4
Insurance benefits

4.1 Summary

In this chapter we develop formulae for the valuation of traditional insurance benefits. In particular, we consider whole life, term and endowment insurance. For each of these benefits we identify the random variables representing the present values of the benefits and we derive expressions for moments of these random variables. The functions we develop for traditional benefits will also be useful when we move to modern variable contracts.

We develop valuation functions for benefits based on the continuous future lifetime random variable, T_x, and the curtate future lifetime random variable, K_x, from Chapter 2. We introduce a new random variable, $K_x^{(m)}$, which we use to value benefits which depend on the number of complete periods of length $1/m$ years lived by a life (x). We explore relationships between the expected present values of different insurance benefits.

We also introduce the actuarial notation for the expected values of the present value of insurance benefits.

4.2 Introduction

In the previous two chapters, we have looked at models for future lifetime. The main reason that we need these models is to apply them to the valuation of payments which are dependent on the death or survival of a policyholder or pension plan member. Because of the dependence on death or survival, the timing and possibly the amount of the benefit are uncertain, so the present value of the benefit can be modelled as a random variable. In this chapter we combine survival models with time value of money functions to derive the distribution of the present value of an uncertain, life contingent future benefit.

We generally assume in this chapter (and in the following five chapters) that the interest rate is constant and fixed. This is appropriate, for example, if the premiums for an insurance policy are invested in risk-free bonds, all yielding the same interest rate, so that the term structure is flat. In Chapter 11 we introduce more realistic term structures, and consider some models of interest that allow for uncertainty.

For the development of present value functions, it is generally easier, mathematically, to work in continuous time. In the case of a death benefit, working in continuous time means that we assume that the death payment is paid at the exact time of death. In the case of an annuity, a continuous benefit of, say, \$1 per year would be paid in infinitesimal units of $\$dt$ in every interval $(t, t+dt)$. Clearly both assumptions are impractical; it will take time to process a payment after death, and annuities will be paid at most weekly, not every moment (though the valuation of weekly payments is usually treated as if the payments were continuous, as the difference is very small). In practice, insurers and pension plan actuaries work in discrete time, often with cash flow projections that are, perhaps, monthly or quarterly. In addition, when the survival model being used is in the form of a life table with annual increments (that is, l_x for integer x), it is simplest to use annuity and insurance present value functions that assume payments are made at integer durations only. We work in continuous time in the first place because the mathematical development is more transparent, more complete and more flexible. It is then straightforward to adapt the results from continuous time analysis to discrete time problems.

4.3 Assumptions

To perform calculations in this chapter, we require assumptions about mortality and interest. We use the term **basis** to denote a set of assumptions used in life insurance or pension calculations, and we will meet further examples of bases when we discuss premium calculation in Chapter 6, policy values in Chapter 7 and pension liability valuation in Chapter 10.

Throughout this chapter we illustrate the results with examples using the following survival model which was introduced in Example 3.13.

The Standard Ultimate Survival Model

$$\text{Makeham's law with } \quad A = 0.00022$$
$$B = 2.7 \times 10^{-6}$$
$$c = 1.124.$$

We call this an ultimate model to differentiate it from the standard select model that we will use in later chapters.

We also assume that interest rates are constant. As discussed above, this interest assumption can be criticized as unrealistic. However, it is a convenient assumption from a pedagogical point of view, is often accurate enough for practical purposes (but not always) and we relax the assumption in later chapters.

It is convenient to work with interest theory functions that are in common actuarial and financial use. We review some of these here.

Given an effective annual rate of interest i, we use $v = 1/(1+i)$, so that the present value of a payment of S which is to be paid in t years' time is Sv^t. The force of interest per year is denoted δ where

$$\delta = \log(1+i), \quad 1+i = e^{\delta}, \quad \text{and} \quad v = e^{-\delta};$$

δ is also known as the continuously compounded rate of interest. In financial mathematics and corporate finance contexts, and in particular if the rate of interest is assumed risk free, the common notation for the continuously compounded rate of interest is r.

The nominal rate of interest compounded p times per year is denoted $i^{(p)}$ where

$$i^{(p)} = p\left((1+i)^{1/p} - 1\right) \Leftrightarrow 1+i = \left(1 + i^{(p)}/p\right)^p.$$

The effective rate of discount per year is d where

$$d = 1 - v = iv = 1 - e^{-\delta},$$

and the nominal rate of discount compounded p times per year is $d^{(p)}$ where

$$d^{(p)} = p\left(1 - v^{1/p}\right) \Leftrightarrow (1 - d^{(p)}/p)^p = v.$$

4.4 Valuation of insurance benefits

4.4.1 Whole life insurance: the continuous case, $\bar{A}_x$

For a whole life insurance policy, the time at which the benefit will be paid is unknown until the policyholder actually dies and the policy becomes a claim. Since the present value of a future payment depends on the payment date, the present value of the benefit payment is a function of the time of death, and is therefore modelled as a random variable. Given a survival model and an interest rate we can derive the distribution of the present value random variable for a life contingent benefit, and can therefore compute quantities such as the mean and variance of the present value.

We start by considering the value of a benefit of amount \$1 payable following the death of a life currently aged x. Using a benefit of \$1 allows us to develop valuation functions per unit of sum insured, then we can multiply these by the actual sum insured for different benefit amounts.

We first assume that the benefit is payable immediately on the death of (x). This is known as the continuous case since we work with the continuous future lifetime random variable T_x. Although in practice there would normally be a short delay between the date of a person's death and the time at which an insurance company would actually pay a death benefit (due to notification of death to the insurance company and legal formalities) the effect is slight and we will ignore that delay here.

For our life (x), the present value of a benefit of \$1 payable immediately on death is a random variable, Z, say, where

$$Z = v^{T_x} = e^{-\delta T_x}.$$

We are generally most interested in the expected value of the present value random variable for some future payment. We refer to this as the Expected Present Value or EPV. It is also commonly referred to as the Actuarial Value or Actuarial Present Value.

The EPV of the whole life insurance benefit payment with sum insured \$1 is $E[e^{-\delta T_x}]$. In actuarial notation, we denote this expected value by $\bar{A}_x$, where the bar above A denotes that the benefit is payable immediately on death.

As T_x has probability density function $f_x(t) = {}_tp_x\,\mu_{x+t}$, we have

$$\boxed{\bar{A}_x = \mathrm{E}[e^{-\delta T_x}] = \int_0^\infty e^{-\delta t}\,{}_tp_x\,\mu_{x+t}dt.} \tag{4.1}$$

It is worth looking at the intuition behind this formula. In Figure 4.1 we use the time-line format that was introduced in Section 2.4.

Consider time s, where $x \le x + s < \omega$. The probability that (x) is alive at time s is ${}_sp_x$, and the probability that (x) dies between ages $x + s$ and $x + s + ds$, having survived to age $x + s$, is, loosely, $\mu_{x+s}\,ds$, provided that ds is very small. In this case, the present value of the death benefit of \$1 is $e^{-\delta s}$.

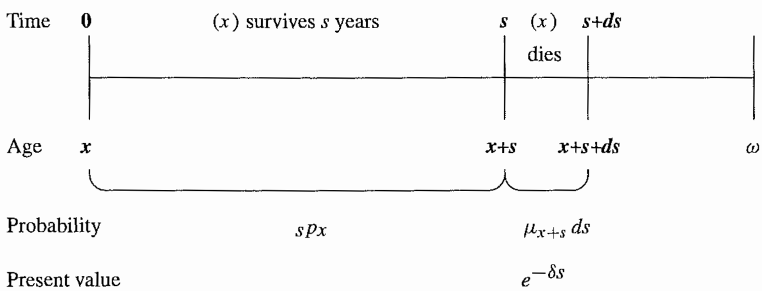

Figure 4.1 Time-line diagram for continuous whole life insurance.

Now we can integrate (that is, sum the infinitesimal components of) the product of present value and probability over all the possible death intervals s to $s + ds$ to obtain the EPV of the death benefit that will be paid in exactly one of these intervals.

Similarly, the second moment (about zero) of the present value of the death benefit is

$$\begin{aligned} \mathrm{E}[Z^2] = \mathrm{E}[(e^{-\delta T_x})^2] &= \mathrm{E}[e^{-2\delta T_x}] \\ &= \int_0^\infty e^{-2\delta t} \, {}_tp_x \, \mu_{x+t} dt \\ &= {}^2\bar{A}_x \end{aligned} \tag{4.2}$$

where the superscript ^{2} indicates that the calculation is at force of interest 2δ, or, equivalently, at rate of interest j, where $1 + j = e^{2\delta} = (1 + i)^2$.

The variance of the present value of a unit benefit payable immediately on death is

$$\mathrm{V}[Z] = \mathrm{V}[e^{-\delta T_x}] = \mathrm{E}[Z^2] - \mathrm{E}[Z]^2 = {}^2\bar{A}_x - (\bar{A}_x)^2. \tag{4.3}$$

Now, if we introduce a more general sum insured, S, say, then the EPV of the death benefit is

$$E[SZ] = E[Se^{-\delta T_x}] = S\bar{A}_x$$

and the variance is

$$\mathrm{V}[SZ] = \mathrm{V}[Se^{-\delta T_x}] = S^2 \left({}^2\bar{A}_x - \bar{A}_x^2\right).$$

In fact we can calculate any probabilities associated with the random variable Z from the probabilities associated with T_x. Suppose we are interested in the probability $\Pr[Z \le 0.5]$, for example. We can rearrange this into a probability for T_x:

$$\begin{aligned} \Pr[Z \le 0.5] &= \Pr[e^{-\delta T_x} \le 0.5] \\ &= \Pr[-\delta T_x \le \log(0.5)] \\ &= \Pr[\delta T_x > -\log(0.5)] \\ &= \Pr[\delta T_x > \log(2)] \\ &= \Pr[T_x > \log(2)/\delta] \\ &= {}_up_x \end{aligned}$$

where $u = \log(2)/\delta$. We note that low values of Z are associated with high values of T_x. This makes sense because the benefit is more expensive to the insurer if it is paid early, as there has been little opportunity to earn interest. It is less expensive if it is paid later.

4.4.2 Whole life insurance: the annual case, A_x

Suppose now that the benefit of \$1 is payable at the end of the year of death of (x), rather than immediately on death. To value this we use the curtate future lifetime random variable, K_x, introduced in Chapter 2. Recall that K_x measures the number of complete years of future life of (x). The time to the end of the year of death of (x) is then $K_x + 1$. For example, if (x) lived for 25.6 years from the issue of the insurance policy, the observed value of K_x would be 25, and the death benefit payable at the end of the year of death would be payable 26 years from the policy's issue.

We again use Z to denote the present value of the whole life insurance benefit of \$1, so that Z is the random variable

$$Z = v^{K_x+1}.$$

The EPV of the benefit, E$[Z]$, is denoted by A_x in actuarial notation.

In Chapter 2 we derived the probability function for K_x, $\Pr[K_x = k] = {}_{k|}q_x$, so the EPV of the benefit is

$$A_x = \mathrm{E}\left[v^{K_x+1}\right] = \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_x = vq_x + v^2\, {}_{1|}q_x + v^3\, {}_{2|}q_x + \cdots . \tag{4.4}$$

Each term on the right-hand side of this equation represents the EPV of a death benefit of \$1, payable at time k conditional on the death of (x) in $(k-1, k]$.

In fact, we can always express the EPV of a life contingent benefit by considering each time point at which the benefit can be paid, and summing over all possible payment times the product of

(1) the amount of the benefit,

(2) the appropriate discount factor, and

(3) the probability that the benefit will be paid at that time.

We will justify this more rigorously in Section 4.6. We illustrate the process for the whole life insurance example in Figure 4.2.

The second moment of the present value is

$$\sum_{k=0}^{\infty} v^{2(k+1)}\, {}_{k|}q_x = \sum_{k=0}^{\infty} (v^2)^{(k+1)}\, {}_{k|}q_x = (v^2)q_x + (v^2)^2\, {}_{1|}q_x + (v^2)^3\, {}_{2|}q_x + \cdots .$$

Just as in the continuous case, we can calculate the second moment about zero of the present value by an adjustment in the rate of interest from i to $(1+i)^2 - 1$.

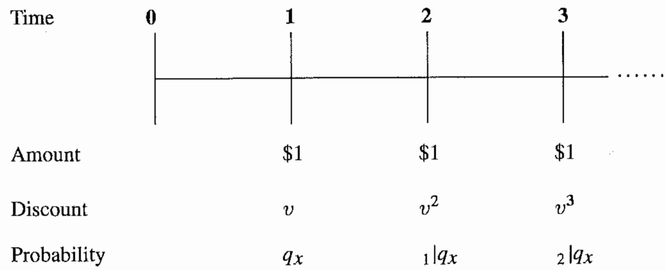

Figure 4.2 Time-line diagram for discrete whole life insurance.

We define

$$^2A_x = \sum_{k=0}^{\infty} v^{2(k+1)} \, {}_{k|}q_x, \tag{4.5}$$

and so the variance of the present value of a benefit of S payable at the end of the year of death is

$$S^2 \left({}^2A_x - (A_x)^2 \right). \tag{4.6}$$

4.4.3 Whole life insurance: the 1/mthly case, $A_x^{(m)}$

In Chapter 2 we introduced the random variable K_x, representing the curtate future lifetime of (x), and we saw in Section 4.4.2 that the present value of an insurance benefit payable at the end of the year of death can be expressed in terms of K_x.

We now define the $1/m$thly curtate future lifetime random variable $K_x^{(m)}$, where $m > 1$ is an integer, to be the future lifetime of (x) in years rounded to the lower $\frac{1}{m}$th of a year. The most common values of m are 2, 4 and 12, corresponding to half years, quarter years and months. Thus, for example, $K_x^{(4)}$ represents the future lifetime of (x), rounded down to the lower 1/4.

Symbolically, if we let $\lfloor\, \rfloor$ denote the integer part (or floor) function, then

$$K_x^{(m)} = \frac{1}{m} \lfloor m T_x \rfloor . \tag{4.7}$$

For example, suppose (x) lives exactly 23.675 years. Then

$K_x = 23$, $K_x^{(2)} = 23.5$, $K_x^{(4)} = 23.5$, and $K_x^{(12)} = 23\frac{8}{12} = 23.6667$.

Note that $K_x^{(m)}$ is a discrete random variable. $K_x^{(m)} = k$ indicates that the life (x) dies in the interval $[k, k + 1/m)$, for $k = 0, \frac{1}{m}, \frac{2}{m}, \ldots$

The probability function for $K_x^{(m)}$ can be derived from the associated probabilities for T_x. For $k = 0, \frac{1}{m}, \frac{2}{m}, \ldots,$

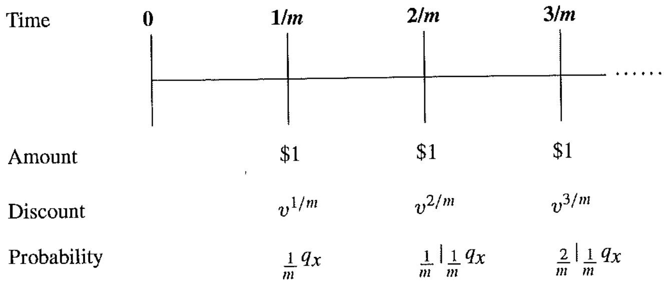

Figure 4.3 Time-line diagram for $1/m$thly whole life insurance.

$$\Pr[K_x^{(m)} = k] = \Pr\left[k \le T_x < k + \tfrac{1}{m}\right] = {}_{k|\frac{1}{m}}q_x = {}_k p_x - {}_{k+\frac{1}{m}}p_x.$$

In Figure 4.3 we show the time-line for the $1/m$thly benefit. At the end of each $1/m$ year period, we show the amount of benefit due, conditional on the death of the insured life in the previous $1/m$ year interval, the probability that the insured life dies in the relevant interval, and the appropriate discount factor.

Suppose, for example, that $m = 12$. A whole life insurance benefit payable at the end of the month of death has present value random variable Z where

$$Z = v^{K_x^{(12)}+1/12}.$$

We let $A_x^{(12)}$ denote the EPV of this benefit, so that

$$E[Z] = A_x^{(12)} = v^{1/12}\,{}_{\frac{1}{12}}q_x + v^{2/12}\,{}_{\frac{1}{12}|\frac{1}{12}}q_x + v^{3/12}\,{}_{\frac{2}{12}|\frac{1}{12}}q_x + v^{4/12}\,{}_{\frac{3}{12}|\frac{1}{12}}q_x + \cdots .$$

Similarly, for any m,

$$A_x^{(m)} = v^{1/m}\,{}_{\frac{1}{m}}q_x + v^{2/m}\,{}_{\frac{1}{m}|\frac{1}{m}}q_x + v^{3/m}\,{}_{\frac{2}{m}|\frac{1}{m}}q_x + v^{4/m}\,{}_{\frac{3}{m}|\frac{1}{m}}q_x + \cdots \tag{4.8}$$

$$= \sum_{k=0}^{\infty} v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x. \tag{4.9}$$

As for the continuous and annual cases, we can derive the variance of the present value of the $1/m$thly whole life benefit by adjusting the interest rate for the first term in the variance. We have

$$\mathrm{E}[Z^2] = \mathrm{E}\left[v^{2(K_x^{(12)}+1/12)}\right] = \mathrm{E}\left[(v^2)^{K_x^{(12)}+1/12}\right] = {}^2A_x^{(12)},$$

so the variance is

$${}^2A_x^{(12)} - (A_x^{(12)})^2.$$

4.4.4 Recursions

In practice, it would be very unusual for an insurance policy to provide the death benefit at the end of the year of death. Nevertheless, the annual insurance function A_x is still useful. We are often required to work with annual life tables, such as those in Chapter 3, in which case we would start by calculating the annual function A_x, then adjust for a more appropriate frequency using the relationships and assumptions we develop later in this chapter.

Using the annual life table in a spreadsheet, we can calculate the values of A_x using **backwards recursion**. To do this, we start from the highest age in the table, ω. We assume all lives expire by age ω, so that $q_{\omega-1} = 1$. If the life table does not have a limiting age, we choose a suitably high value for ω so that $q_{\omega-1}$ is as close to 1 as we like. This means that any life attaining age $\omega - 1$ may be treated as certain to die before age ω, in which case we know that $K_{\omega-1} = 0$ and so

$$A_{\omega-1} = \mathrm{E}\left[v^{K_{\omega-1}+1}\right] = v.$$

Now, working from the summation formula for A_x we have

$$\begin{aligned} A_x &= \sum_{k=0}^{\omega-x-1} v^{k+1}\,{}_kp_x\,q_{x+k} \\ &= v\,q_x + v^2\,p_x\,q_{x+1} + v^3\,{}_2p_x\,q_{x+2} + \cdots \\ &= v\,q_x + v\,p_x\left(v\,q_{x+1} + v^2\,p_{x+1}\,q_{x+2} + v^3\,{}_2p_{x+1}\,q_{x+3} + \cdots\right), \end{aligned}$$

giving the important recursion formula

$$\boxed{A_x = v\,q_x + v\,p_x\,A_{x+1}.} \tag{4.10}$$

This formula can be used in spreadsheet format to calculate A_x backwards from $A_{\omega-1}$ back to A_{x_0}, where x_0 is the minimum age in the table.

The intuition for equation (4.10) is that we separate the EPV of the whole life insurance into the value of the benefit due in the first year, followed by the value at age $x + 1$ of all subsequent benefits, multiplied by p_x to allow for the probability of surviving to age $x + 1$, and by v to discount the value back from age $x + 1$ to age x.

We can use the same approach for $1/m$thly benefits; now the recursion will give $A_x^{(m)}$ in terms of $A^{(m)}_{x+\frac{1}{m}}$. Again, we split the benefit into the part payable in the first period – now of length $1/m$ years – followed by the EPV of the insurance beginning after $1/m$ years. We have

$$\begin{aligned} A_x^{(m)} &= v^{1/m}\,{}_{\frac{1}{m}}q_x + v^{2/m}\,{}_{\frac{1}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{1}{m}} + v^{3/m}\,{}_{\frac{2}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{2}{m}} + \cdots \\ &= v^{1/m}\,{}_{\frac{1}{m}}q_x + v^{1/m}\,{}_{\frac{1}{m}}p_x\left(v^{1/m}\,{}_{\frac{1}{m}}q_{x+\frac{1}{m}} + v^{2/m}\,{}_{\frac{1}{m}}p_{x+\frac{1}{m}}\,{}_{\frac{1}{m}}q_{x+\frac{2}{m}} + \cdots\right), \end{aligned}$$

Table 4.1 *Sample values of A_x using the Standard Ultimate Survival Model, Example 4.1.*

x	A_x	x	A_x	x	A_x
30	0.07698	50	0.18931	98	0.85177
31	0.08054	51	0.19780	99	0.86153
32	0.08427	52	0.20664	100	0.87068
⋮	⋮	⋮	⋮	⋮	⋮

giving the recursion formula

$$A_x^{(m)} = v^{1/m}\ {}_{\frac{1}{m}}q_x + v^{1/m}\ {}_{\frac{1}{m}}p_x\ A^{(m)}_{x+\frac{1}{m}}.$$

Example 4.1 Using the Standard Ultimate Survival Model from Section 4.3, and an interest rate of 5% per year effective, construct a spreadsheet of values of A_x for $x = 20, 21, \ldots, 100$. Assume that $A_{129} = v$.

Solution 4.1 The survival model for the Standard Ultimate Survival Model is the ultimate part of the model used in Example 3.13 and so values of ${}_tp_x$ can be calculated as explained in the solution to that example. The value of q_{129} is 0.99996, which is indeed close to 1. We can use the formula

$$A_x = vq_x + vp_xA_{x+1}$$

to calculate recursively $A_{128}, A_{127}, \ldots, A_{20}$, starting from $A_{129} = v$. Values for $x = 20, 21, \ldots, 80$, are given in Appendix D, Table D.3. Some excerpts are shown in Table 4.1. □

Example 4.2 Using the Standard Ultimate Survival Model from Section 4.3, and an interest rate of 5% per year effective, develop a spreadsheet of values of $A_x^{(12)}$ for x starting at age 20, in steps of 1/12.

Solution 4.2 For this example, we follow exactly the same process as for the previous example, except that we let the ages increase by 1/12 year in each row. We construct a column of values of ${}_{\frac{1}{12}}p_x$ using

$${}_{\frac{1}{12}}p_x = \exp\left\{-A/12 - Bc^x(c^{1/12} - 1)/\log(c)\right\}.$$

We again use 130 as the limiting age of the table. Then set $A^{(12)}_{129\frac{11}{12}} = v^{1/12}$, and for all the other values of $A_x^{(12)}$ use the recursion

$$A_x^{(12)} = v^{1/12}\ {}_{\frac{1}{12}}q_x + v^{1/12}\ {}_{\frac{1}{12}}p_x\ A^{(12)}_{x+\frac{1}{12}}.$$

Table 4.2 *Sample values of $A_x^{(12)}$ using the Standard Ultimate Survival Model, Example 4.2.*

x	${}_{\frac{1}{12}}p_x$	${}_{\frac{1}{12}}q_x$	$A_x^{(12)}$
20	0.999979	0.000021	0.05033
$20\frac{1}{12}$	0.999979	0.000021	0.05051
$20\frac{2}{12}$	0.999979	0.000021	0.05070
$20\frac{3}{12}$	0.999979	0.000021	0.05089
⋮	⋮	⋮	⋮
50	0.999904	0.000096	0.19357
$50\frac{1}{12}$	0.999903	0.000097	0.19429
⋮	⋮	⋮	⋮
$129\frac{10}{12}$	0.413955	0.586045	0.99427
$129\frac{11}{12}$			0.99594

The first and last few lines of the spreadsheet are reproduced in Table 4.2. □

It is worth making a remark about the calculations in Examples 4.1 and 4.2. In Example 4.1 we saw that $q_{129} = 0.99996$, which is sufficiently close to 1 to justify us starting our recursive calculation by setting $A_{129} = v$. In Example 4.2, our recursive calculation started from $A_{129\frac{11}{12}} = v^{1/12}$. If we calculate ${}_{\frac{1}{12}}q_{129\frac{11}{12}}$ we find its value is 0.58960, which is certainly not close to 1.

What is happening in these calculations is that, for Example 4.1, we are replacing the exact calculation

$$A_{129} = v\,(q_{129} + p_{129}\,A_{130})$$

by $A_{129} = v$, which is justifiable because A_{130} is close to 1, meaning that $v(q_{129} + p_{129}\,A_{130})$ is very close to v. Similarly, for Example 4.2, we replace the exact calculation

$$A^{(12)}_{129\frac{11}{12}} = v^{1/12}\left({}_{\frac{1}{12}}q_{129\frac{11}{12}} + {}_{\frac{1}{12}}p_{129\frac{11}{12}}\,A^{(12)}_{130}\right)$$

by $A^{(12)}_{129\frac{11}{12}} = v^{1/12}$. As the value of $A^{(12)}_{130}$ is very close to 1, it follows that

$$v^{1/12}\left({}_{\frac{1}{12}}q_{129\frac{11}{12}} + {}_{\frac{1}{12}}p_{129\frac{11}{12}}\,A^{(12)}_{130}\right)$$

can by approximated by $v^{1/12}$.

Example 4.3 Using the Standard Ultimate Survival Model, and an interest rate of 5% per year effective, calculate the mean and standard deviation of the

Table 4.3 *Mean and standard deviation of the present value of a whole life insurance benefit of \$100 000, for Example 4.3.*

	Continuous (a)		Monthly (b)		Annual (c)	
Age, x	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
20	5 043	5 954	5 033	5 942	4 922	5 810
40	12 404	9 619	12 379	9 600	12 106	9 389
60	29 743	15 897	29 683	15 865	29 028	15 517
80	60 764	17 685	60 641	17 649	59 293	17 255
100	89 341	8 127	89 158	8 110	87 068	7 860

present value of a benefit of \$100 000 payable (a) immediately on death, (b) at the end of the month of death, and (c) at the end of the year of death for lives aged 20, 40, 60, 80 and 100, and comment on the results.

Solution 4.3 For part (a), we must calculate $100\,000\bar{A}_x$ and

$$100\,000\sqrt{{}^2\bar{A}_x - (\bar{A}_x)^2}$$

for $x = 20$, 40, 60 and 80, where ${}^2\bar{A}_x$ is calculated at effective rate of interest $j = 10.25\%$. For parts (b) and (c) we replace each $\bar{A}_x$ by $A_x^{(12)}$ and A_x, respectively. The values are shown in Table 4.3. The continuous benefit values in the first column are calculated by numerical integration, and the annual and monthly benefit values are calculated using the spreadsheets from Examples 4.1 and 4.2.

We can make the following observations about these values. First, values for the continuous benefit are larger than the monthly benefit, which are larger than the annual benefit. This is because the death benefit is payable soonest under (a) and latest under (c). Second, as x increases the mean increases for all three cases. This occurs because the smaller the value of x, the longer the expected time until payment of the death benefit. Third, as x increases, the standard deviation decreases relative to the mean, in all three cases. And further, as we get to very old ages, the standard deviation decreases in absolute terms, as the possible range of payout dates is reduced significantly.

It is also interesting to note that the continuous and monthly versions of the whole life benefit are very close. That is to be expected, as the difference arises from the change in the value of money in the period between the moment of death and the end of the month of death, a relatively short period. □

4.4.5 Term insurance

The continuous case, $\bar{A}^{\,1}_{x:\overline{n}|}$

Under a term insurance policy, the death benefit is payable only if the policyholder dies within a fixed term of, say, n years.

In the continuous case, the benefit is payable immediately on death. The present value of a benefit of \$1, which we again denote by Z, is

$$Z = \begin{cases} v^{T_x} = e^{-\delta\, T_x} & \text{if } T_x \le n, \\ 0 & \text{if } T_x > n. \end{cases}$$

The EPV of this benefit is denoted $\bar{A}^{\,1}_{x:\overline{n}|}$ in actuarial notation. The bar above A again denotes that the benefit is payable immediately on death, and the 1 above x indicates that the life (x) must die before the term of n years expires in order for the benefit to be payable.

Then

$$\boxed{\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\; {}_tp_x\,\mu_{x+t}\,dt} \tag{4.11}$$

and, similarly, the expected value of the square of the present value is

$$^2\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n e^{-2\delta t}\; {}_tp_x\,\mu_{x+t}\,dt$$

which, as with the whole life case, is calculated by a change in the rate of interest used.

The annual case, $A^{\,1}_{x:\overline{n}|}$

Next, we consider the situation when a death benefit of 1 is payable at the end of the year of death, provided this occurs within n years. The present value random variable for the benefit is now

$$Z = \begin{cases} v^{K_x+1} & \text{if } K_x \le n-1, \\ 0 & \text{if } K_x \ge n. \end{cases}$$

The EPV of the benefit is denoted $A^{\,1}_{x:\overline{n}|}$ so that

$$\boxed{A^{\,1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\; {}_{k|}q_x .} \tag{4.12}$$

Table 4.4 *EPVs of term insurance benefits.*

x	$\bar{A}^{\ 1}_{x:\overline{10}\rceil}$	$A^{(4)\ 1}_{x:\overline{10}\rceil}$	$A^{\ 1}_{x:\overline{10}\rceil}$
20	0.00214	0.00213	0.00209
40	0.00587	0.00584	0.00573
60	0.04356	0.04329	0.04252
80	0.34550	0.34341	0.33722

The $1/m$thly case, $A^{(m)\ 1}_{x:\overline{n}\rceil}$

We now consider the situation when a death benefit of 1 is payable at the end of the $1/m$th year of death, provided this occurs within n years. The present value random variable for the benefit is

$$Z = \begin{cases} v^{K_x^{(m)}+\frac{1}{m}} & \text{if } K_x^{(m)} \le n - \frac{1}{m}, \\ 0 & \text{if } K_x^{(m)} \ge n. \end{cases}$$

The EPV of the benefit is denoted $A^{(m)1}_{x:\overline{n}\rceil}$ so that

$$A^{(m)1}_{x:\overline{n}\rceil} = \sum_{k=0}^{mn-1} v^{(k+1)/m} \ {}_{\frac{k}{m}|\frac{1}{m}}q_x . \tag{4.13}$$

Example 4.4 Using the Standard Ultimate Survival Model as specified in Section 4.3, with interest at 5% per year effective, calculate $\bar{A}^{\ 1}_{x:\overline{10}\rceil}$, $A^{(4)1}_{x:\overline{10}\rceil}$ and $A^{\ 1}_{x:\overline{10}\rceil}$ for $x = 20, 40, 60$ and 80 and comment on the values.

Solution 4.4 We use formula (4.11) with $n = 10$ to calculate $\bar{A}^{\ 1}_{x:\overline{10}\rceil}$ (using numerical integration), and formulae (4.13) and (4.12), with $m = 4$ and $n = 10$ to calculate $A^{(4)1}_{x:\overline{10}\rceil}$ and $A^{1}_{x:\overline{10}\rceil}$.

The values are shown in Table 4.4, and we observe that values in each case increase as x increases, reflecting the fact that the probability of death in a 10-year period increases with age for the survival model we are using. The ordering of values at each age is the same as in Example 4.3, for the same reason – the ordering reflects the fact that any payment under the continuous benefit will be paid earlier than a payment under the quarterly benefit. The end year benefit is paid later than the quarterly benefit, except when the death occurs in the final quarter of the year, in which case the benefit is paid at the same time. □

4.4.6 Pure endowment

Pure endowment benefits are conditional on the survival of the policyholder at a policy maturity date. For example, a 10-year pure endowment with sum insured \$10 000, issued to (x), will pay \$10 000 in 10 years if (x) is still alive at that time, and will pay nothing if (x) dies before age $x + 10$. Pure endowment benefits are not sold as stand-alone policies, but may be sold in conjunction with term insurance benefits to create the endowment insurance benefit described in the following section. However, pure endowment valuation functions turn out to be very useful.

The pure endowment benefit of \$1, issued to a life aged x, with a term of n years has present value Z, say, where:

$$Z = \begin{cases} 0 & \text{if } T_x < n, \\ v^n & \text{if } T_x \geq n. \end{cases}$$

There are two ways to denote the EPV of the pure endowment benefit using actuarial notation. It may be denoted $A_{\substack{\\x:\overline{n}|}}^{\;\;1}$. The '1' over the term subscript indicates that the term must expire before the life does for the benefit to be paid. This notation is consistent with the term insurance notation, but it can be cumbersome, considering that this is a function which is used very often in actuarial calculations. A more convenient standard actuarial notation for the EPV of the pure endowment is ${}_nE_x$.

If we rewrite the definition of Z above, we have

$$Z = \begin{cases} 0 & \text{with probability } 1 - {}_np_x, \\ v^n & \text{with probability } {}_np_x. \end{cases} \tag{4.14}$$

Then we can see that the EPV is

$$\boxed{A_{x:\overline{n}|}^{\;\;\;1} = {}_nE_x = v^n\,{}_np_x\,.} \tag{4.15}$$

Note that because the pure endowment will be paid only at time n, assuming the life survives, there is no need to specify continuous and discrete time versions; there is only a discrete time version.

We will generally use the more direct notation $v^n\,{}_np_x$ or ${}_nE_x$ for the pure endowment function, rather than the $A_{x:\overline{n}|}^{\;\;\;1}$ notation.

4.4.7 Endowment insurance

An endowment insurance provides a combination of a term insurance and a pure endowment. The sum insured is payable on the death of (x) should (x) die within a fixed term, say n years, but if (x) survives for n years, the sum insured is payable at the end of the nth year.

Traditional endowment insurance policies were popular in Australia, North America and the UK up to the 1990s, but are rarely sold these days in these markets. However, as with the pure endowment, the valuation function turns out to be quite useful in other contexts. Also, companies operating in these territories will be managing the ongoing liabilities under the policies already written for some time to come. Furthermore, traditional endowment insurance is still relevant and popular in some other insurance markets.

We first consider the case when the death benefit (of amount 1) is payable immediately on death. The present value of the benefit is Z, say, where

$$\begin{aligned} Z &= \begin{cases} v^{T_x} = e^{-\delta T_x} & \text{if } T_x < n, \\ v^n & \text{if } T_x \geq n \end{cases} \\ &= v^{\min(T_x,n)} = e^{-\delta \min(T_x,n)}. \end{aligned}$$

Thus, the EPV of the benefit is

$$\begin{aligned} \mathrm{E}[Z] &= \int_0^n e^{-\delta t}\,{}_tp_x\mu_{x+t}dt + \int_n^\infty e^{-\delta n}\,{}_tp_x\mu_{x+t}dt \\ &= \int_0^n e^{-\delta t}\,{}_tp_x\mu_{x+t}dt + e^{-\delta n}\,{}_np_x \\ &= \bar{A}^{\,1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;\;\,1} \end{aligned}$$

and in actuarial notation we write

$$\boxed{\bar{A}_{x:\overline{n|}} = \bar{A}^{\,1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;\;\,1}.} \tag{4.16}$$

Similarly, the expected value of the squared present value of the benefit is

$$\int_0^n e^{-2\delta t}\,{}_tp_x\mu_{x+t}dt + e^{-2\delta n}\,{}_np_x$$

which we denote ${}^2\bar{A}_{x:\overline{n|}}$.

In the situation when the death benefit is payable at the end of the year of death, the present value of the benefit is

$$\begin{aligned} Z &= \begin{cases} v^{K_x+1} & \text{if } K_x \leq n-1, \\ v^n & \text{if } K_x \geq n \end{cases} \\ &= v^{\min(K_x+1,n)}. \end{aligned}$$

The EPV of the benefit is then

$$\sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x + v^n\,P[K_x \geq n] = A^{\,1}_{x:\overline{n|}} + v^n\,{}_np_x, \tag{4.17}$$

Table 4.5 *EPVs of endowment insurance benefits.*

x	$\bar{A}_{x:\overline{10}\rvert}$	$A^{(4)}_{x:\overline{10}\rvert}$	$A_{x:\overline{10}\rvert}$
20	0.61438	0.61437	0.61433
40	0.61508	0.61504	0.61494
60	0.62220	0.62194	0.62116
80	0.68502	0.68292	0.67674

and in actuarial notation we write

$$\boxed{A_{x:\overline{n}\rvert} = A^{\;1}_{x:\overline{n}\rvert} + A_{x:\overline{n}\rvert}^{\;\;\;\;1}.} \tag{4.18}$$

Similarly, the expected value of the squared present value of the benefit is

$$^{2}A_{x:\overline{n}\rvert} = \sum_{k=0}^{n-1} v^{2(k+1)}\;{}_{k|}q_x + v^{2n}\;{}_{n}p_x.$$

Finally, when the death benefit is payable at the end of the $1/m$th year of death, the present value of the benefit is

$$Z = \begin{cases} v^{K_x^{(m)}+\frac{1}{m}} & \text{if } K_x^{(m)} \le n - \frac{1}{m}, \\ v^n & \text{if } K_x^{(m)} \ge n \end{cases}$$
$$= v^{\min(K_x^{(m)}+\frac{1}{m},\,n)}.$$

The EPV of the benefit is

$$\sum_{k=0}^{mn-1} v^{(k+1)/m}\;{}_{\frac{k}{m}|\frac{1}{m}}q_x + v^n\,P[K_x^{(m)} \ge n] = A^{(m)\,1}_{\;\;x:\overline{n}\rvert} + v^n\;{}_{n}p_x,$$

and in actuarial notation we write

$$\boxed{A^{(m)}_{x:\overline{n}\rvert} = A^{(m)\,1}_{\;\;x:\overline{n}\rvert} + A_{x:\overline{n}\rvert}^{\;\;\;\;1}.} \tag{4.19}$$

Example 4.5 Using the Standard Ultimate Survival Model as specified in Section 4.3, with interest at 5% per year effective, calculate $\bar{A}_{x:\overline{10}\rvert}$, $A^{(4)}_{x:\overline{10}\rvert}$ and $A_{x:\overline{10}\rvert}$ for $x = 20$, 40, 60 and 80 and comment on the values.

Solution 4.5 We can obtain values of $\bar{A}_{x:\overline{10}\rvert}$, $A^{(4)}_{x:\overline{10}\rvert}$ and $A_{x:\overline{10}\rvert}$ by adding $A_{x:\overline{10}\rvert}^{\;\;\;\;1} = v^{10}\;{}_{10}p_x$ to the values of $\bar{A}^{\;1}_{x:\overline{10}\rvert}$, $A^{(4)\,1}_{\;\;x:\overline{10}\rvert}$, and $A^{\;1}_{x:\overline{10}\rvert}$ in Example 4.4. The values are shown in Table 4.5.

The actuarial values of the 10-year endowment insurance functions do not vary greatly with x, unlike the values of the 10-year term insurance functions. The reason for this is that the probability of surviving 10 years is large (${}_{10}p_{20} = 0.9973$, ${}_{10}p_{60} = 0.9425$) and so for each value of x, the benefit is payable after 10 years with a high probability. Note that $v^{10} = 0.6139$, and as time 10 years is the latest possible payment date for the benefit, the values of $\bar{A}_{x:\overline{10}|}$, $A^{(4)}_{x:\overline{10}|}$ and $A_{x:\overline{10}|}$ must be greater than this for any age x.

□

4.4.8 Deferred insurance benefits

Deferred insurance refers to insurance which does not begin to offer death benefit cover until the end of a deferred period. Suppose a benefit of \$1 is payable immediately on the death of (x) provided that (x) dies between ages $x + u$ and $x + u + n$. The present value random variable is

$$Z = \begin{cases} 0 & \text{if } T_x < u \text{ or } T_x \geq u + n, \\ e^{-\delta T_x} & \text{if } u \leq T_x < u + n. \end{cases}$$

This random variable describes the present value of a deferred term insurance. We can, similarly, develop random variables to value deferred whole life or endowment insurance.

The actuarial notation for the EPV of the deferred term insurance benefit is ${}_{u|}\bar{A}^{\;1}_{x:\overline{n}|}$. Thus

$${}_{u|}\bar{A}^{\;1}_{x:\overline{n}|} = \int_u^{u+n} e^{-\delta t} \, {}_t p_x \mu_{x+t} dt. \tag{4.20}$$

Changing the integration variable to $s = t - u$ gives

$$\begin{aligned} {}_{u|}\bar{A}^{\;1}_{x:\overline{n}|} &= \int_0^n e^{-\delta(s+u)} \, {}_{s+u}p_x \, \mu_{x+s+u} \, ds \\ &= e^{-\delta u} \, {}_u p_x \int_0^n e^{-\delta s} \, {}_s p_{x+u} \, \mu_{x+s+u} \, ds \\ &= e^{-\delta u} \, {}_u p_x \, \bar{A}^{\;1}_{x+u:\overline{n}|} = v^u \, {}_u p_x \, \bar{A}^{\;1}_{x+u:\overline{n}|} = {}_u E_x \, \bar{A}^{\;1}_{x+u:\overline{n}|}. \end{aligned} \tag{4.21}$$

A further expression for ${}_{u|}\bar{A}^{\;1}_{x:\overline{n}|}$ is

$$\boxed{{}_{u|}\bar{A}^{\;1}_{x:\overline{n}|} = \bar{A}^{\;1}_{x:\overline{u+n}|} - \bar{A}^{\;1}_{x:\overline{u}|}} \tag{4.22}$$

which follows from formula (4.20) since

$$\int_u^{u+n} e^{-\delta t} \, {}_t p_x \mu_{x+t} dt = \int_0^{u+n} e^{-\delta t} \, {}_t p_x \mu_{x+t} dt - \int_0^u e^{-\delta t} \, {}_t p_x \mu_{x+t} dt.$$

Thus, the EPV of a deferred term insurance benefit can be found by differencing the EPVs of term insurance benefits for terms $u + n$ and u.

Note the role of the pure endowment term ${}_uE_x = v^u\,{}_up_x$ in equation (4.21). This acts similarly to a discount function. If the life survives u years, to the end of the deferred period, then the EPV at that time of the term insurance is $\bar{A}^{\;1}_{x+u:\overline{n}|}$. Multiplying by $v^u\,{}_up_x$ converts this to the EPV at the start of the deferred period.

Our main interest in this EPV is as a building block. We observe, for example, that an n-year term insurance can be decomposed as the sum of n deferred term insurance policies, each with a term of one year, and we can write

$$\begin{aligned}\bar{A}^{\;1}_{x:\overline{n}|} &= \int_0^n e^{-\delta t}\;{}_tp_x\mu_{x+t}dt \\ &= \sum_{r=0}^{n-1}\int_r^{r+1} e^{-\delta t}\;{}_tp_x\mu_{x+t}dt \\ &= \sum_{r=0}^{n-1} {}_{r|}\bar{A}^{\;1}_{x:\overline{1}|}. \end{aligned} \tag{4.23}$$

A similar decomposition applies to a whole life insurance policy and we can write

$$\bar{A}_x = \sum_{r=0}^{\infty} {}_{r|}\bar{A}^{\;1}_{x:\overline{1}|}.$$

We can derive similar results for the deferred benefit payable at the end of the year of death, with EPV denoted ${}_{u|}A^{\;1}_{x:\overline{n}|}$.

In particular, it is useful to note that

$$\boxed{A_x = A^{\;1}_{x:\overline{n}|} + {}_{n|}A_x}$$

where ${}_{n|}A_x$ is the EPV of a benefit of 1 payable at the end of the year of death of (x) if death occurs after time n, so that

$$\begin{aligned}A^{\;1}_{x:\overline{n}|} &= A_x - {}_{n|}A_x \\ &= A_x - v^n\;{}_np_x\;A_{x+n}.\end{aligned}$$

This relationship can be used to calculate $A^{\;1}_{x:\overline{n}|}$ for integer x and n given a table of values of A_x and l_x.

4.5 Relating $\bar{A}_x$, A_x and $A_x^{(m)}$

We mentioned in the introduction to this chapter that, even though insurance contracts with death benefits payable at the end of the year of death are very unusual, functions like A_x are still useful. The reason for this is that we can approximate $\bar{A}_x$ or $A_x^{(m)}$ from A_x, and we might wish to do this if the only

Table 4.6 *Ratios of* $A_x^{(4)}$ *to* A_x *and* $\bar{A}_x$ *to* A_x, *Standard Ultimate Survival Model.*

x	$A_x^{(4)}/A_x$	$\bar{A}_x/A_x$
20	1.0184	1.0246
40	1.0184	1.0246
60	1.0184	1.0246
80	1.0186	1.0248
100	1.0198	1.0261
120	1.0296	1.0368

information we had was a life table, with integer age functions only, rather than a formula for the force of mortality that could be applied for all ages.

In Table 4.6 we show values of the ratios of $A_x^{(4)}$ to A_x and $\bar{A}_x$ to A_x, using the Standard Ultimate Survival Model from Section 4.3, with interest at 5% per year effective.

We see from Table 4.6 that, over a very wide range of ages, the ratios of $A_x^{(4)}$ to A_x and $\bar{A}_x$ to A_x are remarkably stable, giving the appearance of being independent of x. In the following section we show how we can approximate values of $A_x^{(m)}$ and $\bar{A}_x$ using values of A_x.

4.5.1 Using the uniform distribution of deaths assumption

The difference between $\bar{A}_x$ and A_x depends on the lifetime distribution between ages y and $y+1$ for all $y \geq x$. If we do not have information about this, for example, because we have mortality information only at integer ages, we can approximate the relationship between the continuous function $\bar{A}_x$ and the discrete function A_x using the fractional age assumptions that we introduced in Section 3.3. The most convenient fractional age assumption for this purpose is the uniform distribution of deaths assumption, or UDD.

Recall, from equation (3.9), that under UDD, we have for $0 \leq s < 1$, and for integer y, ${}_sp_y\,\mu_{y+s} = q_y$. Using this assumption

$$\begin{aligned}
\bar{A}_x &= \int_0^\infty e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt \\
&= \sum_{k=0}^{\infty} {}_kp_x v^{k+1} \int_0^1 e^{(1-s)\delta}\,{}_sp_{x+k}\,\mu_{x+k+s}\,ds
\end{aligned}$$

$$= \sum_{k=0}^{\infty} {}_kp_x \, q_{x+k} v^{k+1} \int_0^1 e^{(1-s)\delta} \, ds \quad \text{using UDD}$$

$$= A_x \, \frac{e^\delta - 1}{\delta}.$$

Because $e^\delta = 1 + i$, under the assumption of UDD we have

$$\bar{A}_x = \frac{i}{\delta} A_x. \tag{4.24}$$

This exact result under the UDD assumption gives rise to the approximation

$$\boxed{\bar{A}_x \approx \frac{i}{\delta} A_x.} \tag{4.25}$$

The same approximation applies to term insurance and deferred insurance, which we can show by changing the limits of integration in the proof above.

We may also want to derive a $1/m$thly death benefit EPV, such as $A_x^{(m)}$, from the annual function A_x.

Under the UDD assumption we find that

$$\boxed{A_x^{(m)} = \frac{i}{i^{(m)}} A_x,} \tag{4.26}$$

and the right-hand side is used as an approximation to $A_x^{(m)}$. The proof of formula (4.26) is left as an exercise for the reader.

We stress that these approximations apply only to death benefits. The endowment insurance combines the death and survival benefits, so we need to split off the death benefit before applying one of the approximations. That is, under the UDD approach,

$$\boxed{\bar{A}_{x:\overline{n}|} \approx \tfrac{i}{\delta} A^{\;1}_{x:\overline{n}|} + v^n \, {}_np_x.} \tag{4.27}$$

4.5.2 Using the claims acceleration approach

The claims acceleration approach is a more heuristic way of deriving an approximate relationship between the annual death benefit EPV, A_x, and the $1/m$thly or continuous EPVs, $A_x^{(m)}$ and $\bar{A}_x$. The only difference between these benefits is the timing of the payment. Consider, for example, A_x and $A_x^{(4)}$. The insured life, (x), dies in the year of age $x+K_x$ to $x+K_x+1$. Under the end year of death benefit (valued by A_x), the sum insured is paid at time K_x+1. Under the end of quarter-year of death benefit (valued by $A_x^{(4)}$), the benefit will be paid either at $K_x+1/4$, $K_x+2/4$, $K_x+3/4$ or K_x+1 depending on the quarter year in which the death occurred. If the deaths occur evenly over the year (the same assumption as we use in the UDD approach), then, on average, the benefit is paid at time $K_x + 5/8$, which is $3/8$ years earlier than the end of year of death benefit.

Similarly, suppose the benefit is paid at the end of the month of death. Assuming deaths occur uniformly over the year, then on average the benefit is paid at $K_x + 13/24$, which is $11/24$ years earlier than the end year of death benefit.

In general, for an $1/m$thly death benefit, assuming deaths are uniformly distributed over the year of age, the average time of payment of the death benefit is $(m+1)/2m$ in the year of death.

So we have the resulting approximation

$$A_x^{(m)} \approx q_x\, v^{\frac{m+1}{2m}} + {}_1|q_x\, v^{1+\frac{m+1}{2m}} + {}_2|q_x\, v^{2+\frac{m+1}{2m}} + \cdots$$
$$= \sum_{k=0}^{\infty} {}_k|q_x\, v^{k+\frac{m+1}{2m}}$$
$$= (1+i)^{\frac{m-1}{2m}} \sum_{k=0}^{\infty} {}_k|q_x\, v^{k+1}.$$

That is

$$\boxed{A_x^{(m)} \approx (1+i)^{\frac{m-1}{2m}} A_x.} \tag{4.28}$$

For the continuous benefit EPV, $\bar{A}_x$, we let $m \to \infty$ in equation (4.28), to give the approximation

$$\boxed{\bar{A}_x \approx (1+i)^{1/2} A_x.} \tag{4.29}$$

This is explained by the fact that, if the benefit is paid immediately on death, and lives die uniformly through the year, then, on average, the benefit is paid half-way through the year of death, which is half a year earlier than the benefit valued by A_x.

As with the UDD approach, these approximations apply only to death benefits. Hence, for an endowment insurance using the claims acceleration approach we have

$$\boxed{\bar{A}_{x:\overline{n}|} \approx (1+i)^{1/2} A^{\,1}_{x:\overline{n}|} + v^n\, {}_np_x.} \tag{4.30}$$

Note that both the UDD and the claims acceleration approaches give values for $A_x^{(m)}$ or $\bar{A}_x$ such that the ratios $A_x^{(m)}/A_x$ and $\bar{A}_x/A_x$ are independent of x. Note also that for $i = 5\%$, $i/i^{(4)} = 1.0186$ and $i/\delta = 1.0248$, whilst $(1+i)^{3/8} = 1.0185$ and $(1+i)^{1/2} = 1.0247$. The values in Table 4.6 show that both approaches give good approximations in these cases.

4.6 Variable insurance benefits

For all the insurance benefits studied in this chapter the EPV of the benefit can be expressed as the sum over all the possible payment dates of the product of three terms:

- the amount of benefit paid,
- the appropriate discount factor for the payment date, and
- the probability that the benefit will be paid at that payment date.

This approach works for the EPV of any traditional benefit – that is, where the future lifetime is the sole source of uncertainty. It will not generate higher moments or probability distributions.

The approach can be justified technically using **indicator random variables**. Consider a life contingent event E – for example, E is the event that a life aged x dies in the interval $(k, k+1]$. The indicator random variable is

$$I(E) = \begin{cases} 1 & \text{if } E \text{ is true,} \\ 0 & \text{if } E \text{ is false.} \end{cases}$$

In this example, $\Pr[E \text{ is True}] = {}_{k|}q_x$, so the expected value of the indicator random variable is

$$\mathrm{E}[I(E)] = 1({}_{k|}q_x) + 0(1 - {}_{k|}q_x) = {}_{k|}q_x,$$

and, in general, the expected value of an indicator random variable is the probability of the indicator event.

Consider, for example, an insurance that pays \$1000 after 10 years if (x) has died by that time, and \$2000 after 20 years if (x) dies in the second 10-year period, with no benefit otherwise.

We can write the present value random variable as

$$1\,000\, I(T_x \leq 10)v^{10} + 2\,000\, I(10 < T_x \leq 20)v^{20}$$

and the EPV is then

$$1\,000\, {}_{10}q_x\, v^{10} + 2\,000\, {}_{10|10}q_x\, v^{20}.$$

Indicator random variables can also be used for continuous benefits. Here we consider indicators of the form

$$I(t < T_x \leq t + dt)$$

for infinitesimal dt, with associated probability

$$\begin{aligned} \mathrm{E}[I(t < T_x \leq t + dt)] &= \Pr[t < T_x \leq t + dt] \\ &= \Pr[T_x > t]\Pr[T_x < t + dt | T_x > t] \\ &\approx {}_t p_x\, \mu_{x+t}\, dt. \end{aligned}$$

Consider, for example, an increasing insurance policy with a death benefit of T_x payable at the moment of death. That is, the benefit is exactly equal to the number of years lived by an insured life from age x to his or her death. This is a continuous whole life insurance under which the benefit is a linearly increasing function.

To find the EPV of this benefit, we note that the payment may be made at any time, so we consider all the infinitesimal intervals $(t, t + dt)$, and we sum over all these intervals by integrating from $t = 0$ to $t = \infty$.

First, we identify the amount, discount factor and probability for a benefit payable in the interval $(t, t + dt)$. The amount is t, the discount factor is $e^{-\delta t}$. The probability that the benefit is paid in the interval $(t, t+dt)$ is the probability that the life survives from x to $x + t$, and then dies in the infinitesimal interval $(t, t + dt)$, which gives an approximate probability of ${}_tp_x\, \mu_{x+t}\, dt$.

So, we can write the EPV of this benefit as

$$\int_0^\infty t\, e^{-\delta t}\, {}_tp_x\, \mu_{x+t} dt. \tag{4.31}$$

In actuarial notation we write this as $(\bar{I}\bar{A})_x$. The I here stands for 'increasing' and the bar over the I denotes that the increases are continuous.

An alternative approach to deriving equation (4.31) is to identify the present value random variable for the benefit, denoted by Z, say, in terms of the future lifetime random variable,

$$Z = T_x\, e^{-\delta T_x}.$$

Then any moment of Z can be found from

$$\mathrm{E}[Z^k] = \int_0^\infty \left(t\, e^{-\delta t}\right)^k {}_tp_x\, \mu_{x+t}\, dt.$$

The advantage of the first approach is that it is very flexible and generally quick, even for very complex benefits.

If the policy term ceases after a fixed term of n years, the EPV of the death benefit is

$$(\bar{I}\bar{A})^{\ 1}_{x:\overline{n}|} = \int_0^n t\, e^{-\delta t}\, {}_tp_x\, \mu_{x+t} dt.$$

There are a number of other increasing or decreasing benefit patterns that are fairly common. We present several in the following examples.

Example 4.6 Consider an n-year term insurance policy issued to (x) under which the death benefit is $k+1$ if death occurs between ages $x+k$ and $x+k+1$, for $k = 0, 1, 2, \ldots, n-1$. Assume that the benefit is paid at the end of the year of death.

(a) Derive a formula for the EPV of the benefit using the first approach described, that is multiplying together the amount, the discount factor and the probability of payment, and summing for each possible payment date.

(b) Derive a formula for the variance of the present value of the benefit.

Solution 4.6 (a) If the benefit is paid at time $k+1$, the benefit amount is $(k+1)$ and the discount factor is v^{k+1}. The probability that the benefit is paid at that date is the probability that the policyholder died in the year $(k, k+1]$, which is ${}_{k|}q_x$, so the EPV of the death benefit is

$$\sum_{k=0}^{n-1} v^{k+1}(k+1)\, {}_{k|}q_x.$$

In actuarial notation the above EPV is denoted $(IA)^{\;1}_{x:\overline{n}|}$.

If the term n is infinite, so that this is a whole life version of the increasing annual policy, with benefit $k+1$ following death in the year k to $k+1$, the EPV of the death benefit is denoted $(IA)_x$ where

$$(IA)_x = \sum_{k=0}^{\infty} v^{k+1}(k+1)\, {}_{k|}q_x.$$

(b) We must go back to first principles. First, we identify the random variable as

$$Z = \begin{cases} (K_x+1)v^{K_x+1} & \text{if } K_x < n, \\ 0 & \text{if } K_x \geq n. \end{cases}$$

So

$$\mathrm{E}[Z^2] = \sum_{k=0}^{n-1} v^{2(k+1)}(k+1)^2\, {}_{k|}q_x,$$

and the variance is

$$\mathrm{V}[Z] = \sum_{k=0}^{n-1} v^{2(k+1)}(k+1)^2\, {}_{k|}q_x - \left((IA)^{\;1}_{x:\overline{n}|}\right)^2.$$

□

Example 4.7 A whole life insurance policy offers an increasing death benefit payable at the end of the quarter year of death. If (x) dies in the first year of the contract, then the benefit is 1, in the second year it is 2, and so on. Derive an expression for the EPV of the death benefit.

Solution 4.7 First, we note that the possible payment dates are 1/4, 2/4, 3/4, ... Next, if (x) dies in the first year, then the benefit payable is 1, if death occurs in the second year the benefit payable is 2, and so on. Third, corresponding to the possible payment dates, the discount factors are $v^{1/4}$, $v^{2/4}$, ...

The probabilities associated with the payment dates are ${}_{\frac{1}{4}}q_x$, ${}_{\frac{1}{4}|\frac{1}{4}}q_x$, ${}_{\frac{2}{4}|\frac{1}{4}}q_x$, ${}_{\frac{3}{4}|\frac{1}{4}}q_x$, ...

Hence, the EPV, which is denoted $(IA^{(4)})_x$, can be calculated as

$$\begin{aligned}
&{}_{\frac{1}{4}}q_x\, v^{\frac{1}{4}} + {}_{\frac{1}{4}|\frac{1}{4}}q_x\, v^{\frac{2}{4}} + {}_{\frac{2}{4}|\frac{1}{4}}q_x\, v^{\frac{3}{4}} + {}_{\frac{3}{4}|\frac{1}{4}}q_x\, v^{1} \\
&\quad + 2\left({}_{1|\frac{1}{4}}q_x\, v^{1\frac{1}{4}} + {}_{1\frac{1}{4}|\frac{1}{4}}q_x\, v^{1\frac{2}{4}} + {}_{1\frac{2}{4}|\frac{1}{4}}q_x\, v^{1\frac{3}{4}} + {}_{1\frac{3}{4}|\frac{1}{4}}q_x\, v^{2}\right) \\
&\quad + 3\left({}_{2|\frac{1}{4}}q_x\, v^{2\frac{1}{4}} + {}_{2\frac{1}{4}|\frac{1}{4}}q_x\, v^{2\frac{2}{4}} + {}_{2\frac{2}{4}|\frac{1}{4}}q_x\, v^{2\frac{3}{4}} + {}_{2\frac{3}{4}|\frac{1}{4}}q_x\, v^{3}\right) + \cdots \\
&\quad = A^{(4)\,1}_{x:\overline{1|}} + 2\,{}_{1|}A^{(4)\,1}_{x:\overline{1|}} + 3\,{}_{2|}A^{(4)\,1}_{x:\overline{1|}} + \cdots .
\end{aligned}$$

□

We now consider the case when the amount of the death benefit increases in geometric progression. This is important in practice because compound reversionary bonuses will increase the sum insured as a geometric progression.

Example 4.8 Consider an n-year term insurance issued to (x) under which the death benefit is paid at the end of the year of death. The benefit is 1 if death occurs between ages x and $x+1$, $1+j$ if death occurs between ages $x+1$ and $x+2$, $(1+j)^2$ if death occurs between ages $x+2$ and $x+3$, and so on. Thus, if death occurs between ages $x+k$ and $x+k+1$, the death benefit is $(1+j)^k$ for $k = 0, 1, 2, \ldots, n-1$. Derive a formula for the EPV of this death benefit.

Solution 4.8 The amount of benefit is 1 if the benefit is paid at time 1, $(1+j)$ if the benefit is paid at time 2, $(1+j)^2$ if the benefit is paid at time 3, and so on, up to time n. The EPV of the death benefit is then

$$\begin{aligned}
&v\, q_x + (1+j)v^2\, {}_{1|}q_x + (1+j)^2\, v^3\, {}_{2|}q_x + \cdots + (1+j)^{n-1}\, v^n\, {}_{n-1|}q_x \\
&\quad = \sum_{k=0}^{n-1} v^{k+1}(1+j)^k\, {}_{k|}q_x \\
&\quad = \frac{1}{1+j}\sum_{k=0}^{n-1} v^{k+1}(1+j)^{k+1}\, {}_{k|}q_x \\
&\quad = \frac{1}{1+j} A^{\;1}_{x:\overline{n|}\,i^*} \qquad (4.32)
\end{aligned}$$

where

$$i^* = \frac{1+i}{1+j} - 1 = \frac{i-j}{1+j}.$$

□

The notation $A^{\;1}_{x:\overline{n}|\,i^*}$ indicates that the EPV is calculated using the rate of interest i^*, rather than i. In most practical situations, $i > j$ so that $i^* > 0$.

Example 4.9 Consider an insurance policy issued to (x) under which the death benefit is $(1 + j)^t$ if death occurs at age $x + t$, with the death benefit being payable immediately on death.

(a) Derive an expression for the EPV of the death benefit if the policy is an n-year term insurance.
(b) Derive an expression for the EPV of the death benefit if the policy is a whole life insurance.

Solution 4.9 (a) The present value of the death benefit is $(1 + j)^{T_x} v^{T_x}$ if $T_x < n$, and is zero otherwise, so that the EPV of the death benefit is

$$\int_0^n (1+j)^t v^t \; {}_tp_x \mu_{x+t} dt = \bar{A}^{\;1}_{x:\overline{n}|\,i^*}$$

where

$$i^* = \frac{1+i}{1+j} - 1.$$

(b) Similarly, if the policy is a whole life insurance rather than a term insurance, then the EPV of the death benefit would be

$$\int_0^\infty (1+j)^t v^t \; {}_tp_x \mu_{x+t} dt = (\bar{A}_x)_{\,i^*}$$

where

$$i^* = \frac{1+i}{1+j} - 1.$$

□

4.7 Functions for select lives

Throughout this chapter we have developed results in terms of lives subject to ultimate mortality. We have taken this approach simply for ease of presentation. All of the above development equally applies to lives subject to select mortality.

For example, $\bar{A}_{[x]}$ denotes the EPV of a benefit of 1 payable immediately on the death of a select life (x). Similarly, $A_{[x]:\overline{n}|}$ denotes the EPV of a benefit of 1 payable at the end of the year of death within n years, of a newly selected life age x, or at age $x + n$ if (x) survives.

4.8 Notes and further reading

The Standard Ultimate Survival Model incorporates Makeham's law as its survival model. A feature of Makeham's law is that we can integrate the force of mortality analytically and hence we can evaluate, for example, ${}_tp_x$ analytically, as in Exercise 2.11. This in turn means that the EPV of an insurance benefit payable immediately on death, for example $\bar{A}_x$, can be written as an integral where the integrand can be evaluated directly, as follows

$$\bar{A}_x = \int_0^\infty e^{-\delta t}\, {}_tp_x\, \mu_{x+t}\, dt.$$

This integral cannot be evaluated analytically but can be evaluated numerically. In many practical situations, the force of mortality cannot be integrated analytically, for example if μ_x is a GM(r, s) function with $s \geq 2$, from Section 2.7. In such cases, ${}_tp_x$ can be evaluated numerically but not analytically. Functions such as $\bar{A}_x$ can still be evaluated numerically but, since the integrand has to be evaluated numerically, the procedure may be a little more complicated. See Exercise 4.22 for an example. The survival model in Exercise 4.22 has been derived from data for UK whole life and endowment insurance policyholders (non-smokers), 1999–2002. See CMI (2006, Table 1).

4.9 Exercises

Exercise 4.1 You are given the following table of values for l_x and A_x, assuming an effective interest rate of 6% per year.

x	l_x	A_x
35	100 000.00	0.151375
36	99 737.15	0.158245
37	99 455.91	0.165386
38	99 154.72	0.172804
39	98 831.91	0.180505
40	98 485.68	0.188492

Calculate

(a) ${}_5E_{35}$,

(b) $A^{\;1}_{35:\overline{5|}}$,

(c) ${}_{5|}A_{35}$, and

(d) $\bar{A}_{35:\overline{5|}}$ assuming UDD.

Exercise 4.2 Using tables in Appendix D and interest at 5% per year effective, calculate

(a) $A^{\;1}_{30:\overline{20|}}$,
(b) $\bar{A}_{40:\overline{20|}}$ assuming UDD,
(c) ${}_{10|}A_{25}$.

Exercise 4.3 Assuming a uniform distribution of deaths over each year of age, show that $A_x^{(m)} = (i/i^{(m)})A_x$.

Exercise 4.4 A whole life insurance policy issued to a life aged exactly 30 has an increasing sum insured. In the tth policy year, $t = 1, 2, 3, \ldots$, the sum insured is \$100 000 (1.03^{t-1}). Using the Standard Ultimate Survival Model, with interest at 5% per year, calculate the EPV of this benefit.

Exercise 4.5 (a) Show that

$$A_{x:\overline{n|}} = \sum_{k=0}^{n-2} v^{k+1}\,{}_{k|}q_x + v^n\,{}_{n-1}p_x.$$

(b) Compare this formula with formula (4.17) and comment on the differences.

Exercise 4.6 Show that

$$(IA^{(m)})_x = A_x^{(m)} + v p_x A_{x+1}^{(m)} + v^2\,{}_2p_x A_{x+2}^{(m)} + \cdots$$

and explain this result intuitively.

Exercise 4.7 (a) Derive the following recursion formula for an n-year increasing term insurance:

$$(IA)^{\;1}_{x:\overline{n|}} = vq_x + vp_x\left((IA)^{\;\;1}_{x+1:\overline{n-1|}} + A^{\;\;1}_{x+1:\overline{n-1|}}\right).$$

(b) Give an intuitive explanation of the formula in part (a).
(c) You are given that $(IA)_{50} = 4.99675$, $A^{\;1}_{50:\overline{1|}} = 0.00558$, $A_{51} = 0.24905$ and $i = 0.06$. Calculate $(IA)_{51}$.

Exercise 4.8 You are given that $A_x = 0.25$, $A_{x+20} = 0.40$, $A_{x:\overline{20|}} = 0.55$ and $i = 0.03$. Calculate $10\,000\bar{A}_{x:\overline{20|}}$ using

(a) claims acceleration, and
(b) UDD.

Exercise 4.9 Show that

$$(IA)^{1}_{x:\overline{n}|} = (n+1)A^{1}_{x:\overline{n}|} - \sum_{k=1}^{n} A^{1}_{x:\overline{k}|}$$

and explain this result intuitively.

Exercise 4.10 Assuming a uniform distribution of deaths over each year of age, find an expression for $(\bar{I}\bar{A})_x$ in terms of A_x and $(IA)_x$.

Exercise 4.11 Show that $\bar{A}_x$ is a decreasing function of i, and explain this result by general reasoning.

Exercise 4.12 Calculate A_{70} given that

$$A_{50:\overline{20}|} = 0.42247, \quad A^{1}_{50:\overline{20}|} = 0.14996, \quad A_{50} = 0.31266.$$

Exercise 4.13 Under an endowment insurance issued to a life aged x, let X denote the present value of a unit sum insured, payable at the moment of death or at the end of the n-year term.

Under a term insurance issued to a life aged x, let Y denote the present value of a unit sum insured, payable at the moment of death within the n-year term.

Given that

$$\mathrm{V}[X] = 0.0052, \quad v^n = 0.3, \quad {}_np_x = 0.8, \quad \mathrm{E}[Y] = 0.04,$$

calculate $\mathrm{V}[Y]$.

Exercise 4.14 Show that if $\nu_y = -\log p_y$ for $y = x, x+1, x+2, \ldots$, then under the assumption of a constant force of mortality between integer ages,

$$\bar{A}_x = \sum_{t=0}^{\infty} v^t \, {}_tp_x \, \frac{\nu_{x+t}(1 - vp_{x+t})}{\delta + \nu_{x+t}}.$$

Exercise 4.15 Let Z_1 denote the present value of an n-year term insurance benefit, issued to (x). Let Z_2 denote the present value of a whole of life insurance benefit, issued to the same life.

Express the covariance of Z_1 and Z_2 in actuarial functions, simplified as far as possible.

Exercise 4.16 You are given the following excerpt from a select life table.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	$x+4$
[40]	100 000	99 899	99 724	99 520	99 288	44
[41]	99 802	99 689	99 502	99 283	99 033	45
[42]	99 597	99 471	99 268	99 030	98 752	46
[43]	99 365	99 225	99 007	98 747	98 435	47
[44]	99 120	98 964	98 726	98 429	98 067	48

Assuming an interest rate of 6% per year, calculate

(a) $A_{[40]+1:\overline{4}|}$,
(b) the standard deviation of the present value of a four-year term insurance, deferred one year, issued to a newly selected life aged 40, with sum insured \$100 000, payable at the end of the year of death, and
(c) the probability that the present value of the benefit described in (b) is less than or equal to \$85 000.

Exercise 4.17 (a) Describe **in words** the insurance benefits with the present values given below.

(i) $$Z_1 = \begin{cases} 20\, v^{T_x} & \text{if } T_x \le 15, \\ 10\, v^{T_x} & \text{if } T_x > 15. \end{cases}$$

(ii) $$Z_2 = \begin{cases} 0 & \text{if } T_x \le 5, \\ 10\, v^{T_x} & \text{if } 5 < T_x \le 15, \\ 10\, v^{15} & \text{if } T_x > 15. \end{cases}$$

(b) Write down in integral form the formula for the expected value for (i) Z_1 and (ii) Z_2.
(c) Derive expressions in terms of standard actuarial functions for the expected values of Z_1 and Z_2.
(d) Derive an expression in terms of standard actuarial functions for the covariance of Z_1 and Z_2.

Exercise 4.18 (a) Describe in words the insurance benefits with present value given by

$$Z = \begin{cases} T_{30}\, v^{T_{30}} & \text{if } T_{30} \le 25, \\ 25\, v^{T_{30}} & \text{if } T_{30} > 25. \end{cases}$$

(b) Write down an expression in terms of standard actuarial functions for E[Z].

Exercise 4.19 Using the tables in Appendix D and interest at 5% per year effective, calculate the standard deviation of the present value of a payment of \$100 000 at the end of the year of death of a life now aged 30 who is subject to ultimate mortality, if payment is contingent on death occurring

(a) at any age, and

(b) before age 50.

Exercise 4.20 Suppose that Makeham's law applies with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$. Assume also that the effective rate of interest is 6% per year.

(a) Use Excel and backward recursion in parts (i) and (ii).
   (i) Construct a table of values of A_x for integer ages, starting at $x = 50$.
   (ii) Construct a table of values of $A_x^{(4)}$ for $x = 50, 50.25, 50.5, \ldots$. (Do not use UDD for this.)
   (iii) Hence, write down the values of A_{50}, A_{100}, $A_{50}^{(4)}$ and $A_{100}^{(4)}$.

(b) Use your values for A_{50} and A_{100} to estimate $A_{50}^{(4)}$ and $A_{100}^{(4)}$ using the UDD assumption.

(c) Compare your estimated values for the $A^{(4)}$ functions (from (b)) with your accurate values (from (a)). Comment on the differences.

Exercise 4.21 A life insurance policy issued to a life aged 50 pays \$2000 at the end of the quarter year of death before age 65 and \$1000 at the end of the quarter year of death after age 65. Use the Standard Ultimate Survival Model, with interest at 5% per year, in the following.

(a) Calculate the EPV of the benefit.

(b) Calculate the standard deviation of the present value of the benefit.

(c) The insurer charges a single premium of \$500. Assuming that the insurer invests all funds at exactly 5% per year effective, what is the probability that the policy benefit has greater value than the accumulation of the single premium?

Exercise 4.22 The force of mortality for a survival model is given by

$$\mu_x = A + BC^x D^{x^2},$$

where

$$A = 3.5 \times 10^{-4}, \; B = 5.5 \times 10^{-4}, \; C = 1.00085, \; D = 1.0005.$$

(a) Calculate ${}_t p_{60}$ for $t = 0, 1/40, 2/40, \ldots, 2$.
Hint: Use the repeated Simpson's rule.

(b) Calculate $\bar{A}^{\,1}_{60:\overline{2|}}$ using an effective rate of interest of 5% per year.
Hint: Use the repeated Simpson's rule.

Answers to selected exercises

4.1 (a) 0.735942
(b) 0.012656
(c) 0.138719
(d) 0.748974

4.2 (a) 0.00645
(b) 0.38163
(c) 0.05907

4.4 \$33 569.47

4.7 (c) 5.07307

4.8 (a) 5 507.44
(b) 5 507.46

4.12 0.59704

4.13 0.01

4.16 (a) 0.79267
(b) \$7519.71
(c) 0.99825

4.19 (a) 7 186
(b) 6 226

4.20 (a) (iii) 0.33587, 0.87508, 0.34330, 0.89647
(b) 0.34333, 0.89453

4.21 (a) \$218.83
(b) \$239.73
(c) 0.04054

4.22 (a) Selected values are ${}_{1/4}p_{60} = 0.999031$, $p_{60} = 0.996049$ and ${}_{2}p_{60} = 0.991903$
(b) 0.007725