x \right) = \tfrac{1}{2} + \sum_{j=1}^{\infty} {}_j p_x .$$

Thus, we have an approximation that is frequently applied in practice, namely

$$\mathring{e}_x \approx e_x + \tfrac{1}{2}. \qquad (2.25)$$

In Chapter 5 we will meet a refined version of this approximation. Table 2.2 shows values of $\mathring{e}_x$ and e_x for a range of values of x when the survival model is Gompertz' law, with $B = 0.0003$ and $c = 1.07$. Values of e_x were calculated by applying formula (2.24) with an upper limit of summation of $130 - x$, and values of $\mathring{e}_x$ are as in Table 2.1. This table illustrates that formula (2.25) is a very good approximation in this particular case for younger ages, but is less accurate at very old ages. This observation is true for most realistic survival models.

2.7 Notes and further reading

Although laws of mortality such as Gompertz' law are appealing due to their simplicity, they rarely represent mortality over the whole span of human ages.

Table 2.2 *Values of* e_x *and* $\mathring{e}_x$ *for Gompertz' law with* $B = 0.0003$ *and* $c = 1.07$.

x	e_x	$\mathring{e}_x$
0	71.438	71.938
10	61.723	62.223
20	52.203	52.703
30	42.992	43.492
40	34.252	34.752
50	26.192	26.691
60	19.052	19.550
70	13.058	13.555
80	8.354	8.848
90	4.944	5.433
100	2.673	3.152

A simple extension of Gompertz' law is **Makeham's law** (Makeham, 1860), which models the force of mortality as

$$\mu_x = A + Bc^x. \tag{2.26}$$

This is very similar to Gompertz' law, but adds a fixed term that is not age related, that allows better for accidental deaths. The extra term tends to improve the fit of the model to mortality data at younger ages. See Exercise 2.11.

In recent times, the Gompertz–Makeham approach has been generalized further to give the GM(r, s) (Gompertz–Makeham) formula,

$$\mu_x = h_r^1(x) + \exp\{h_s^2(x)\},$$

where h_r^1 and h_s^2 are polynomials in x of degree r and s, respectively. A discussion of this formula can be found in Forfar *et al.* (1988). Both Gompertz' law and Makeham's law are special cases of the GM formula.

In Section 2.3, we noted the importance of the force of mortality. A further significant point is that when mortality data are analysed, the force of mortality is a natural quantity to estimate, whereas the lifetime distribution is not. The analysis of mortality data is a huge topic and is beyond the scope of this book. An excellent summary article on this topic is Macdonald (1996). For more general distributions, the quantity $f_0(x)/S_0(x)$, which actuaries call the force of mortality at age x, is known as the **hazard rate** in survival analysis and the **failure rate** in reliability theory.

2.8 Exercises

Exercise 2.1 Let $F_0(t) = 1 - (1 - t/105)^{1/5}$ for $0 \le t \le 105$. Calculate

(a) the probability that a newborn life dies before age 60,
(b) the probability that a life aged 30 survives to at least age 70,
(c) the probability that a life aged 20 dies between ages 90 and 100,
(d) the force of mortality at age 50,
(e) the median future lifetime at age 50,
(f) the complete expectation of life at age 50,
(g) the curtate expectation of life at age 50.

Exercise 2.2 The function

$$G(x) = \frac{18\,000 - 110x - x^2}{18\,000}$$

has been proposed as the survival function $S_0(x)$ for a mortality model.

(a) What is the implied limiting age ω?
(b) Verify that the function G satisfies the criteria for a survival function.
(c) Calculate ${}_{20}p_0$.
(d) Determine the survival function for a life aged 20.
(e) Calculate the probability that a life aged 20 will die between ages 30 and 40.
(f) Calculate the force of mortality at age 50.

Exercise 2.3 Calculate the probability that a life aged 0 will die between ages 19 and 36, given the survival function

$$S_0(x) = \frac{1}{10}\sqrt{100 - x}, \quad 0 \le x \le 100 \ (= \omega).$$

Exercise 2.4 Let

$$S_0(x) = \exp\left\{-\left(Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}\right)\right\}$$

where A, B, C and D are all positive.

(a) Show that the function S_0 is a survival function.
(b) Derive a formula for $S_x(t)$.
(c) Derive a formula for μ_x.
(d) Now suppose that

$$A = 0.00005, \quad B = 0.0000005, \quad C = 0.0003, \quad D = 1.07.$$

(i) Calculate ${}_tp_{30}$ for $t = 1, 5, 10, 20, 50, 90$.
(ii) Calculate ${}_tq_{40}$ for $t = 1, 10, 20$.

(iii) Calculate ${}_{t|10}q_{30}$ for $t = 1, 10, 20$.
(iv) Calculate e_x for $x = 70, 71, 72, 73, 74, 75$.
(v) Calculate $\overset{\circ}{e}_x$ for $x = 70, 71, 72, 73, 74, 75$, using numerical integration.

Exercise 2.5 Let $F_0(t) = 1 - e^{-\lambda t}$, where $\lambda > 0$.

(a) Show that $S_x(t) = e^{-\lambda t}$.
(b) Show that $\mu_x = \lambda$.
(c) Show that $e_x = (e^{\lambda} - 1)^{-1}$.
(d) What conclusions do you draw about using this lifetime distribution to model human mortality?

Exercise 2.6 Given $p_x = 0.99$, $p_{x+1} = 0.985$, ${}_3p_{x+1} = 0.95$ and $q_{x+3} = 0.02$, calculate

(a) p_{x+3},
(b) ${}_2p_x$,
(c) ${}_2p_{x+1}$,
(d) ${}_3p_x$,
(e) ${}_{1|2}q_x$.

Exercise 2.7 Given

$$F_0(x) = 1 - \frac{1}{1+x} \quad \text{for } x \geq 0,$$

find expressions for (a), (b), (c) below, simplifying as far as possible,

(a) $S_0(x)$,
(b) $f_0(x)$,
(c) $S_x(t)$,

and calculate:

(d) p_{20}, and
(e) ${}_{10|5}q_{30}$.

Exercise 2.8 Given

$$S_0(x) = e^{-0.001x^2} \quad \text{for } x \geq 0,$$

find expressions for (a) and (b), simplifying as far as possible,

(a) $f_0(x)$, and
(b) μ_x.

Exercise 2.9 Show that

$$\frac{d}{dx}\,{}_tp_x = {}_tp_x\,(\mu_x - \mu_{x+t}).$$

Exercise 2.10 Suppose that Gompertz' law applies with $\mu_{30} = 0.000130$ and $\mu_{50} = 0.000344$. Calculate ${}_{10}p_{40}$.

Exercise 2.11 A survival model follows Makeham's law, so that

$$\mu_x = A + Bc^x \qquad \text{for } x \geq 0.$$

(a) Show that under Makeham's law

$$ {}_tp_x = s^t g^{c^x(c^t-1)}, \tag{2.27}$$

where $s = e^{-A}$ and $g = \exp\{-B/\log c\}$.

(b) Suppose you are given the values of ${}_{10}p_{50}$, ${}_{10}p_{60}$ and ${}_{10}p_{70}$. Show that

$$c = \left(\frac{\log({}_{10}p_{70}) - \log({}_{10}p_{60})}{\log({}_{10}p_{60}) - \log({}_{10}p_{50})} \right)^{0.1}.$$

Exercise 2.12 (a) Construct a table of p_x for Makeham's law with parameters $A = 0.0001$, $B = 0.00035$ and $c = 1.075$, for integer x from age 0 to age 130, using Excel or other appropriate computer software. You should set the parameters so that they can be easily changed, and you should keep the table, as many exercises and examples in future chapters will use Makeham's law.

(b) Use the table to determine the age last birthday at which a life currently aged 70 is most likely to die.

(c) Use the table to calculate e_{70}.

(d) Using a numerical approach, calculate $\mathring{e}_{70}$.

Exercise 2.13 A life insurer assumes that the force of mortality of smokers at all ages is twice the force of mortality of non-smokers.

(a) Show that, if * represents smokers' mortality, and the 'unstarred' function represents non-smokers' mortality, then

$${}_tp_x^* = ({}_tp_x)^2 .$$

(b) Calculate the difference between the life expectancy of smokers and non-smokers aged 50, assuming that non-smokers mortality follows Gompertz' law, with $B = 0.0005$ and $c = 1.07$.

(c) Calculate the variance of the future lifetime for a non-smoker aged 50 and for a smoker aged 50 under Gompertz' law.

Hint: You will need to use numerical integration for parts (b) and (c).

Exercise 2.14 (a) Show that

$$\mathring{e}_x \leq \mathring{e}_{x+1} + 1.$$

(b) Show that

$$\mathring{e}_x \geq e_x.$$

(c) Explain (in words) why

$$\mathring{e}_x \approx e_x + \frac{1}{2}.$$

(d) Is $\mathring{e}_x$ always a non-increasing function of x?

Exercise 2.15 (a) Show that

$$\mathring{e}_x = \frac{1}{S_0(x)} \int_x^{\infty} S_0(t)dt,$$

where $S_0(t) = 1 - F_0(t)$, and hence, or otherwise, prove that

$$\frac{d}{dx}\mathring{e}_x = \mu_x \mathring{e}_x - 1.$$

Hint: $\frac{d}{dx}\left\{\int_a^x g(t)dt\right\} = g(x)$. What about $\frac{d}{dx}\left\{\int_x^a g(t)dt\right\}$?

(b) Deduce that

$$x + \mathring{e}_x$$

is an increasing function of x, and explain this result intuitively.

Exercise 2.16 Show that for integer n,

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x\,.$$

Answers to selected exercises

2.1 (a) 0.1559
(b) 0.8586
(c) 0.1394
(d) 0.0036
(e) 53.28
(f) 45.83
(g) 45.18

2.2 (a) 90
(c) 0.8556
(d) $1 - 3x/308 - x^2/15\,400$
(e) 0.1169
(f) 0.021

2.3 0.1

2.4 (d) (i) 0.9976, 0.9862, 0.9672, 0.9064, 0.3812, 3.5×10^{-7}
(ii) 0.0047, 0.0629, 0.1747
(iii) 0.0349, 0.0608, 0.1082
(iv) 13.046, 12.517, 12.001, 11.499, 11.009, 10.533
(v) 13.544, 13.014, 12.498, 11.995, 11.505, 11.029

2.6 (a) 0.98
(b) 0.97515
(c) 0.96939
(d) 0.95969
(e) 0.03031

2.7 (d) 0.95455
(e) 0.08218

2.10 0.9973

2.12 (b) 73
(c) 9.339
(d) 9.834

2.13 (b) 6.432
(c) 125.89 (non-smokers), 80.11 (smokers)