Applications of First Order Differential Equation

Dr Mansoor Alshehri

King Saud University

Applications of First Order Differential Equation

- Orthogonal Trajectories
  - How to Find Orthogonal Trajectories
- Growth and Decay
- Newton's Law of cooling

Suppose that we have a family of curves given by

$$F(x, y, c) = 0, \tag{1}$$

and another family of curves given by

$$G(x, y, k) = 0, \tag{2}$$

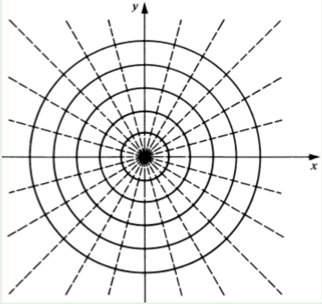
such that at any intersection of a curve of the family $F(x, y, c)$ with a curve of the family $G(x, y, k) = 0$, the tangents of the curves are perpendicular.

Therefore, are two families of curves that always intersect perpendicularly.

Orthogonal trajectories,

Example

The family of circles represented by $x^2 + y^2 = c$, with center at the origin, and the family $y = kx$ of straight lines through the origin, are **orthogonal trajectories** of each other, as shown in the figure.



How to Find Orthogonal Trajectories

To find the orthogonal trajectories of the family

$$F(x, y, c) = 0, \tag{3}$$

Step1: Differentiate (3) implicitly with respect to x to get a relation of the form (3)

$$g\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, c\right); \tag{4}$$

Step2: Eliminate the parameter c from (3), and (4) to obtain the differential equation

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{5}$$

corresponding to the first family (3);
Step3: Replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (5) to obtain the differential equation

$$H\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{6}$$

of the orthogonal trajectories (as shown in the figure below);

Step4: General solution of (6) gives the required orthogonal trajectories.

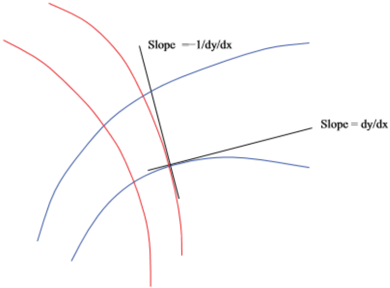


Figure: Orthogonal trajectories

Example (1)

Find the orthogonal trajectories of family of straight lines through the origin.

Solution: The family of straight lines through the origin is given by

$$y = kx, \tag{7}$$

To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (7) implicitly with respect to x, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = k, \tag{8}$$

Step2: Eliminate the parameter k from (7), and (8), we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}, \tag{9}$$

This gives the differential equation of the family (7).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (9) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}, \tag{10}$$

Step4: Solving differential equation (10), we obtain

$$x^2 + y^2 = c. \tag{11}$$

Thus, the orthogonal trajectories of family of straight lines through the origin is given by (11). Note that (11) is the family of circles with centre at the origin.

Example (2)

Find the orthogonal trajectories of the family

$$cx^2 - y^2 = 1, \tag{12}$$

Solution: To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (12) implicitly with respect to x, we obtain

$$2cx - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{13}$$

Step2: Eliminate the parameter c. From (12) we have

$$c = \frac{1+y^2}{x^2}$$

Thus, we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y^2}{xy}, \tag{14}$$

This gives the differential equation of the family (12).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (14) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-xy}{1+y^2}. \tag{15}$$

Step4: Solving differential equation (15) by method of separation of variables, we obtain

$$\int \frac{1+y^2}{y}\ dy = -\int x\ dx$$

$$\int \left(\frac{1}{y} + y\right)\ dy = -\int x\ dx$$

$$\ln y + (y^2/2) = (-x^2/2) + c_1$$

$$2\ln y + y^2 + x^2 = c_1. \tag{16}$$

Thus, the required equation of orthogonal trajectories is given by (16).

Example (3)

Find the orthogonal trajectories of the family

$$y^2 = cx^3, \tag{17}$$

Solution: To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (17) implicitly with respect to x, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3cx^2}{2y}. \tag{18}$$

Step2: Eliminate the parameter c. From (17) we have

$$c = \frac{y^2}{x^3}$$

Thus, we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y}{2x} = f(x, y), \tag{19}$$

This gives the differential equation of the family (17).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (14) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{3y}. \tag{20}$$

Step4: Solving differential equation (20) by method of separation of variables, we obtain

$$\int 3y\ dy = -\int 2x\ dx$$

$$\frac{3}{2}y^2 + x^2 = c_1. \tag{21}$$

Thus, the required equation of orthogonal trajectories is given by (21).

Example (4)

Find the orthogonal trajectories of the family

$$x^3 + 3xy^2 = c \tag{22}$$

Solution: To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (22) implicitly with respect to x, we obtain

$$3x^2 + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{23}$$

Step2: Equation (23) has no parameter, so, we will go to the next step.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2 + y^2}{2xy} = f(x, y), \tag{24}$$

This gives the differential equation of the family (22).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (24) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 + y^2}. \tag{25}$$

This gives the differential equation of the orthogonal trajectories. Now we have to solve (25) to get the required equation of orthogonal trajectories.

Step4: Solving differential equation (25). We see that the equation (25) is a homogeneous differential equation

$$(x^2 + y^2)dy - 2xydx = 0,$$

by substituting $x = vy \Rightarrow 1 = v\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}v}{\mathrm{d}x} \Rightarrow dx = vdy + ydv$.
After completing the solution of this homogeneous differential equation, we obtained the equation of family of orthogonal trajectories

$$(y^2 - x^2) = cy.$$

Exercises

1. Find the orthogonal trajectories of the circles

$$x^2 + (y - c)^2 = c^2, \quad c \neq 0$$

2. Find the orthogonal trajectories of the family of curves

$$y = x + ce^{-x}$$

3. Find the orthogonal trajectories of the family of curves

$$2x^2 + y^2 = 6cx$$

4. Find the orthogonal trajectories of the family of curves

$$2y + x + ce^{2y} = 0$$

Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For example, if $y = y(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $y'(t)$ is proportional to the population $y(t)$; that is, $y'(t) = ky(t)$ for some constant k.

The mathematical model given by the equation $y'(t) = ky(t)$ can be predicted what actually happens fairly accurately under ideal conditions (unlimited environment, adequate nutrition, immunity to disease).
Also, we can see many examples in nuclear physics, chemistry and finance.

In general, if y(t) is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size y(t) at any time, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \tag{26}$$

where k is a constant, and Equation (26) is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). Thus, the law of Exponential Growth and Decay can be written as

$$y = ce^{kt},$$

c is the initial value and can be found from the initial condition $y(t_0) = y_0$
k is the constant of proportionality, which is can be found from an additional condition which might be given in the problem.

Note

If $k > 0$ the exponential growth occurs, and if $k < 0$ the exponential decay occurs.

To proof that let us take some initial time quantity is known and is $y(t_0) = y_0$. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

is separable differential equation and we can solve it.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

$$\frac{dy}{y} = kdt$$

$$\int \frac{dy}{y} = \int kdt$$

$$\ln y = kt + c$$

$$e^{\ln y} = e^{kt+c}$$

$$y = e^c e^{kt}$$

$$y = c_1 e^{kt};\ c_1 = \pm e^c$$

Using the initial condition $y(0) = y_0$, i.e $t_0 = 0$, $y = y_0$

$$y_0 = c_1 e^0 \;\Rightarrow\; y_0 = c_1$$

$$y = y_0 e^{kt}.$$

To find the additional constant k we need additional condition which might be given in the problem.

Example (1)

A certain culture of bacteria grows at rate proportional to its size. If the size doubles in 4 days, find the time required for the culture to increase to 10 times to its original size.

Solution Let $p(t)$ be the size of the culture after t days.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp$$

we will use the initial condition

$$p(0) = p_0$$

to find the arbitrary constant c, and we will find the additional constant k by using the additional condition

$$p(4) = 2p_0.$$

We have

$$p = ce^{kt}$$

from the initial condition $p(0) = p_0$ i.e. $t = 0$, $p = p_0$ we will have arbitrary constant c, thus

$$p_0 = ce^0 \;\Rightarrow\; c = p_0,$$

hence, we have

$$p = p_0 e^{kt}.$$

Now by using the the additional condition $p(4) = 2p_0$ i.e. $t = 4$, we we can find the additional constant k

$$2p_0 = p_0 e^{4k}$$

$$e^{4k} = 2$$

$$\ln\left(e^{4k}\right) = \ln 2$$

$$4k = \ln 2 \;\Rightarrow\; k = \frac{\ln 2}{4} \simeq 0.173$$

Thus, the time is required for the culture to increase 10 times to its original size can be found from

$$10p_0 = p_0 e^{0.173t}$$

$$e^{0.173t} = 10$$

$$\ln\left(e^{0.173t}\right) = \ln 10$$

$$t = \frac{\ln 10}{0.173}$$

$$t \simeq 13.31 \text{ days.}$$

Example (2)

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate k? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution We measure the population $p(t)$ in millions of people. We have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp \Rightarrow p = ce^{kt}$$

and we have the initial condition

$$p(t_0) = p_0 \Rightarrow p(0) = 2560,$$

thus, we can find the arbitrary constant c

$$p = ce^{kt} \Rightarrow p(0) = ce^0 \Rightarrow 2560 = c.$$

Now we will find the additional constant k (the relative growth rate) by using the additional condition

$$p(10) = 3040.$$

$$p = ce^{kt}$$

$$3040 = 2560e^{10k}$$

$$e^{10k} = \frac{3040}{2560}$$

$$\ln\left(e^{10k}\right) = \ln 1.1875$$

$$10k = \ln 1.1875 \ \Rightarrow \ k = \frac{\ln 1.1875}{10} \simeq 0.01785.$$

The relative growth rate is about 1.7% per year and the model is

$$p(t) = 2560e^{0.017185t}.$$

We estimate that the world population in 1993 was by using the model

$$p(t) = 2560e^{0.017185t}.$$

$$p(43) = 2560e^{0.017185(43)} \simeq 5360 \text{ million}.$$

The model predicts that the population in 2020 will be

$$p(70) = 2560e^{0.017185(70)} \simeq 8524 \text{ million}.$$

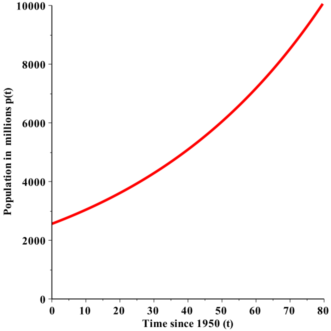


Figure: A model for world population growth in the 2^{nd} half of the 20th century

Exercise

The population of a town at a rate proportional to the population size at any time. Its initial population of 1000 increases by 10% in 5 years. What will be the population after 50 years?

(Hint: $p(0) = 1000$, $p(5) = 1000 + (1000/10) = 1100$)

Example (3)

A radio active material has an initial mass 100mg. After two years it is left to 75mg. Find the amount of the material at any time. What is the period of its half-life?

Solution We measure the amount of the material present at any time t. We have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \Rightarrow y = ce^{kt}$$

and we have the initial condition

$$y(t_0) = y_0 \Rightarrow y(0) = 100,$$

thus, we can find the arbitrary constant c

$$y = ce^{kt} \Rightarrow y(0) = ce^0 \Rightarrow 100 = c.$$

$$y = 100e^{kt}$$

Now we will find the additional constant k by using the additional condition

$$y(2) = 75.$$

$$75 = 100e^{2k}$$

$$e^{2k} = \frac{75}{100}$$

$$\ln\left(e^{10k}\right) = \ln 0.75$$

$$2k = \ln 0.75 \;\Rightarrow\; k = \frac{\ln 0.75}{2} \simeq -0.1438.$$

Thus, we have

$$y(t) = 100e^{-0.1438t}.$$

$$y(t) = 100e^{-0.1438t},$$

from the latest equation we will find the Half-life of the material which is the time when $y = 50\text{mg}$.

$$50 = 100e^{-0.1438t}$$

$$e^{-0.1438t} = 0.5$$

$$\ln e^{-0.1438t} = \ln 0.5$$

$$-0.1438t = \ln 0.5$$

$$t = \frac{\ln 0.5}{-0.1438} \simeq 4.82 \text{ years.}$$

Exercise

Initially there were 100 milligrams (mg.) of a radioactive substance present after 6 hours the mass decreased by 3%. If the rate of decay is proportional to the amount remaining after 24 hours. Determine the half-life of radioactive substance.

Newton's Law of cooling

Newtons Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large.

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newtons Law of Cooling as a differential equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s), \tag{27}$$

where k is a constant of proportionality.

We could solve equation (27) as a separable differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s)$$

$$\frac{dT}{T - T_s} = kdt$$

$$\int \frac{dT}{T - T_s} = \int kdt$$

$$\ln(T - T_s) = kt + c_1$$

$$e^{\ln(T - T_s)} = e^{kt + c_1}$$

$$T - T_s = e^{kt}e^{c_1} = ce^{kt},$$

thus,

$$T = T_s + ce^{kt}. \tag{28}$$

The constant of integration c can be calculated by using the initial condition and the constant of proportionality k can be calculated by using an additional condition.

Example (1)

A glass of a hot water has an initial temperature 80°C, placed in a room where the temperature is 30°C. After one minute the water temperature drops to 70°C. What will be the temperature after 3 minutes? At what time the water cools down to 40°C?

Solution We have the Newtons Law of Cooling is given from

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s) \Rightarrow T = T_s + ce^{kt}.$$

Also we have the initial condition $T(0) = 80^\circ$, the temperature of the surrounding $T_s = 30^\circ$, and the additional condition $T(1) = 70^\circ$

$$T = T_s + ce^{kt} \Rightarrow T(0) = 30 + ce^0 \Rightarrow 80 = 30 + c \Rightarrow c = 50.$$

Thus,

$$T = 30 + 50e^{kt}.$$

Now we will find the additional constant k by using the additional condition $T(1) = 70^\circ$

$$T = 30 + 50e^{kt},$$

$$T(1) = 30 + 50e^{k} \;\Rightarrow\; 70 = 30 + 50e^{k} \;\Rightarrow\; e^{k} = \frac{40}{50}$$

$$\ln\left(e^{k}\right) = \ln 0.8$$

$$k = \ln(0.8).$$

Thus, the temperature of the water at any time is given by

$$T(t) = 30 + 50e^{\ln(0.8)t},$$

so, when $t = 3$ we have the temperature of the water

$$T(3) = 30 + 50e^{3\ln(0.8)}$$

$$T(3) = 30 + 25.6 = 55.6^\circ\mathrm{C}.$$

Now we will find the time t when the water cools down to 40°

$$T = 30 + 50e^{kt},$$

$$40 = 30 + 50e^{kt}$$

$$40 = 30 + 50e^{\ln(0.8)t}$$

$$e^{\ln(0.8)t} = \frac{10}{50}$$

$$\ln\left(e^{\ln(0.8)t}\right) = \ln 0.2$$

$$\ln(0.8)t = \ln 0.2$$

$$t = \frac{\ln 0.2}{\ln 0.8}$$

$$t \simeq 7.2\text{mins}.$$

Example (2)

A bottle of soda at room temperature $72°$F is placed in a refrigerator where the temperature is $44°$F. After half an hour the soda has cooled to $61°$F.

(a) What is the temperature of the soda after another half hour?

(b) How long does it take for the soda to cool to $50°$F?

Solution We have the Newtons Law of Cooling is given from

$$\frac{dT}{dt} = k(T - T_s) \Rightarrow T = T_s + ce^{kt}.$$

Also we have the initial condition $T(0) = 72°$, the temperature of the surrounding $T_s = 44°$, and the additional condition $T(30) = 61°$

$$T = T_s + ce^{kt} \Rightarrow T(0) = 44 + ce^0 \Rightarrow 72 = 44 + c \Rightarrow c = 28.$$

Thus,

$$T = 44 + 28e^{kt}.$$

Now we will find the additional constant k by using the additional condition $T(1) = 70^\circ$

$$T = 44 + 50e^{kt},$$

$$T(30) = 44 + 28e^{30k} \Rightarrow 61 = 44 + 28e^{30k} \Rightarrow e^{30k} = \frac{17}{28}$$

$$\ln\left(e^{30k}\right) = \ln 0.607$$

$$30k = \ln(0.607) \Rightarrow k = -0.0166$$

Thus, the temperature of the water at any time is given by

$$T(t) = 44 + 28e^{-0.0166t}.$$

(a) When $t = 60$ we have the temperature of the water

$$T(60) = 44 + 28e^{60\times(-0.0166)}$$

$$T(60) \simeq 54.3$$

(b) We have $T(t) = 50$ when

$$50 = 44 + 28e^{-0.0166t},$$

$$e^{-0.0166t} = \frac{6}{28}$$

$$\ln\left(e^{-0.0166t}\right) = \ln$$

$$-0.0166t = \ln 0.214$$

$$t = \frac{\ln 0.214}{-0.0166}$$

$$t \simeq 92.9 \text{ mins.}$$

Thus, the soda cools to $50°$F after about 1 hour 33 minutes.

Exercise

A small metal bar, whose initial temperature was $20°C$, is dropped in to large container of boiling water. How long will it take the bar to reach $90°C$ if it is known that its temperature increase $2°C$ per second? How will it take the bar to reach $98°C$.