MOMENTS AND MOMENT GENERATING FUNCTION(mgf)

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Central Moments: the rth moment of a R.V. x about its mean μ (called rth central moment) is defined as

$$\mu_r = E\left[(X-\mu_X)^r\right]$$

- **Central moments** are the expected value of the difference of a random variable and its expected value (or mean) to a power. It is also called **moment about the mean.**
- **Second central moment** or variance $E\left((X-E(X))^2\right)$
- **Third central moment** $E\left((X-E(X))^3\right)$ and so on...

Clearly,
$\mu_0=1$,
$\mu_1=0$,
$\mu_2=\sigma^2$, the variance value of random variable

Non- Central Moments: the rth moment of a R.V. x about 0 called **r^{th} moment** or called r^{th} non-central moment is defined as

$$\mu_r` = E(X^r)$$

Clearly, **$\mu`_0=1$,**
$\mu_x = \mu`_1$:the mean or expected value of random variable
$\sigma_x^2 = \mu`_2 - (\mu`_1)^2$: the variance value of random variable

Moment Generating function MGF:

Definition

In probability theory and statistics, the **moment-generating function** of a random variable *X* is

$$M(t) = M_X(t) = \mathrm{E}(e^{tX}).$$

Where The series expansion of e^{tX} is

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^nX^n}{n!} + \cdots.$$

Hence,

$$M_X(t) = E(e^{tX}) = 1 + tm_1 + \frac{t^2m_2}{2!} + \frac{t^3m_3}{3!} + \cdots + \frac{t^nm_n}{n!} + \cdots,$$

where m_n is the *n*th moment = $\boldsymbol{\mu_n` = E(X^r)}$

Notes a bout mgf's

- Moment generating function uniquely determine a distribution.
- If X and Y are independent r.v.'s then $\mathbf{M_{X+Y}(t) = M_X(t)\, M_Y(t)}$

and if X and Y are i.i.d. r.v.'s then $\mathbf{M_{X+Y}(t) = [M(t)]^2}$ where M (t) is the common mgf

Example

The following example shows how the mgf of an exponential random variable is calculated:

Example Let X be a continuous random variable with support

$$R_X = [0, \infty)$$

and probability density function

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \in R_X \\ 0 & \text{if } x \notin R_X \end{cases}$$

where λ is a strictly positive number. The expected value $\mathrm{E}[\exp(tX)]$ can be computed as follows:

$$\begin{aligned} \mathrm{E}[\exp(tX)] &= \int_{-\infty}^{\infty} \exp(tx) f_X(x) dx \\ &= \int_0^{\infty} \exp(tx) \lambda \exp(-\lambda x) dx \\ &= \lambda \int_0^{\infty} \exp((t-\lambda)x) dx \quad \textit{(which is finite only if } t < \lambda \textit{)} \\ &= \lambda \left[\frac{1}{t-\lambda} \exp((t-\lambda)x) \right]_0^{\infty} \\ &= \lambda \left[0 - \frac{1}{t-\lambda} \right] \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

Deriving moments with the mgf

If a moment-generating function of a random variable X *does exist*, it can be used to generate all the moments of that variable

Proposition If a random variable X possesses a mgf $M_X(t)$, then the n-th moment of X, denoted by $\mu_X(n)$, exists and is finite for any $n \in \mathbb{N}$. Furthermore:

$$\mu_X(n) = \mathrm{E}[X^n] = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$$

where $\left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$ is the n-th derivative of $M_X(t)$ with respect to t, evaluated at the point $t = 0$.

Proof:

The intuition, however, is straightforward: since the expected value is a linear operator and differentiation is a linear operation, under appropriate conditions one can differentiate through the expected value, as follows:

$$\frac{d^n M_X(t)}{dt^n} = \frac{d^n}{dt^n}\mathrm{E}[\exp(tX)] = \mathrm{E}\left[\frac{d^n}{dt^n}\exp(tX)\right] = \mathrm{E}[X^n \exp(tX)]$$

which, evaluated at the point $t = 0$, yields:

$$\left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0} = \mathrm{E}[X^n \exp(0 \cdot X)] = \mathrm{E}[X^n] = \mu_X(n)$$

Example Continuing the example above, the mgf of an exponential random variable is:

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

The expected value of X can be computed by taking the first derivative of the mgf:

$$\frac{dM_X(t)}{dt} = \frac{\lambda}{(\lambda - t)^2}$$

and evaluating it at $t = 0$:

$$E[X] = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \frac{\lambda}{(\lambda - 0)^2} = \frac{1}{\lambda}$$

The second moment of X can be computed by taking the second derivative of the mgf:

$$\frac{d^2M_X(t)}{dt^2} = \frac{2\lambda}{(\lambda - t)^3}$$

and evaluating it at $t = 0$:

$$E[X^2] = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \frac{2\lambda}{(\lambda - 0)^3} = \frac{2}{\lambda^2}$$

And so on for the higher moments.

Exercise

3.63 Find the mgf of the random variable X: $X = \begin{cases} \frac{1}{2} & \text{with probabilty } \frac{1}{2} \\ \frac{-1}{2} & \text{with probabilty } \frac{1}{2} \end{cases}$

Find the first four raw moments.

3.64:Find the mgf of R.V. 'X' given by

$$f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the first 4 moments. Also find the mgf of y=3-2X

Moment generating function of a linear transformation

If the random variable X has a mgf , $M_X(t)$, then the linear transform y =a+bX has the mgf

$$let\ y = a + bX$$

$$M_Y(t) = e^{at} M_X(bt)$$

Proof

$$M_Y(t) = E(e^{tY}) = E(e^{t(a+bX)}) = E(e^{at} \times e^{btX})$$

$$= E(e^{at}) E(e^{(bt)X}) \quad \text{but } e^{at} \text{ is constant}$$

$$= e^{at} M_X(bt)$$

while if Y = bX, then

$$\begin{aligned} g_Y(t) &= E(e^{tY}) \\ &= E(e^{tbX}) \\ &= g_X(bt) . \end{aligned}$$

In particular, if

$$X^* = \frac{X - \mu}{\sigma} ,$$

$$g_{x^*}(t) = e^{-\mu t/\sigma} g_X\left(\frac{t}{\sigma}\right)$$

Exercise 1.1

Let X be a discrete random variable having a Bernoulli distribution. Its support R_X is:

$$R_X = \{0, 1\}$$

and its probability mass function $p_X(x)$ is:

$$p_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{if } x \notin R_X \end{cases}$$

where $p \in (0, 1)$ is a constant. Derive the moment generating function of X, if it exists.

Solution

Using the definition of moment generating function:

$$\begin{aligned} M_X(t) &= \mathrm{E}[\exp(tX)] \\ &= \sum_{x \in R_X} \exp(tx) p_X(x) \\ &= \exp(t \cdot 1) \cdot p_X(1) + \exp(t \cdot 0) \cdot p_X(0) \\ &= \exp(t) \cdot p + 1 \cdot (1 - p) \\ &= 1 - p + p \exp(t) \end{aligned}$$

Exercise 2.1:

Derive the variance of X, where X is a random variable with moment generating function

$$M_X(t) = \frac{1}{2}(1 + \exp(t))$$

Solution:

We can use the following formula for computing the variance:

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

The expected value of X is computed by taking the first derivative of the moment generating function:

$$\frac{dM_X(t)}{dt} = \frac{1}{2}\exp(t)$$

and evaluating it at $t = 0$:

$$\mathrm{E}[X] = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \frac{1}{2}\exp(0) = \frac{1}{2}$$

The second moment of X is computed by taking the second derivative of the moment generating function:

$$\frac{d^2M_X(t)}{dt^2} = \frac{1}{2}\exp(t)$$

and evaluating it at $t = 0$:

$$\mathrm{E}[X^2] = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \frac{1}{2}\exp(0) = \frac{1}{2}$$

Therefore:

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 \\ &= \frac{1}{2} - \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{4}\end{aligned}$$

Moment generating function of a sum of mutually independent random variables

Let , X1, …Xn   be mutualy independent random variables . Let be their sum

$$Z = \sum_{i=1}^{n} X_i$$

Then, the mgf of Z is the product of the mgfs of X_1 , ..., X_n :

$$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

Proof:

$$\begin{aligned}
M_Z(t) &= \mathrm{E}[\exp(tZ)] \\
&= \mathrm{E}\left[\exp\left(t\sum_{i=1}^{n} X_i\right)\right] \\
&= \mathrm{E}\left[\exp\left(\sum_{i=1}^{n} tX_i\right)\right] \\
&= \mathrm{E}\left[\prod_{i=1}^{n} \exp(tX_i)\right] \\
&= \prod_{i=1}^{n} \mathrm{E}[\exp(tX_i)] && \text{(by mutual independence)} \\
&= \prod_{i=1}^{n} M_{X_i}(t) && \text{(by the definition of mgf)}
\end{aligned}$$

Exercise 1.3

A random variable X is said to have a Chi-square distribution with n degrees of freedom if its moment generating function is defined for any $t < \frac{1}{2}$ and it is equal to:

$$M_X(t) = (1-2t)^{-n/2}$$

Define

$$Y = X_1 + X_2$$

where X_1 and X_2 are two independent random variables having Chi-square distributions with n_1 and n_2 degrees of freedom respectively. Prove that Y has a Chi-square distribution with $n_1 + n_2$ degrees of freedom.

Solution

The moment generating functions of X_1 and X_2 are:

$$M_{X_1}(t) = (1-2t)^{-n_1/2}$$
$$M_{X_2}(t) = (1-2t)^{-n_2/2}$$

The moment generating function of a sum of independent random variables is just the product of their moment generating functions:

$$M_Y(t) = (1-2t)^{-n_1/2}(1-2t)^{-n_2/2}$$
$$= (1-2t)^{-(n_1+n_2)/2}$$

Therefore, $M_Y(t)$ is the moment generating function of a Chi-square random variable with $n_1 + n_2$ degrees of freedom. As a consequence, Y has a Chi-square distribution with $n_1 + n_2$ degrees of freedom.

Example If X and Y are independent discrete random variables with the non-negative integers $\{0, 1, 2, 3, \ldots\}$ as range, and with geometric distribution function

$$p_X(j) = p_Y(j) = q^j p \ ,$$

then

$$g_X(t) = g_Y(t) = \frac{p}{1 - qe^t} \ ,$$

and if $Z = X + Y$, then $g_Z(t) = g_X(t)g_Y(t) = \dfrac{p^2}{1 - 2qe^t + q^2e^{2t}}$.

Problem 3 The moment-generating functions are unique; that is, two random variables that have the same moment-generating function have the same probability distributions as well. This statement is

(a) True

(b) False

Problem 5 Let X has a gamma distribution with $M_X(t) = (1 - \beta t)^{-\alpha}$. Then $E(X)$ is

(a) α

(b) $\alpha\beta^2$

(c) $\alpha^2\beta$

(d) $\alpha\beta$