Chapter 2
Mathematical Expectation:

2.1 Mean of a Random Variable:

**Definition 1:**
Let $X$ be a random variable with a probability distribution $f(x)$. The mean (or expected value) of $X$ is denoted by $\mu_X$ (or $E(X)$) and is defined by:

$$
E(X) = \mu_X = \begin{cases} 
\sum_{x} x f(x) ; & \text{if } X \text{ is discrete} \\
\int_{-\infty}^{\infty} x f(x) \, dx ; & \text{if } X \text{ is continuous}
\end{cases}
$$

**Example 1:** (Example 4 in ch1)
A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the expected number of defective computers purchased.
Solution:

Let $X$ = the number of defective computers purchased. In this example, we found that the probability distribution of $X$ is:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = p(X=x)$</td>
<td>$\frac{10}{28}$</td>
<td>$\frac{15}{28}$</td>
<td>$\frac{3}{28}$</td>
</tr>
</tbody>
</table>

or:

$$f(x) = P(X = x) = \begin{cases} \frac{\binom{3}{x} \times 5}{\binom{8}{2}} & ; \ x = 0, 1, 2 \\ 0 ; \ otherwise \end{cases}$$
The expected value of the number of defective computers purchased is the mean (or the expected value) of X, which is:

\[ E(X) = \mu_X = \sum_{x=0}^{2} x f(x) \]

\[ = (0) f(0) + (1) f(1) + (2) f(2) \]

\[ = (0) \frac{10}{28} + (1) \frac{15}{28} + (2) \frac{3}{28} \]

\[ = \frac{15}{28} + \frac{6}{28} = \frac{21}{28} = 0.75 \text{ (computers)} \]

**Example 2:**
Let X be a continuous random variable that represents the life (in hours) of a certain electronic device. The pdf of X is given by:

\[ f(x) = \begin{cases} 
20,000 \\ x^3 & ; x > 100 \\
0 & ; \text{elsewhere}
\end{cases} \]

Find the expected life of this type of devices.
Solution:

\[ E(X) = \mu_X = \int_{-\infty}^{\infty} x f(x) \, dx \]

\[ = \int_{100}^{\infty} x \frac{20000}{x^3} \, dx \]

\[ = 20000 \int_{100}^{\infty} \frac{1}{x^2} \, dx \]

\[ = 20000 \left[ -\frac{1}{x} \right]_{x=100}^{x=\infty} \]

\[ = -20000 \left[ 0 - \frac{1}{100} \right] = 200 \text{ (hours)} \]

Therefore, we expect that this type of electronic devices to last, on average, 200 hours.
**Theorem 2.1:**
Let \( X \) be a random variable with a probability distribution \( f(x) \), and let \( g(X) \) be a function of the random variable \( X \). The mean (or expected value) of the random variable \( g(X) \) is denoted by \( \mu_{g(X)} \) (or \( E[g(X)] \)) and is defined by:

\[
E[g(X)] = \mu_{g(X)} = \begin{cases} 
\sum_{all \ x} g(x) f(x); & \text{if } X \text{ is discrete} \\
\int_{-\infty}^{\infty} g(x) f(x) \, dx; & \text{if } X \text{ is continuous}
\end{cases}
\]

**Example 3:**
Let \( X \) be a discrete random variable with the following probability distribution

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( \frac{10}{28} )</td>
<td>( \frac{15}{28} )</td>
<td>( \frac{3}{28} )</td>
</tr>
</tbody>
</table>

Find \( E[g(X)] \), where \( g(X) = (X - 1)^2 \).
Solution:

\[ g(X) = (X - 1)^2 \]

\[ E[g(X)] = \mu_{g(x)} = \sum_{x=0}^{2} g(x) f(x) = \sum_{x=0}^{2} (x - 1)^2 f(x) \]

\[ = (0-1)^2 f(0) + (1-1)^2 f(1) + (2-1)^2 f(2) \]

\[ = (-1)^2 \frac{10}{28} + (0)^2 \frac{15}{28} + (1)^2 \frac{3}{28} \]

\[ = \frac{10}{28} + 0 + \frac{3}{28} = \frac{10}{28} \]
Example 4:
In Example 2, find \( \mathbf{E}\left( \frac{1}{X} \right) \).

Solution:

\[
f(x) = \begin{cases} 
\frac{20,000}{x^3} & ; \ x > 100 \\
0 & ; \text{elsewhere}
\end{cases}
\]

\[
g(X) = \frac{1}{X}
\]

\[
\mathbf{E}\left( \frac{1}{X} \right) = \mathbf{E}[g(X)] = \mu_{g(X)} = \int_{-\infty}^{\infty} g(x) f(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{x} f(x) \, dx
\]

\[
= \int_{100}^{\infty} \frac{20000}{x^3} \, dx = 20000 \int_{100}^{\infty} \frac{1}{x^4} \, dx = 20000 \left[ \frac{1}{x^3} \right]_{x = \infty}^{x = 100}
\]

\[
= \frac{-20000}{3} \left[ 0 - \frac{1}{1000000} \right] = 0.0067
\]
2.2 Variance (of a Random Variable):

The most important measure of variability of a random variable $X$ is called the variance of $X$ and is denoted by $\text{Var}(X)$ or $\sigma^2_X$.

**Definition 2:**
Let $X$ be a random variable with a probability distribution $f(x)$ and mean $\mu$. The variance of $X$ is defined by:

\[
V(x) = \sigma^2 = E((x - \mu)^2) = \sum_{x} (x - \mu)^2 f(x) = E(X^2) - (E(X))^2 \quad \text{if } x \text{ is discrete} \tag{2}
\]

\[
V(x) = \sigma^2 = E((x - \mu)^2) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = E(X^2) - (E(X))^2 \quad \text{if } x \text{ is continuous} \tag{3}
\]

\[
E(x^2) = \begin{cases} 
\sum x^2 f(x) & \text{if } x \text{ is discrete} \\
\int_{-\infty}^{\infty} x^2 f(x) dx & \text{if } x \text{ is continuous}
\end{cases}
\]

**Definition 3:**
The positive square root of the variance of $X$, $\sigma_X = \sqrt{\sigma^2_X}$, is called the **standard deviation** of $X$.

**Note:**
$\text{Var}(X) = E[g(X)]$, where $g(X) = (X - \mu)^2$
**Theorem 2.2:**
The variance of the random variable $X$ is given by:

$$\text{Var}(X) = \sigma_X^2 = \text{E}(X^2) - \mu^2$$

where

$$\text{E}(X^2) = \begin{cases} 
\sum x^2 f(x) ; & \text{if } X \text{ is discrete} \\
\int_{-\infty}^{\infty} x^2 f(x) \, dx ; & \text{if } X \text{ is continuous}
\end{cases}$$

**Example 5:**
Let $X$ be a discrete random variable with the following probability distribution:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.15</td>
<td>0.38</td>
<td>0.10</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Find $\text{Var}(X) = \sigma^2_X$. 
Solution:
\[ \mu = \sum_{x=0}^{3} x f(x) = (0) f(0) + (1) f(1) + (2) f(2) + (3) f(3) \]
\[ = (0)(0.51) + (1)(0.38) + (2)(0.10) + (3)(0.01) = 0.61 \]

1. First method:
\[ \text{Var}(X) = \sigma_X^2 = \sum_{x=0}^{3} (x - \mu)^2 f(x) \]
\[ = \sum_{x=0}^{3} (x - 0.61)^2 f(x) \]
\[ = (0-0.61)^2 f(0) + (1-0.61)^2 f(1) + (2-0.61)^2 f(2) + (3-0.61)^2 f(3) \]
\[ = (0-0.61)^2 (0.51) + (0.39)^2 (0.38) + (1.39)^2 (0.10) + (2.39)^2 (0.01) \]
\[ = 0.4979 \]

2. Second method:
\[ \text{Var}(X) = \sigma_X^2 = \text{E}(X^2) - \mu^2 \]
\[ \text{E}(X^2) = \sum_{x=0}^{3} x^2 f(x) = (0^2) f(0) + (1^2) f(1) + (2^2) f(2) + (3^2) f(3) \]
\[ = (0)(0.51) + (1)(0.38) + (4)(0.10) + (9)(0.01) = 0.87 \]
\[ \text{Var}(X) = \sigma_X^2 = \text{E}(X^2) - \mu^2 = 0.87 - (0.61)^2 = 0.4979 \]
Example 6:
Let $X$ be a continuous random variable with the following pdf:

$$f(x) = \begin{cases} 
2(x-1) & ; \ 1 < x < 2 \\
0 & ; \text{elsewhere}
\end{cases}$$

Find the mean and the variance of $X$.

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx = \int_{1}^{2} x \cdot [2(x-1)] \, dx = 2 \int_{1}^{2} x (x-1) \, dx = \frac{5}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx = \int_{1}^{2} x^2 \cdot [2(x-1)] \, dx = 2 \int_{1}^{2} x^2 (x-1) \, dx = \frac{17}{6}$$

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{8}$$
2.3 Means and Variances of Linear Combinations of Random Variables:

If $X_1, X_2, \ldots, X_n$ are $n$ random variables and $a_1, a_2, \ldots, a_n$ are constants, then the random variable:

$$Y = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

is called a linear combination of the random variables $X_1, X_2, \ldots, X_n$.

**Theorem 2.3:**
If $X$ is a random variable with mean $\mu = \text{E}(X)$, and if $a$ and $b$ are constants, then:

$$\text{E}(aX \pm b) = a \text{E}(X) \pm b$$

$$\iff$$

$$\mu_{aX \pm b} = a \mu_X \pm b$$

**Corollary 1:** $\text{E}(b) = b$ \hspace{1cm} (a=0 in Theorem 4.5)

**Corollary 2:** $\text{E}(aX) = a \text{E}(X)$ \hspace{1cm} (b=0 in Theorem 4.5)
Example 7:
Let X be a random variable with the following probability density function:

\[
f(x) = \begin{cases} 
  \frac{1}{3} \frac{x^2}{3} & ; -1 < x < 2 \\
  0 & ; \text{elsewhere}
\end{cases}
\]

Find \( E(4X+3) \).

Solution:
\[
\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-1}^{2} x \left[ \frac{1}{3} x^2 \right] \, dx = \frac{1}{3} \int_{-1}^{2} x^3 \, dx = \frac{1}{3} \left[ \frac{1}{4} x^4 \right]_{x=-1}^{x=2} = \frac{5}{4}
\]

\[
E(4X+3) = 4 E(X) + 3 = 4(\frac{5}{4}) + 3 = 8
\]

Another solution:
\[
E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx \quad ; \quad g(X) = 4X+3
\]

\[
E(4X+3) = \int_{-\infty}^{\infty} (4x + 3) f(x) \, dx = \int_{-1}^{2} (4x + 3) \left[ \frac{1}{3} x^2 \right] \, dx = \cdots = 8
\]
**Theorem 2.4:**
If $X_1, X_2, \ldots, X_n$ are $n$ random variables and $a_1, a_2, \ldots, a_n$ are constants, then:
\[
E(a_1X_1+a_2X_2+ \ldots +a_nX_n) = a_1E(X_1)+ a_2E(X_2)+ \ldots+a_nE(X_n)
\]
\[\iff\]
\[
E(\sum_{i=1}^{n} a_iX_i) = \sum_{i=1}^{n} a_iE(X_i)
\]

If $X$ and $Y$ are **independent** then for any functions $h$ and $g$
\[
E[h(X) \cdot g(Y)] = E(h(X)) \cdot E(g(Y))
\]

**Corollary:** If $X$, and $Y$ are random variables, then:
\[
E(X \pm Y) = E(X) \pm E(Y)
\]

**Theorem 2.5:**
If $X$ is a random variable with variance $Var(X) = \sigma_X^2$ and if $a$ and $b$ are constants, then:
\[
Var(aX \pm b) = a^2 \text{ Var}(X)
\]
\[\iff\]
\[
\sigma_{aX+b}^2 = a^2 \sigma_X^2
\]
**Theorem 2.6:**
If $X_1, X_2, \ldots, X_n$ are $n$ independent random variables and $a_1, a_2, \ldots, a_n$ are constants, then:

\[
\text{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2) + \ldots + a_n^2 \text{Var}(X_n)
\]

\[
\iff
\text{Var}\left(\sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i)
\]

\[
\iff
\sigma^2_{a_1X_1 + a_2X_2 + \ldots + a_nX_n} = a_1^2 \sigma^2_{X_1} + a_2^2 \sigma^2_{X_2} + \ldots + a_n^2 \sigma^2_{X_n}
\]

**Corollary:**
If $X$, and $Y$ are independent random variables, then:
- $\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
- $\text{Var}(aX - bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
- $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y)$
Example 8:

Let $X$, and $Y$ be two independent random variables such that $E(X)=2$, $\text{Var}(X)=4$, $E(Y)=7$, and $\text{Var}(Y)=1$. Find:
1. $E(3X+7)$ and $\text{Var}(3X+7)$
2. $E(5X+2Y−2)$ and $\text{Var}(5X+2Y−2)$.

Solution:
1. $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$
   $\text{Var}(3X+7)= (3)^2 \text{Var}(X)=(3)^2 (4) = 36$
2. $E(5X+2Y−2)= 5E(X) + 2E(Y) −2= (5)(2) + (2)(7) − 2= 22$
   $\text{Var}(5X+2Y−2)= \text{Var}(5X+2Y)= 5^2 \text{Var}(X) + 2^2 \text{Var}(Y)$
   $\quad = (25)(4)+(4)(1) = 104$
Example 9:

The probability distribution for company A is given by:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
</tr>
</tbody>
</table>

and for company B is given by:

<table>
<thead>
<tr>
<th>Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(y)</td>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Show that the variance of the probability distribution for company B is greater than that of company A.
Solution:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>∑</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>xf(x)</td>
<td>0.3</td>
<td>0.8</td>
<td>0.9</td>
<td>2</td>
</tr>
<tr>
<td>f(x)x^2</td>
<td>0.3</td>
<td>1.6</td>
<td>2.7</td>
<td>4.6</td>
</tr>
</tbody>
</table>

\[ \sigma^2 = E(x^2) - (E(x))^2 = 4.6 - 4 = 0.6, \sigma = .77 \]
$\sigma^2 = E(y^2) - (E(y))^2 = 5.6 - 4 = 1.6, \sigma = 1.26$

Note that $\sigma_y^2$ is greater than $\sigma_x^2$.

**Problem 4** Let $X$ have a mixed distribution $F(X)$ written uniquely as

$$F(X) = cF_1(X) + (1 - c)F_2(X)$$

where $F_1$ is the distribution function of a discrete random variable $X_1$ and $F_2$ is the distribution function of a continuous random variable $X_2$. Then $E(X^2)$ is

(a) $cE(X_1) + (1 - c)E(X_2)$

(b) $E(X_1) + E(X_2)$

(c) $cE(X_1^2) + (1 - c)E(X_2^2)$

(d) $E(X_1^2) + E(X_2^2)$
Solved Problems

4.15 The density function of the continuous random variable \( X \), the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, is given in Exercise 3.7 on page 88 as
Find the average number of hours per year that families run their vacuum cleaners.

**Solution**

\[
E(X) = \int_0^1 x^2 \, dx + \int_1^2 x(2 - x) \, dx = 1. \quad \text{Therefore, the average number of hours per year is} \quad (1)(100) = 100 \, \text{hours.}
\]

4.34 Let \( X \) be a random variable with the following probability distribution:
Find the standard deviation of \( X \).

**Solution**

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.3</td>
<td>0.2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

\[
\mu = (-2)(0.3) + (3)(0.2) + (5)(0.5) = 2.5 \quad \text{and} \quad E(X^2) = (-2)^2(0.3) + (3)^2(0.2) + (5)^2(0.5) = 15.5.
\]

So, \( \sigma^2 = E(X^2) - \mu^2 = 9.25 \) and \( \sigma = 3.041 \).
4.36 Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that 0, 1, 2, or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable $X$ representing the number of power failures striking this subdivision.

**Solution**

4.36 $\mu = (0)(0.4) + (1)(0.3) + (2)(0.2) + (3)(0.1) = 1.0,$
and $E(X^2) = (0)^2(0.4) + (1)^2(0.3) + (2)^2(0.2) + (3)^2(0.1) = 2.0.$
So, $\sigma^2 = 2.0 - 1.0^2 = 1.0.$

4.43 The length of time, in minutes, for an airplane to obtain clearance for take off at a certain airport is a random variable $Y = 3X - 2$, where $X$ has the density function

$$f(x) = \begin{cases} 
\frac{1}{4}e^{-x/4}, & x > 0 \\
0, & \text{elsewhere}
\end{cases}$$

Find the mean and variance of the random variable $Y$.

**Solution**

4.43 $\mu_Y = E(3X - 2) = \frac{1}{4} \int_0^{\infty} (3x - 2)e^{-x/4} \, dx = 10.$ So
$\sigma^2_Y = E\{(3X - 2) - 10\}^2\} = \frac{9}{4} \int_0^{\infty} (x - 4)^2e^{-x/4} \, dx = 144.$
4.50 On a laboratory assignment, if the equipment is working, the density function of the observed outcome, $X$, is

$$
\begin{align*}
2(1-x), & \quad 0 < x < 1 \\
0, & \quad \text{otherwise}
\end{align*}
$$

Find the variance and standard deviation of $X$.

**Solution**

$$
\begin{align*}
E(X) &= 2 \int_0^1 x(1-x) \, dx = 2 \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^1 = \frac{1}{3} \\
E(X^2) &= 2 \int_0^1 x^2(1-x) \, dx = 2 \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1 = \frac{1}{6}.
\end{align*}
$$

Hence,

$$
\begin{align*}
Var(X) &= \frac{1}{6} - \left( \frac{1}{3} \right)^2 = \frac{1}{18}, \quad \text{and} \quad \sigma = \sqrt{1/18} = 0.2357.
\end{align*}
$$
2.4 Chebyshev's Theorem:
* Suppose that $X$ is any random variable with mean $E(X)=\mu$ and variance $\text{Var}(X)=\sigma^2$ and standard deviation $\sigma$.
* Chebyshev's Theorem gives a conservative estimate of the probability that the random variable $X$ assumes a value within $k$ standard deviations ($k\sigma$) of its mean $\mu$, which is $P(\mu-k\sigma < X < \mu+k\sigma)$.
* $P(\mu-k\sigma < X < \mu+k\sigma) \approx 1 - \frac{1}{k^2}$

area = $P(\mu-k\sigma < X < \mu+k\sigma) \geq 1 - \frac{1}{k^2}$
**Theorem 2.7:** *(Chebyshev's Theorem)*

Let $X$ be a random variable with mean $E(X)=\mu$ and variance $\text{Var}(X)=\sigma^2$, then for $k>1$, we have:

$$P(\mu-k\sigma<X<\mu+k\sigma) \geq 1-\frac{1}{k^2} \iff P(|X-\mu|<k\sigma) \geq 1-\frac{1}{k^2}$$

---

**Example 10**

Let $X$ be a random variable having an unknown distribution with mean $\mu=8$ and variance $\sigma^2=9$ (standard deviation $\sigma=3$). Find the following probability:

(a) $P(-4<X<20)$

(b) $P(|X-8| \geq 6)$
Solution:
(a) $P(-4 < X < 20) = ??$

$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$

$(-4 < X < 20) = (\mu - k\sigma < X < \mu + k\sigma)$

\[-4 = \mu - k\sigma \iff -4 = 8 - k(3) \quad \text{or} \quad 20 = \mu + k\sigma \iff 20 = 8 + k(3)\]

\[-4 = 8 - 3k \iff 3k = 12 \quad \quad \iff 20 = 8 + 3k \iff 3k = 12 \quad \iff k = 4\]

\[1 - \frac{1}{k^2} = 1 - \frac{1}{16} = \frac{15}{16}\]

Therefore, $P(-4 < X < 20) \geq \frac{15}{16}$, and hence, $P(-4 < X < 20) \approx \frac{15}{16}$ (approximately)
(b) \( P(|X - 8| \geq 6) = \) ??

\[
P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6)
\]

\[P(|X - 8| < 6) = ??\]

\[
P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}
\]

\(|X - 8| < 6) = (|X - \mu| < k\sigma)

\[6 = k\sigma \iff 6 = 3k \iff k = 2\]

\[
1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4}
\]

\[
P(|X - 8| < 6) \geq \frac{3}{4} \iff 1 - P(|X - 8| < 6) \leq 1 - \frac{3}{4}
\]

\[
\iff 1 - P(|X - 8| < 6) \leq \frac{1}{4}
\]

\[
\iff P(|X - 8| \geq 6) \leq \frac{1}{4}
\]

Therefore, \( P(|X - 8| \geq 6) \approx \frac{1}{4} \) (approximately)
Another solution for part (b):

\[ P(|X-8| < 6) = P(-6 < X - 8 < 6) \]
\[ = P(-6 + 8 < X < 6 + 8) \]
\[ = P(2 < X < 14) \]

\[ (2 < X < 14) = (\mu - k\sigma < X < \mu + k\sigma) \]

\[ 2 = \mu - k\sigma \iff 2 = 8 - k(3) \iff 2 = 8 - 3k \iff 3k = 6 \iff k = 2 \]

\[ 1 - \frac{1}{k^2} = 1 - \frac{3}{4} = \frac{1}{4} \]

\[ P(2 < X < 14) \geq \frac{3}{4} \iff P(|X-8| < 6) \geq \frac{3}{4} \]
\[ \iff 1 - P(|X-8| < 6) \leq 1 - \frac{3}{4} \]
\[ \iff 1 - P(|X-8| < 6) \leq \frac{1}{4} \]
\[ \iff P(|X-8| \geq 6) \leq \frac{1}{4} \]

Therefore, \( P(|X-8| \geq 6) \approx \frac{1}{4} \) (approximately)