MOMENTS AND MOMENT GENERATING FUNCTION (mgf)
Central Moments: the $r$th moment of a R.V. $X$ about its mean $\mu$ (called $r$th central moment) is defined as

$$\mu_r = E[(X - \mu_X)^r]$$

- **Central moments** are the expected value of the difference of a random variable and its expected value (or mean) to a power. It is also called moment about the mean.
- **Second central moment** or variance $E\left((X - E(X))^2\right)$
- **Third central moment** $E\left((X - E(X))^3\right)$ and so on...

Clearly, $\mu_0=1$, $\mu_1=0$, $\mu_2=\sigma^2$, the variance value of random variable

Non-Central Moments: the $r$th moment of a R.V. $X$ about 0 called $r$th moment or called $r$th non-central moment is defined as

$$\mu_r = E(X^r)$$

Clearly, $\mu'_0=1$, $\mu'_1 = \mu'_1$ : the mean or expected value of random variable

$$\sigma^2 = \mu'_2 - (\mu'_1)^2$$ : the variance value of random variable
**Moment Generating function MGF:**

**Definition**

In probability theory and statistics, the **moment-generating function** of a random variable $X$ is

$$M(t) = M_X(t) = E(e^{tX}).$$

Where The series expansion of $e^{tX}$ is

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^nX^n}{n!} + \cdots.$$

Hence,

$$M_X(t) = E(e^{tX}) = 1 + tm_1 + \frac{t^2m_2}{2!} + \frac{t^3m_3}{3!} + \cdots + \frac{t^nm_n}{n!} + \cdots,$$

where $m_n$ is the $n$th **moment** $= \mu_n = E(X^n)$

**Notes about mgf’s**

- Moment generating function uniquely determine a distribution.
- If $X$ and $Y$ are independent r.v.’s then $M_{X+Y}(t) = M_X(t) M_Y(t)$

and if $X$ and $Y$ are i.i.d. r.v.’s then $M_{X+Y}(t) = [M(t)]^2$ where $M(t)$ is the common mgf
Example

The following example shows how the mgf of an **exponential random variable** is calculated:

**Example** Let $X$ be a continuous random variable with support $R_X = [0, \infty)$

and probability density function

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \in R_X \\ 0 & \text{if } x \notin R_X \end{cases}$$

where $\lambda$ is a strictly positive number. The expected value $E[\exp(tX)]$ can be computed as follows:

$$E[\exp(tX)] = \int_{-\infty}^{\infty} \exp(tx)f_X(x)\,dx$$
$$= \int_{0}^{\infty} \exp(tx)\lambda \exp(-\lambda x)\,dx$$
$$= \lambda \int_{0}^{\infty} \exp((t-\lambda)x)\,dx \quad \text{(which is finite only if } t < \lambda)$$
$$= \lambda \left[ \frac{1}{t-\lambda} \exp((t-\lambda)x) \right]_{0}^{\infty}$$
$$= \lambda \left[ 0 - \frac{1}{t-\lambda} \right]$$
$$= \frac{\lambda}{\lambda - t}$$
Deriving moments with the mgf

If a moment-generating function of a random variable $X$ does exist, it can be used to generate all the moments of that variable.

**Proposition** If a random variable $X$ possesses a mgf $M_X(t)$, then the $n$-th moment of $X$, denoted by $\mu_X(n)$, exists and is finite for any $n \in \mathbb{N}$. Furthermore:

$$\mu_X(n) = \mathbb{E}[X^n] = \left. \frac{d^n M_X(t)}{dt^n} \right|_{t=0}$$

where $\left. \frac{d^n M_X(t)}{dt^n} \right|_{t=0}$ is the $n$-th derivative of $M_X(t)$ with respect to $t$, evaluated at the point $t = 0$.

**Proof:**

The intuition, however, is straightforward: since the expected value is a linear operator and differentiation is a linear operation, under appropriate conditions one can differentiate through the expected value, as follows:

$$\frac{d^n M_X(t)}{dt^n} = \left. \frac{d^n \mathbb{E}[\exp(tX)]}{dt^n} \right|_{t=0} = \mathbb{E}\left[ \left. \frac{d^n}{dt^n} \exp(tX) \right|_{t=0} \right] = \mathbb{E}[X^n \exp(tX)]$$

which, evaluated at the point $t = 0$, yields:

$$\left. \frac{d^n M_X(t)}{dt^n} \right|_{t=0} = \mathbb{E}[X^n \exp(0 \cdot X)] = \mathbb{E}[X^n] = \mu_X(n)$$
Example  Continuing the example above, the mgf of an exponential random variable is:

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

The expected value of $X$ can be computed by taking the first derivative of the mgf:

$$\frac{dM_X(t)}{dt} = \frac{\lambda}{(\lambda - t)^2}$$

and evaluating it at $t = 0$:

$$E[X] = \frac{dM_X(t)}{dt} \bigg|_{t=0} = \frac{\lambda}{(\lambda - 0)^2} = \frac{1}{\lambda}$$

The second moment of $X$ can be computed by taking the second derivative of the mgf:

$$\frac{d^2M_X(t)}{dt^2} = \frac{2\lambda}{(\lambda - t)^3}$$

and evaluating it at $t = 0$:

$$E[X^2] = \frac{d^2M_X(t)}{dt^2} \bigg|_{t=0} = \frac{2\lambda}{(\lambda - 0)^3} = \frac{2}{\lambda^2}$$

And so on for the higher moments.
Exercise

3.63  Find the mgf of the random variable $X$:

$$x = \begin{cases} 
\frac{1}{2} & \text{with probability } \frac{1}{2} \\
-\frac{1}{2} & \text{with probability } \frac{1}{2}
\end{cases}$$

Find the first four raw moments.

3.64: Find the mgf of R.V. ‘X’ given by

$$f(x) = \begin{cases} 
\frac{x}{2} & 0 \leq x \leq 2 \\
0 & \text{otherwise}
\end{cases}$$

Find the first 4 moments. Also find the mgf of $y=3-2X$
If the random variable $X$ has a mgf, $M_X(t)$, then the linear transform $y = a + bX$ has the mgf

\[
M_Y(t) = e^{at}M_X(bt)
\]

Proof:

\[
M_Y(t) = E(e^{iY}) = E(e^{(a+bX)}) = E(e^{at} \times e^{btX}) \\
= E(e^{at})E(e^{btX}) \quad \text{but } e^{at} \text{ is constant} \\
= e^{at}M_X(bt)
\]

while if $Y = bX$, then

\[
g_Y(t) = E(e^{tY}) = E(e^{tbX}) = g_X(bt).
\]

In particular, if

\[
X^* = \frac{X - \mu}{\sigma},
\]

\[
g_{x^*}(t) = e^{-\mu t/\sigma}g_X\left(\frac{t}{\sigma}\right)
\]
Exercise 1.1

Let $X$ be a discrete random variable having a Bernoulli distribution. Its support $R_X$ is:

$$R_X = \{0, 1\}$$

and its probability mass function $p_X(x)$ is:

$$p_X(x) = \begin{cases} 
  p & \text{if } x = 1 \\
  1 - p & \text{if } x = 0 \\
  0 & \text{if } x \notin R_X 
\end{cases}$$

where $p \in (0, 1)$ is a constant. Derive the moment generating function of $X$, if it exists.

Solution

Using the definition of moment generating function:

$$M_X(t) = E[\exp(tX)]$$

$$= \sum_{x \in R_X} \exp(tx)p_X(x)$$

$$= \exp(t \cdot 1) \cdot p_X(1) + \exp(t \cdot 0) \cdot p_X(0)$$

$$= \exp(t) \cdot p + 1 \cdot (1 - p)$$

$$= 1 - p + p \exp(t)$$
Exercise 2.1:
Derive the variance of $X$, where $X$ is a random variable with moment generating function

$$M_X(t) = \frac{1}{2}(1 + \exp(t))$$

Solution:

We can use the following formula for computing the variance:

$$\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2$$

The expected value of $X$ is computed by taking the first derivative of the moment generating function:

$$\frac{dM_X(t)}{dt} = \frac{1}{2} \exp(t)$$

and evaluating it at $t = 0$:

$$\text{E}[X] = \frac{dM_X(t)}{dt} \bigg|_{t=0} = \frac{1}{2} \exp(0) = \frac{1}{2}$$

The second moment of $X$ is computed by taking the second derivative of the moment generating function:

$$\frac{d^2M_X(t)}{dt^2} = \frac{1}{2} \exp(t)$$

and evaluating it at $t = 0$:

$$\text{E}[X^2] = \frac{d^2M_X(t)}{dt^2} \bigg|_{t=0} = \frac{1}{2} \exp(0) = \frac{1}{2}$$

Therefore:

$$\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{4}$$
Let \(X_1, \ldots, X_n\) be mutually independent random variables. Let be their sum

\[ Z = \sum_{i=1}^{n} X_i \]

Then, the mgf of \(Z\) is the product of the mgfs of \(X_1, \ldots, X_n\):

\[ M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t) \]

**Proof:**

\[
M_Z(t) = E[\exp(tZ)] \\
= E\left[\exp\left(t \sum_{i=1}^{n} X_i\right)\right] \\
= E\left[\exp\left(\sum_{i=1}^{n} tX_i\right)\right] \\
= E\left[\prod_{i=1}^{n} \exp(tX_i)\right] \\
= \prod_{i=1}^{n} E[\exp(tX_i)] \quad \text{(by mutual independence)} \\
= \prod_{i=1}^{n} M_{X_i}(t) \quad \text{(by the definition of mgf)}
\]
Exercise 1.3

A random variable $X$ is said to have a **Chi-square distribution** with $n$ degrees of freedom if its moment generating function is defined for any $t < \frac{1}{2}$ and it is equal to:

$$M_X(t) = (1 - 2t)^{-n/2}$$

Define

$$Y = X_1 + X_2$$

where $X_1$ and $X_2$ are two independent random variables having Chi-square distributions with $n_1$ and $n_2$ degrees of freedom respectively. Prove that $Y$ has a Chi-square distribution with $n_1 + n_2$ degrees of freedom.

The moment generating functions of $X_1$ and $X_2$ are:

$$M_{X_1}(t) = (1 - 2t)^{-n_1/2}$$
$$M_{X_2}(t) = (1 - 2t)^{-n_2/2}$$

The moment generating function of a sum of independent random variables is just the product of their moment generating functions:

$$M_Y(t) = (1 - 2t)^{-n_1/2}(1 - 2t)^{-n_2/2}$$
$$= (1 - 2t)^{-(n_1 + n_2)/2}$$

Therefore, $M_Y(t)$ is the moment generating function of a Chi-square random variable with $n_1 + n_2$ degrees of freedom. As a consequence, $Y$ has a Chi-square distribution with $n_1 + n_2$ degrees of freedom.
Example  If $X$ and $Y$ are independent discrete random variables with the non-negative integers $\{0, 1, 2, 3, \ldots\}$ as range, and with geometric distribution function

$$p_X(j) = p_Y(j) = q^j p ,$$

then

$$g_X(t) = g_Y(t) = \frac{p}{1 - q e^t} ,$$

and if $Z = X + Y$, then

$$g_Z(t) = g_X(t)g_Y(t) = \frac{p^2}{1 - 2q e^t + q^2 e^{2t}} .$$

Problem 3 The moment-generating functions are unique; that is, two random variables that have the same moment-generating function have the same probability distributions as well. This statement is

(a) True
(b) False

Problem 5 Let $X$ has a gamma distribution with $M_X(t) = (1 - \beta t)^{-\alpha}$. Then $E(X)$ is

(a) $\alpha$

(b) $\alpha \beta^2$

(c) $\alpha^2 \beta$

(d) $\alpha \beta$