Linear Differential Equations of Higher Order

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Linear Differential Equations of Higher Order

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**General Solution of Homogeneous Linear Differential Equations**

**Definition**

The general linear differential equations of order \( n \) is an equation that can be written

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x),
\]

where \( a_n(x), a_{n-1}(x), a_1(x) \) and \( a_0(x) \) are functions of \( x \in I = (a, b) \), and they are called **coefficients**.

Equation (1) is called homogeneous linear differential equation if the function \( g(x) \) is zero for all \( x \in (a, b) \).

If \( g(x) \) is not equal to zero on \( I \), the equation (1) is called non-homogeneous linear differential equation.
Initial-Value Problem (IVP)

An \( n \)-th order initial-value problem associate with (1) takes the form:

Solve:

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x),
\]

subject to:

\[
y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \quad \ldots, \quad y^{n-1}(x_0) = y_{n-1}.
\]  

(2)

Here (2) is a set of initial conditions.
Boundary-Value Problem (BVP)

Remark (Initial vs. Boundary Conditions)
Initial Conditions: all conditions are at the same $x = x_0$. Boundary Conditions: conditions can be at different $x$.

Remark (Number of Initial/Boundary Conditions)
Usually a $n$-th order ODE requires $n$ initial/boundary conditions to specify an unique solution.

Remark (Order of the derivatives in the conditions)
Initial/boundary conditions can be the value or the function of 0-th to $(n - 1)$-th order derivatives, where $n$ is the order of the ODE.
Consider the following second-order ODE

\[ a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \quad (3) \]

- IVP: Solve (3) s.t. \( y(x_0) = y_0; y'(x_0) = y_1 \).
- BVP: Solve (3) s.t. \( y(a) = y_0; y(b) = y_1 \).
- BVP: Solve (3) s.t. \( y(b) = y_0; y(a) = y_1 \).
Existence and Uniqueness of the Solution to an IVP

Theorem

For the given linear differential equations of order $n$

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x), \quad (4)$$

which is normal on an interval $I$. Subject to

$$y(x_0) = y_0, \ y'(x_0) = y_1, \ y''(x_0) = y_2, \ \ldots, \ y^{n-1}(x_0) = y_{n-1}. \quad (5)$$

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $R(x)$ are all continuous on an interval $I$, $a_n(x)$ is not a zero function on $I$, and the initial point $x_0 \in I$, then the above IVP has a unique solution in $I$. 

Example (1)

Discuss the Existence of unique solution of IVP

\[
\begin{aligned}
(x^2 + 1)y'' + x^2y' + 5y &= \cos(x) \\
y(3) &= 2, \quad y'(3) = 1.
\end{aligned}
\]

Solution

The functions

\[ a_2(x) = x^2 + 1, \quad a_1(x) = x^2, \quad a_0(x) = 5. \]

and

\[ R(x) = \cos(x). \]

are continuous on \( I = \mathbb{R} = (-\infty, \infty) \) and \( a_2(x) \neq 0 \) for all \( x \in \mathbb{R} \), the point \( x_0 = 3 \in I \). Then the previous Theorem assures that the IVP has a unique solution on \( \mathbb{R} \).
Example (2)

Find an interval \( I \) for which the initial values problem (IVP)

\[
\begin{align*}
\left\{ \begin{array}{l}
x^2 y'' + \frac{x}{\sqrt{2-x}} y' + \frac{2}{\sqrt{x}} y = 0 \\
y(1) = 0, \quad y'(1) = 1.
\end{array} \right.
\end{align*}
\]

has a unique solution around \( x_0 = 1 \).

Solution

The function

\[ a_2(x) = x^2, \]

is continuous on \( \mathbb{R} \) and \( a_2(x) \neq 0 \) if \( x > 0 \) or \( x < 0 \). But \( x_0 = 1 \in I_1 = (0, \infty) \). The function

\[ a_1(x) = \frac{x}{\sqrt{2-x}}, \]

is continuous on \( I_2 = (-\infty, 2) \) and the function

\[ a_0(x) = \frac{2}{\sqrt{x}}, \]
is continuous on $I_1 = (0, \infty)$.

Then the (IVP ) has a unique solution on $I_1 \cap I_2 = (0, 2) = I$. We can take any interval $I_3 \subset (0, 2)$ such that $x_0 = 1 \in I_3$. So $I$ is that the largest interval for which the (IVP) has a unique solution.
Example (3)

Find an interval $I$ for which the IVP

\[
\begin{align*}
(x - 1)(x - 3)y'' + xy' + y &= x^2 \\
y(2) &= 1, \quad y'(2) = 0
\end{align*}
\]

has a unique solution about $x_0 = 2$.

Solution  The functions

\[
a_2(x) = (x - 1)(x - 3) , \quad a_1(x) = x , \quad a_0(x) = 1 , \quad R(x) = x^2,
\]

are continuous on $\mathbb{R}$. But $a_2(x) \neq 0$ if $x \in (-\infty, 1)$ or $x \in (1, 3)$ or $x \in (3, \infty)$. As $x_0 = 2$ so we take $I = (1, 3)$. Then the IVP has a unique solution on $I = (1, 3)$.
Exercises

1. Discuss the Existence of unique solution of IVP

\[
\begin{cases}
(|x| + 3)y'' + x^3y' + 5y = \sin(x) \\
y(2) = 1, \quad y'(2) = 0.
\end{cases}
\]

2. Find an interval \( I \) for which the IVP

\[
\begin{cases}
(x - 2)y'' + 3y = x \\
y(0) = 0, \quad y'(0) = 1.
\end{cases}
\]

has a unique solution about \( x_0 = 0 \).

3. Find an interval \( I \) for which the IVP

\[
\begin{cases}
y'' + (\tan x)y = e^x \\
y(0) = 1, \quad y'(0) = 0.
\end{cases}
\]

has a unique solution about \( x_0 = 0 \).
Linear Dependence and Independence of Functions

**Definition**

A set of functions \( \{f_1(x), f_2(x), \ldots, f_n(x)\} \) are **linearly dependent** on an interval \( I \) if \( \exists c_1, c_2, \ldots, c_n \) not all zero i.e. \( (c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0) \) such that

\[
c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall x \in I
\]

that is, the linear combination is a zero function.

If the set of functions is not linearly dependent, it is **linearly independent**, i.e. when \( c_1, c_2, \ldots, c_n \) all zero i.e. \( (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0) \).
Example (1)

Show that \( f_1(x) = \cos(2x) \), \( f_2(x) = 1 \), \( f_3(x) = \cos^2(x) \) are linearly dependent on \( \mathbb{R} \).

Solution  We know that

\[
 f_3(x) = \cos^2(x) = \frac{1 + \cos(2x)}{2} = \frac{1}{2} f_2(x) + \frac{1}{2} f_1(x)
\]

for all \( x \in \mathbb{R} \). Then there exist \( c_1 = c_2 = \frac{1}{2} \) and \( c_3 = -1 \) such that

\[
 c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x \in \mathbb{R}.
\]

So \( f_1 \), \( f_2 \) and \( f_3 \) are linearly dependent on \( \mathbb{R} \).
Example (2)

Show that

\[ f_1(x) = 1, \quad f_2(x) = \sec^2(x) \quad \text{and} \quad f_3(x) = \tan^2(x) \]

are linearly dependent on \((0, \frac{\pi}{2})\).

Solution  We know that

\[ f_2(x) = \sec^2(x) = 1 + \tan^2(x) = f_1(x) + f_3(x) \]

hence

\[ f_1(x) - f_2(x) + f_3(x) = 0 \quad \text{for all} \quad x \in \left(0, \frac{\pi}{2}\right). \]

So there exist \(c_1 = c_3 = 1\) and \(c_2 = -1\) such that

\[ c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all} \quad x \in \left(0, \frac{\pi}{2}\right). \]

So \(f_1, f_2\) and \(f_3\) are linearly dependent on \((0, \frac{\pi}{2})\).
Example (3)

Show that \( f_1(x) = x \) and \( f_2(x) = x^2 \) are linearly independent on \( I = [-1, 1] \).

Solution  Let \( c_1, c_2 \in \mathbb{R} \) such that

\[
c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } x \in I.
\]

We have to prove that \( c_1 = c_2 = 0 \). As

\[
c_1 x + c_2 x^2 = 0 \quad \text{for all } -1 \leq x \leq 1,
\]

then for \( x = 1 \) and \( x = -\frac{1}{2} \) we have

\[
c_1 + c_2 = 0,
\]

and

\[
-\frac{1}{2}c_1 + \frac{1}{4}c_2 = 0,
\]
which implies that $c_1 = c_2 = 0$. Then $f_1$ and $f_2$ are linearly independent on $I$. 
Example (4)

Show that

\[ f_1(x) = \sin(x), \quad f_2(x) = \sin(2x). \]

are linearly independent on \( I = [0, \pi) \).

Solution  \ Let \( c_1, c_2 \in I \) such that

\[ c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all} \ x \in I. \]

We have to show that \( c_1 = c_2 = 0 \). In fact for \( x = \frac{\pi}{4} \) and \( x = \frac{\pi}{3} \) we have

\[
\begin{align*}
\left\{ \begin{array}{l}
c_1 \sin\left( \frac{\pi}{4} \right) + c_2 \sin\left( \frac{\pi}{2} \right) = 0 \\
c_1 \sin\left( \frac{\pi}{3} \right) + c_2 \sin\left( 2 \frac{\pi}{3} \right) = 0
\end{array} \right.
\end{align*}
\]

hence

\[
\frac{1}{\sqrt{2}} c_1 + c_2 = 0, \quad \frac{\sqrt{3}}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 0,
\]
which implies that \( c_1 = c_2 = 0 \). Then \( f_1 \) and \( f_2 \) are linearly independent on \( I \).
Example (5)

Show that

\[ f_1(x) = x^2 \quad \text{and} \quad f_2(x) = x |x| \]

(i) linearly dependent on \([0, 1]\)
(ii) linearly independent on \([-1, 1]\)

Solution

(i) on \([0, 1]\) we have

\[ f_1(x) = f_2(x) = x^2, \]

hence

\[ f_1(x) - f_2(x) = 0 \quad \text{for all} \quad 0 \leq x \leq 1. \]

So there exist \(c_1 = 1, c_2 = -1\) such that

\[ c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all} \quad 0 \leq x \leq 1. \]

Then \(f_1\) and \(f_2\) are linearly dependent on \([0, 1]\).
(ii) Let $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all } -1 \leq x \leq 1,$$

hence

$$c_1 x^2 + c_2 x |x| = 0 \quad \text{for all } -1 \leq x \leq 1.$$

Now for $x = 1$ and $x = -1$ we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which implies that $c_1 = c_2 = 0$. Then $f_1$ and $f_2$ are linearly independent on $[-1, 1]$. 

MATH204-Differential Equations
Exercises

1. Determine whether the functions

\[ f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 4x - x^2. \]

are linearly dependent or independent on \((-\infty, \infty)\).

2. Determine whether the functions

\[ f_1(x) = e^x, \quad f_2(x) = e^{-x}, \quad f_3(x) = \cosh x. \]

are linearly dependent or independent on \((-\infty, \infty)\).

3. Determine whether the functions

\[ f_1(x) = x, \quad f_2(x) = x^2 - 1, \quad f_3(x) = x^2 + 2x + 1. \]

are linearly dependent or independent on \([0, 1]\).
Criterion of Linearly Independent Solutions

Consider the homogeneous linear $n$-th order DE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0,$$

Given $n$ solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, we would like to test if they are independent or not.

Note: In Linear Algebra, to test if $n$ vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent, we can compute the determinant of the matrix.

$$V := [v_1 \ v_2 \ \ldots \ v_n].$$

If the determinant of $V = 0$, they are linearly dependent; if the determinant of $V \neq 0$, they are linearly independent.
**Definition**

For $n$ functions $W(f_1, f_2, \ldots, f_n)$ which are $n - 1$ times differentiable on an interval $I$, the **Wronskian** $W(x, f_1, f_2, \ldots, f_n)$ as a function on $I$ is defined by

\[
W(x, f_1, f_2, \ldots, f_n) = \begin{vmatrix}
 f_1 & f_2 & \cdots & f_n \\
 f_1' & f_2' & \cdots & f_n' \\
 f_1'' & f_2'' & \cdots & f_n'' \\
 \vdots & \vdots & \ddots & \vdots \\
 f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1}
\end{vmatrix}
\]
To test the linear independence of $n$ solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ to

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \quad (6)$$

we can use the following theorem.

**Theorem**

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be $n$ solutions to the homogeneous linear DE (6) on an interval $I$. They are **linearly independent** on $I$ if and only if

$$W(x, f_1, f_2, \ldots, f_n) := \begin{vmatrix}
    f_1 & f_2 & \cdots & f_n \\
    f_1' & f_2' & \cdots & f_n' \\
    f_1'' & f_2'' & \cdots & f_n'' \\
    \vdots & \vdots & \cdots & \vdots \\
    f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1}
\end{vmatrix} \neq 0.$$
Example (1)

Prove that \( f_1(x) = x^2, \; f_2(x) = x^2 \ln(x) \) are linearly independent on \((0, \infty)\).

Solution  We have that

\[
W(f_1, f_2) = \begin{vmatrix} x^2 & x^2 \ln(x) \\ 2x & 2x \ln(x) + x \end{vmatrix} = 2x^3 \ln(x) + x^3 - 2x^3 \ln(x) = x^3 \neq 0
\]

for all \( x \in (0, \infty) \),

then \( f_1 \) and \( f_2 \) are linearly independent on \((0, \infty)\).
Example (2)

It is easy to see that the functions

\[ y_1 = x, \quad y_2 = x^2, \]

and

\[ y_3 = x^3. \]

are solutions of the differential equation

\[ x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0. \]

Show that \( y_1, y_2 \) and \( y_3 \) are linearly independent on \((0, \infty)\).

**Solution**  Here we have \( a_3(x) = x^3 \neq 0 \) for all \( x > 0 \) or \( x < 0 \). By using the Wronskian we have

\[
W(y_1, y_2, y_3) = \begin{vmatrix}
1 & 2x & 3x^2 \\
0 & 2 & 6x \\
x & x^2 & x^3
\end{vmatrix} = 2x^3 \neq 0.
\]
for all $x \in (0, \infty)$ or for all $x \in (-\infty, 0)$. So $y_1$, $y_2$ and $y_3$ are linearly independent on $(0, \infty)$. 
Exercises

1. Show by computing the Wronskian that the functions

\[ f_1(x) = x, \quad f_2(x) = xe^x, \quad f_3(x) = x^2e^x. \]

are linearly dependent or independent on \((0, \infty)\).

2. Show that the functions

\[ y_1 = \cosh(2x) \]

and

\[ y_2 = \sinh(2x) \]

are solutions of the differential equation

\[ y'' - 4y' = 0. \]

Show that \(y_1\) and \(y_2\) are linearly independent on \((-\infty, \infty)\).
Fundamental Set of Solutions

Definition

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0, \quad (7) \]

Any set \( \{f_1(x), f_2(x), \ldots, f_n(x)\} \) of \( n \) linearly independent solutions to the homogeneous linear \( n \)-th order DE (7) on an interval \( I \) is called a fundamental set of solutions.

Theorem

Let \( \{f_1(x), f_2(x), \ldots, f_n(x)\} \) be a fundamental set of solutions to the homogeneous linear \( n \)-th order DE (7) on an interval \( I \). Then the general solution to (7) is

\[ y(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \]

where \( \{c_i \mid (i = 1, 2, \ldots, n)\} \) are arbitrary constants.
Example (1)

Verify that $y_1 = e^{2x}$ and $y_2 = e^{-3x}$ form a fundamental set of solutions of the differential equation

$$y'' + y' - 6y = 0.$$ 

and find the general solution.

Solution

Substituting

$$y_1 = e^{2x}, \quad y_1' = 2e^{2x}, \quad y_1'' = 4e^{2x},$$

in the differential equation we have

$$4e^{2x} + 2e^{2x} - 6e^{2x} = 0.$$ 

Hence $y_1 = e^{2x}$ is a solution of the differential equation. By the same method we can prove that $y_2 = e^{-3x}$ is also a solution of the differential equation.
Now we have

\[ W(e^{2x}, e^{-3x}) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -5e^{-x} \neq 0 \text{ for all } x \in \mathbb{R}. \]

Then \( y_1 \) and \( y_2 \) are linearly independent on \( \mathbb{R} \). From the previous Theorem we deduce the general solution of the differential equation given by

\[ y(x) = c_1 y_1(x) + c_2 y_2(x). \]

where \( c_1, c_2 \in \mathbb{R} \).
Example (2)

It is easy to see that the functions

\[ y_1 = e^x, y_2 = e^{2x}, \text{ and } y_3 = e^{3x} \]

are solutions of the differential equation

\[ y''' - 6y'' + 11y' - 6y = 0. \]

Find the general solution of the differential equation.

Solution  Since

\[ W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0. \]

for all \( x \in \mathbb{R} \).
We deduce that

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$  

is the general solution of the differential equation.
Example (3)

Prove that

\[ y_1 = x^3 e^x, \quad \text{and} \quad y_2 = e^x. \]

are solutions of the differential equation

\[ xy'' - 2(x + 1)y' + (x + 2)y = 0 \]

where \( x > 0 \). Find also the general solution of the differential equation.

Solution

Substituting

\[ y_1 = x^3 e^x, \quad y_1' = 3x^2 e^x + x^3 e^x, \quad y_1'' = 6xe^x + 6x^2 e^x + x^3 e^x, \]

in the differential equation we have

\[ 6x^2 e^x + 6x^3 e^x + x^4 e^x - 6x^3 e^x - 2x^4 e^x e^x - 6x^2 e^x + -2x^3 e^x + x^4 e^x + 2x^3 e^x = 0. \]

Substituting

\[ y_2 = y_2' = y_2'' = e^x, \]
in the differential equation

\[ xe^x - 2xe^x - 2e^x + xe^x + 2e^x = 0. \]

Now we have to show that

\[ y_1 = x^3 e^x, \]

and

\[ y_2 = e^x. \]

are linearly independent on \((0, \infty)\).

In fact

\[
W(x^3 e^x, e^x) = \begin{vmatrix} x^3 e^x & e^x \\ 3x^2 e^x + x^3 e^x & e^x \end{vmatrix} = -3x^2 e^x \neq 0 \text{ for all } x > 0.
\]

Then

\[ y_1 = x^3 e^x, \]
and

\[ y_2 = e^x. \]

are linearly independent on \((0, \infty)\) and we conclude that

\[ y_c = c_1 x^3 e^x + c_2 e^x. \]

is the general solution of the differential equation.
Exercises

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval, then find the general solution of the differential equation.

- \( y'' - y' - 12y = 0; \quad e^{-3x}, \quad e^{4x} \text{ on } (-\infty, \infty) \)
- \( x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0; \quad x, \quad x^{-2}, \quad x^{-2} \ln x \text{ on } (0, \infty) \)
- \( y^{(4)} + y'' = 0; \quad 1, \quad x, \quad \sin x, \quad \cos x \text{ on } (0, \infty) \)
Reduction of order Method (when one solution is given)

It is employed when one solution $y_1(x)$ is known and a second linearly independent solution $y_2(x)$ is desired. The method also applies to $n$-th order equations.

Suppose that $y_1(x)$ is a non-zero solution of the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$  \hspace{1cm} (8)

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous functions defined on interval $I$ such that $a_2(x) \neq 0$ for all $x \in I$.

The method of reduction of order is used to obtain a second linearly independent $y_2(x)$ solution to this differential equation (8) using our one known solution.
We suppose that the solution of (8) is in the form

\[ y = u(x)y_1, \]

where \( u \) is a fountain of \( x \) and which will be determined and satisfies a linear second-order differential equation (8) by using the following method

\[ y = u(x)y_1 \Rightarrow y' = u'y_1 + y_1u \Rightarrow y'' = u''y_1 + 2u'y_1 + y_1u. \]

It is best to describe the procedure with a concrete example.
Example (1)

If

\[ y_1 = \frac{\sin x}{\sqrt{x}}. \]

is a solution of the differential equation

\[ 4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \quad \text{on} \quad 0 < x < \pi. \]

then find the general solution of the differential equation.

Solution

The solution of the differential equation is of the form

\[ y = u(x)y_1 \quad \text{or} \quad y = \frac{\sin x}{\sqrt{x}} u = (\sin x)(x)^{-\frac{1}{2}} u, \]

hence

\[ y' = (\cos x)(x)^{-\frac{1}{2}} u - \frac{1}{2} \sin x(x)^{-\frac{3}{2}} u + \sin x(x)^{-\frac{1}{2}} u', \]
\[
\begin{align*}
y'' &= -\sin x(x)^{-\frac{1}{2}}u - \cos x(x)^{-\frac{3}{2}}u + 2\cos x(x)^{-\frac{1}{2}}u' \\
&\quad + \frac{3}{4}\sin x(x)^{-\frac{5}{2}}u - \sin x(x)^{-\frac{3}{2}}u' + \sin x(x)^{-\frac{1}{2}}u''
\end{align*}
\]

we substitute \(y\), \(y'\), and \(y''\) in the arbitrary constant we obtain

\[
4x^\frac{3}{2}\sin xu'' + \left(8x^\frac{3}{2}\cos x\right)u' = 0,
\]

hence

\[
\sin xu'' + 2\cos xu' = 0.
\]

To solve this differential equation we put \(w = u'\), then we have \(w' = u''\).

Then

\[
\int \frac{dw}{w} dx + \int \frac{2\cos x}{\sin x} dx = 0,
\]
hence

\[ u' = w = \frac{c_1}{\sin^2 x}, \]

where \( c_1 \neq 0 \) is an arbitrary constant. So we have \( u = -c_1 \cot x + c_2 \), hence

\[ y = y_1 u = \frac{\sin x}{\sqrt{x}} (-c_1 \cot x + c_2), \]

or

\[ y = c_3 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}, \]

finally we have

\[ y = c_2 y_1 + c_3 y_2, \]

where \( c_3 = -c_1 \) and \( c_2 \) are arbitrary constants, is the general solution of the differential equation and we can prove that

\[ y_1 = \frac{\sin x}{\sqrt{x}} \text{ and } y_2 = \frac{\cos x}{\sqrt{x}} \]

are linearly independent on solutions \((0, \pi)\).
General case of Equation (8)

Equation

\[ a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \]

can be written as the form

\[ y'' + p(x)y' + q(x)y = 0, \] \hspace{1cm} (9)

where

\[ p(x) = \frac{a_1(x)}{a_2(x)}, \]

and

\[ q(x) = \frac{a_0(x)}{a_2(x)}. \]

Also, let us suppose that \( y_1 \) is a known solution of (9) on \( I \) and \( y_1(x) \neq 0 \) for all \( x \in I \).
Thus the second solution of (9) $y_2$ can be given from

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} \, dx.$$  \hspace{1cm} (10)
Example (1)

If
\[ y_1 = \frac{\sin x}{\sqrt{x}} \]
is a solution of the differential equation
\[ 4x^2 y'' + 4xy' + (4x^2 - 1)y = 0 \quad \text{on} \quad (0, \pi), \]
then find the second solution.

Solution  As
\[ y'' + \frac{1}{x}y' + \frac{4x^2 - 1}{4x^2}y = 0. \]
then
\[ p(x) = \frac{1}{x}, \]
and
\[ e^{-\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}. \]
We have

\[ y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} \frac{dx}{y_1} = \frac{\sin x}{\sqrt{x}} \int \frac{1}{x} \frac{dx}{\sin^2(x)} \]

\[ = \frac{\sin x}{\sqrt{x}} \int \frac{dx}{\sin^2(x)} \]

\[ = -\frac{\cos x}{\sqrt{x}}. \]

Hence

\[ y_2 = -\frac{\cos x}{\sqrt{x}} \quad \text{or} \quad y_2 = \frac{\cos x}{\sqrt{x}} \]

is the second solution of the differential equation on \((0, \pi)\).
Example (2)

If \( y_1 = e^{3x} \) is a solution of the differential equation

\[
xy'' + (x - 1)y' + (3 - 12x)y = 0 \quad ; \quad x > 0.
\]

Find the general solution.

Solution  We have

\[
y'' + \left(1 - \frac{1}{x}\right)y' + \left(\frac{3}{x} - 12\right)y = 0.
\]

From the formula (10) we can find directly \( y_2 \), where

\[
\int -p(x)dx = \int \left(-1 + \frac{1}{x}\right)dx = -x + \ln x,
\]
hence

\[ y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} \, dx = e^{3x} \int \frac{e^{-x + \ln x}}{e^{6x}} \, dx = e^{3x} \int xe^{-7x} \, dx \]

\[ = e^{3x} \left[ -\frac{1}{7}xe^{-7x} - \frac{1}{49}e^{-7x} \right] \]

\[ = e^{-4x} \left( -\frac{x}{7} - \frac{1}{49} \right) = -\frac{1}{7}e^{-4x}(x + \frac{1}{7}). \]

Then the general solution is

\[ y = c_1y_1 + c_2y_2 = c_1e^{3x} + c_2e^{-4x}(x + \frac{1}{7}) \]

on the interval \((0, \infty)\).
Exercises

Apply the reduction of order method to obtain another linearly independent solution for the following differential equations

1. \( y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}. \)
2. \( x^2 y'' + x^2 y' - (x + 2)y = 0; \quad y_1 = x^{-1}e^{-x}, \quad x > 0. \)
3. \( x^2(1 - \ln x)y'' + xy' - y = 0; \quad y_1 = x, \quad x > e. \)
Homogeneous Linear Differential Equations with Constant Coefficients

The linear differential equations with Constant Coefficients has the general form

\[ a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \quad (11) \]

which is a homogeneous linear DE with \textbf{constant real coefficients}, where each coefficient \( a_i, 1 \leq i \leq n \) is real constant and \( a_n \neq 0 \).

**Definition**

The polynomial

\[ f(m) = a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0, \quad (12) \]

is called the characteristic polynomial for equation (11), and \( f(m) = 0 \) is called the characteristic equation of the linear differential equations with constant coefficients (11).
We conclude that if \( m \) is a root of equation (52), then

\[
y = e^{mx}
\]

is a solution of the differential equation (11). Also, Equation (52) has \( n \) roots.

Let us summarize the method to solve the differential equation (11)

(1) If all the roots of the characteristic equation are real roots then:

(i) If the roots are distinct (i.e. \( m_1 \neq m_2 \neq m_3 \neq \cdots \neq m_n \)), then the solution of the differential equation (11) is given by

\[
y = c_1 e^{m_1x} + c_2 e^{m_2x} + \cdots + c_n e^{m_nx}
\]

(ii) If the roots are equal (i.e. \( m_1 = m_2 = m_3 = \cdots = m_n \)) (i.e. \( m = m_i \) is a root of multiplicity \( n \)), then the solution of the differential equation (11) is given by

\[
y = c_1 e^{mx} + c_2 xe^{mx} + c_3 x^2 e^{mx} + \cdots + c_n x^{n-1} e^{mx}
\]

\[
y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}
\]
Example (1)

Solve the differential equation

\[ y'' - y = 0. \]

Solution  For this, the characteristic equation is \( m^2 - 1 = 0 \) hence \( m = \pm 1 \). Then \( y_1 = e^x \) and \( y_2 = e^{-x} \) form the fundamental set of solutions, hence the general solution is

\[ y = c_1 e^x + c_2 e^{-x}. \]
Example (2)

Find the general solution of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$ 

**Solution**  For this differential equation the characteristic equation is

$$m^3 - 6m^2 + 11m - 6 = (m - 1)(m - 2)(m - 3) = 0.$$ 

Then $m = 1, 2, 3$ and $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ form the fundamental set of solutions, hence the general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$
Example (3)

Solve the differential equation

\[ y'' - 2y' + y = 0. \]

Solution  The characteristic equation for the differential equation is

\[ m^2 - 2m + 1 = 0, \]

so \( m = 1 \) is a root of multiplicity 2, hence the general solution is

\[ y = c_1 e^x + c_2 x e^x. \]
Example (4)

Solve the differential equation

\[ y''' - 3y'' + 3y' - y = 0 \]

Solution  The characteristic equation for the differential equation is

\[ m^3 - 3m^2 + 3m - 1 = (m - 1)^3 = 0, \] so \( m = 1 \) is a root of multiplicity 3

then the general solution is

\[ y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x. \]
Now we see the second case

(2) If the characteristic equation has **complex conjugate roots** such as

\[ m = \alpha \pm i\beta \]

then the solution of the differential equation of second order is given by

\[ y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) \]

Remember:

1) \( \sqrt{-1} = i \)

2) \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

to find the roots of Quadratic equation

\[ ax^2 + bx + c = 0 \]
Example (5)

Solve the differential equation

\[ y'' + 4y' + 5y = 0. \]

Solution  The characteristic (auxiliary) equation for the differential equation is \( m^2 + 4m + 5 = 0 \), now we need to find the roots of this characteristic equation

\[ m = \frac{-4 \mp \sqrt{16 - 20}}{2} \]

then \( m = -2 \mp i \) hence the general solution is

\[ y(x) = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x). \]
Example (6)

Solve the differential equation

\[ y^{(5)} - 3y^{(4)} + 4y''' - 4y'' + 3y' - y = 0. \]

Solution The characteristic for the differential equation is

\[ m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = 0, \]

then

\[ m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = (m - 1)^3(m^2 + 1) = 0. \]

Thus \( m = 1, 1, 1, \mp i \) where \( \sqrt{-1} = i \) and the general solution of the equation has the form

\[ y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x + c_4 \cos x + c_5 \sin x. \]
Example (7)

Solve the initial value problem (IVP)

\[
\begin{align*}
    y'' + y' + y &= 0 \\
    y(0) &= 1, \quad y'(0) = \sqrt{3}.
\end{align*}
\]

Solution The characteristic equation for the differential equation is

\[ m^2 + m + 1 = 0. \]

Hence

\[ m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i. \]

So the general solution of a differential equation is

\[ y = c_1 e^{\frac{-x}{2}} \cos \left( \frac{\sqrt{3}}{2} x \right) + c_2 e^{\frac{-x}{2}} \sin \left( \frac{\sqrt{3}}{2} x \right). \]
from the conditions $y(0) = 1$ and $y'(0) = \sqrt{3}$ we have $c_1 = 1$. and

$$\frac{-c_1}{2} + c_2 \frac{\sqrt{3}}{2} = \sqrt{3}$$

hence $c_1 = 1$ and $c_2 = 2 + \frac{1}{\sqrt{3}}$. So the solution of the IVP is

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right) + \left(2 + \frac{1}{\sqrt{3}}\right)e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right).$$
Exercises

Find the general solution of the following differential equations

- $2y'' + 3y' + y = 0$
- $y'' - y' - 6y = 0$
- $y''' - 4y'' - 5y' = 0$
- $y^{(4)} - 2y'' + y = 0$
- $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

Find the solution of the initial value problems

- $y'' + y' + 2y = 0; \quad y(0) = y'(0) = 0$
- $y''' + 12y'' + 36y' = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -7$
Cauchy-Euler Differential Equation

A Cauchy-Euler differential equation is in the form

\[ a_n x^n \frac{d^ny}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0, \quad (13) \]

where each coefficient \( a_i, 1 \leq i \leq n \) are constants and \( a_n \neq 0 \) i.e. the coefficient \( a_n x^n \) should never be zero. Equation (13) is on the interval either \((0, \infty)\) or \((-\infty, 0)\).

Euler differential equation is probably the simplest type of linear differential equation with variable coefficients.

The most common Cauchy-Euler equation is the second-order equation, appearing in a number of physics and engineering applications, such as when solving Laplace’s equation in polar coordinates.
It is given by the equation

\[ x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = 0 \]  

To solve the Cauchy-Euler differential equation, we assume that \( y = x^m \), where \( x > 0 \) and \( m \) is a root of a polynomial equation.
Example (1)

Solve the Cauchy-Euler differential equation

\[ x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = 0. \]

**Solution**  We substitute

\[ y = x^m \quad \Rightarrow \quad y' = mx^{m-1} \quad \Rightarrow \quad y'' = m(m - 1)x^{m-2} \]

in the differential equation, we obtain

\[ x^2[m(m - 1)x^{m-2}] + ax[mx^{m-1}] + bx^m = 0 \]

\[ x^m(m^2 - m) + amx^m + bx^m = 0 \]

\[ x^m[(m^2 - m) + am + b] = 0 \]

\[ x^m[m^2 + (1 - a)m + b] = 0. \]
Since \( x^m \neq 0 \), then we have

\[
m^2 + (1 - a)m + b = 0
\]

We then can solve for \( m \). There are three particular cases of interest:

**Case 1:** Two distinct roots, \( m_1 \) and \( m_2 \). Thus, the solution is given by

\[
y = c_1 x^{m_1} + c_2 x^{m_2}.
\]

**Case 2:** One real repeated root, \( m \). Thus, the solution is given by

\[
y = c_1 x^m \ln(x) + c_2 x^m.
\]

**Case 3:** Complex roots, \( \alpha \pm i\beta \). Thus, the solution is given by

\[
y = c_1 x^\alpha \cos (\beta \ln(x)) + c_2 x^\alpha \sin (\beta \ln(x)).
\]
Example (2)

Solve the Euler differential equation

\[ 2x^2 y'' - 3xy' - 3y = 0. \] \hspace{1cm} (15)

For \( x > 0 \).

Solution \hspace{1cm} \text{We substitute}

\[ y = x^m \implies y' = mx^{m-1} \implies y'' = m(m - 1)x^{m-2} \]

in the differential equation, we obtain

\[ 2x^2[m(m - 1)x^{m-2}] - 3x[mx^{m-1}] - x^m = 0 \]

\[ x^m(2m^2 - 2m) - 3mx^m - 3x^m = 0 \]

\[ x^m[2m^2 - 2m - 3m - 3] = 0 \]

\[ x^m[2m^2 - 5m - 3] = 0. \]
Since \( x^m \neq 0 \), then we have

\[
2m^2 - 5m - 3 = 0
\]

So the roots of this equation are \( m_1 = -\frac{1}{2} \), \( m_2 = 3 \). Thus, from case 1 we have the solution is given by

\[
y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^3.
\]

which is the general solution.
Example (3)

Find the general of the differential equation

\[ x^2 y'' - 3xy' + 13y = 0 \quad ; \quad x > 0. \]

Solution  
Substituting \( y = x^m \) in the equation, we obtain

\[ m(m - 1) - 3m + 13 = m^2 - 4m + 13 = 0. \]

Then we have two complex roots \( m = 3 \pm 3i \) (case 3), hence the the general of the differential equation is

\[ y = c_1 x^3 \cos(3 \ln x) + c_2 x^3 \sin(3 \ln x) \quad ; \quad x > 0. \]

If we suppose \( x < 0 \), then the general of the differential equation is

\[ y = c_1 (-x)^3 \cos(3 \ln(-x)) + c_2 (-x)^3 \sin(3 \ln(-x)) \quad ; \quad x < 0. \]
Example (4)

Find the general solution of the differential equation

\[ x^4 y^{(4)} - 5x^3 y''' + 3x^2 y'' - 6xy' + 6y = 0 \; ; \; x > 0. \]

Solution  Substituting  \( y = x^m \)  in the equation, we obtain

\[ m(m - 1)(m - 2)(m - 3) - 5m(m - 1)(m - 2) + 3m(m - 1) - 6m + 6 = 0. \]

This implies that

\[ (m - 1)(m - 2)(m^2 - 8m + 3) = 0. \]

The roots of this equation are  \( m = 1 \),  \( m = 2 \), and  \( m = 4 \pm \sqrt{13} \), then the general solution of the differential equation is

\[ y = c_1x + c_2x^2 + c_3x^{4+\sqrt{13}} + c_4x^{4-\sqrt{13}} \; ; \; x > 0. \]
Example (5)

Find the general solution of the differential equation

\[ x^5y^{(5)} - 2x^3y''' + 4x^2y'' = 0 \quad ; \quad x < 0. \]

Solution  Substituting  \( y = x^m \)  in the equation, we obtain

\[ m(m - 1)(m - 2)(m - 3)(m - 4) - 2m(m - 1)(m - 2) + 4m(m - 1) = 0, \]

\[ m(m - 1)(m^3 - 9m^2 + 24m - 20) = m(m - 1)(m - 2)^2(m - 5) = 0. \]

So the roots of this equation are  \( m = 0 \) ,  \( m = 1 \) ,  \( m = 2 \) repeated two times  and  \( m = 5 \) , then the general of the differential equation is

\[ y = c_1 + c_2(-x) + c_3(-x)^2 + c_4(-x)^2 \ln(-x) + c_5(-x)^5. \]
Exercises

Find the general solution of the following differential equations, where we suppose that $x > 0$.

- $x^2 y'' - y = 0$
- $x^2 y'' + 5xy' + 3y = 0$
- $4x^2 y'' + 4xy' - y = 0$
- $x^3 y''' + xy' - y = 0$
- $x^3 y''' + 4x^2 y'' - 8xy' + 8y = 0$
- $(3x + 4)^2 y'' + 10(3x + 4)y' + 9y = 0; \quad x > -\frac{4}{3}$
Nonhomogeneous linear $n$-th order ODE takes the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \quad (16)$$

where $a_n(x)$, $a_{n-1}(x)$, $a_1(x)$ and $a_0(x)$ are functions of $x \in I = (a, b)$, such that $a_n(x) \neq 0$ for all $x \in I$, and $g(x) \neq 0$.

**Idea**

- Find the general solution $y_c$ to the homogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$
Find a solution $y_p$ to the nonhomogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

The general solution $y = y_c + y_p$. 
Undetermined coefficients

Let us take an example

**Example (1)**

Find the general solution of the differential equation:

\[ y'' - y = -2x^2 + 5 + 2e^x. \] (*)

**Solution**

1) First we have to find the general solution of the differential equation:

\[ y'' - y = 0. \]

For, we have \( m^2 - 1 = 0 \), hence \( m = \pm 1 \) then

\[ y_c = c_1 e^x + c_2 e^{-x}. \]
2) The form of the particular solution of

\[ y'' - y = -2x^2 + 5, \]

is

\[ y_{1,p} = Ax^2 + Bx + C, \]

and the form of the particular solution of

\[ y'' - y = 2e^x, \]

is

\[ y_{2,p} = Dxe^x, \]

because \( r = 1 \) is a simple root of the characteristic equation. Thus the particular solution of (*) is

\[ y_p = y_{1,p} + y_{2,p} = Ax^2 + Bx + C + Dxe^x. \]

Now we have to find the constants \( A, B, C \), and \( D \) by substituting \( y_p \) and \( y''_p \) in differential equation (*) and we find

\[ y''_p - y_p = -Ax^2 - Bx + 2A - C + 2De^x = -2x^2 + 5 + 2e^x. \]
Equating coefficients of similar terms (because the functions $x^2, 1$ and are $e^x$ linearly independent on $\mathbb{R}$), we obtain the following system of equation $A = 2$, $B = 0$, $2A - C = 5$, and $2D = 2$. Thus we have $A = 2$, $B = 0$, $C = -1$, and $D = 1$. Then the particular solution of (*) is

$$y_p = 2x^2 - 1 + xe^x,$$

and the general solution of the differential equation of (*) is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-x} + 2x^2 - 1 + xe^x.$$
Some of the Typical forms of the particular integral

<table>
<thead>
<tr>
<th>Function of $x$</th>
<th>Form for $y_p$</th>
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</thead>
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<tr>
<td>$ke^{ax}$</td>
<td>$Ce^{ax}$</td>
</tr>
<tr>
<td>$kx^n$, $n = 0, 1, 2, \ldots$</td>
<td>$\sum_{i=1}^{n} C_i x^i$</td>
</tr>
<tr>
<td>$k \cos(ax)$ or $k \sin(ax)$</td>
<td>$C_1 \cos(ax) + C_2 \sin(ax)$</td>
</tr>
<tr>
<td>$ke^{ax} \cos(bx)$ or $ke^{ax} \sin(bx)$</td>
<td>$e^{ax} (C_1 \cos(bx) + C_2 \sin(bx))$</td>
</tr>
<tr>
<td>$\left( \sum_{i=1}^{n} k_i x^i \right) \cos(ax)$</td>
<td>$\left( \sum_{i=1}^{n} C_i x^i \right) \cos(ax) + \left( \sum_{i=1}^{n} R_i x^i \right) \sin(ax)$</td>
</tr>
<tr>
<td>or $\left( \sum_{i=1}^{n} k_i x^i \right) \sin(ax)$</td>
<td></td>
</tr>
</tbody>
</table>
Exercises

Find the general solution of the following differential equations.

- \( x^2 y'' - y = 0 \)
- \( y'' + 4y = \sin(2x) + e^x \)
- \( y'' - 5y' + 4y = e^{2x}(\cos x + \sin x) \)

Find only the form of the particular solution of the given differential equation by using the method of undetermined coefficients.

- \( y'' - y = e^x + s \sin x \)
- \( y'' - y = x^2 e^x \)
- \( y^{(6)} - 3y^{(3)} = 3x + 1 \)
- \( y''' - y' = x^5 + \cos x \)
Variation of Parameters

This method is used to determine the particular solution \( y_p \) of nonhomogeneous differential equation

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \tag{17}
\]

If we have the nonhomogeneous differential equation

\[
a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{18}
\]

which has the particular solution

\[
y_p = y_1u_1 + y_2u_2,
\]

where \( y_1 \) and \( y_2 \) are the first and the second solution of the homogeneous differential equation, respectively.

\[
a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{19}
\]
Here we will explain the method to find \( u_1 \) and \( u_2 \). So, if we have \( y_1 \) & \( y_2 \), then we will determine as below

\[
W(x, y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1,
\]

\[
W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y'_2 \end{vmatrix} = -y_2 g(x),
\]

\[
W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & g(x) \end{vmatrix} = y_1 g(x).
\]

Thus,

\[
u'_1 = \frac{W_1}{W}
\]

and

\[
u'_2 = \frac{W_2}{W}.
\]
Example (1)

Solve the differential equation

\[ y'' + y = \csc x ; \quad 0 < x < \pi. \]

Solution

1) The general solution of

\[ y'' + y = 0, \]

is

\[ y_c = c_1 \sin x + c_2 \cos x. \]

2) The particular solution of

\[ y'' + y = \csc x, \]

is the form

\[ y_p = y_1 u_1 + y_2 u_2, \]

where

\[ y_1 = \sin x \quad \text{and} \quad y_2 = \cos x. \]
The functions $u_1$ and $u_2$ are determined from the system below

$$W(x, y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

$$W_1 = \begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix} = -\cot x,$$

$$W_2 = \begin{vmatrix} \sin x & 0 \\ \cos x & \csc x \end{vmatrix} = 1,$$

Hence

$$u_1' = \frac{W_1}{W} = \cot x,$$

then

$$u_1 = \ln(\sin x).$$

But

$$u_2' = -1,$$
hence $u_2 = -x$. Therefore we have

$$y_p = y_1 u_1 + y_2 u_2 = \sin x \cdot \ln(\sin x) - x \cos x,$$

and the general solution of the differential equation is

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + \sin x \cdot \ln(\sin x) - x \cos x.$$
Example (2)

Solve the differential equation

\[ y'' - 4y' + 4y = (x + 1)e^{2x}. \]

Solution

1) The general solution of

\[ y'' - 4y' + 4y = 0, \]

is

\[ y_c = c_1e^{2x} + c_2xe^{2x}. \]

2) Let

\[ y_1 = e^{2x} \quad \text{and} \quad y_2 = xe^{2x}. \]

So we have

\[ W(x, y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}, \]
\[ W_1(x, y_1, y_2) = \begin{vmatrix} 0 & xe^{2x} \\ (x + 1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -x(x + 1)e^{4x}, \]

and

\[ W_2(x, y_1, y_2) = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x + 1)e^{2x} \end{vmatrix} = (x + 1)e^{4x}, \]

hence

\[ u'_1 = \frac{W_1}{W} = -x(x + 1) = -x^2 - x, \]

so

\[ u_1 = -\frac{x^3}{3} - \frac{x^2}{2}, \]

But

\[ u'_2 = \frac{W_2}{W} = x + 1, \]

then

\[ u_2 = \frac{x^2}{2} + x. \]
Therefore,

\[ y_p = y_1 u_1 + y_2 u_2 = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + x\left(\frac{x^2}{2} + x\right)e^{2x} = \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}, \]

and The general solution of the differential equation is

\[ y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}. \]

In this example we can use the undetermined coefficients, where

\[ y_p = x^2(A + Bx)e^{2x}. \]
Example (3)

Solve the Differential equation

\[ y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}. \]

Solution

1) The general solution of

\[ y'' - 3y' + 2y = 0. \]

is

\[ y_c = c_1 e^x + c_2 e^{2x}. \]

2) Let

\[ y_1 = e^x \quad \text{and} \quad y_2 = e^{2x}, \]

then

\[ W(x, y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}, \]
\[ W_1(x, y_1, y_2) = \begin{vmatrix} 0 & e^{2x} \\ 1 + e^{-x} & 2e^{2x} \end{vmatrix} = \frac{-e^{2x}}{1 + e^{-x}}, \]

\[ W_2(x, y_1, y_2) = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{1 + e^{-x}} \end{vmatrix} = \frac{e^x}{1 + e^{-x}}, \]

hence

\[ u'_1 = \frac{W_1}{W} = -\frac{e^{-x}}{1 + e^{-x}}, \]

and

\[ u_1(x) = -\int \frac{e^{-x}}{1 + e^{-x}} \, dx = \ln(1 + e^{-x}). \]

But

\[ u'_2 = \frac{W_2}{W} = \frac{e^{-2x}}{1 + e^{-x}}, \]

and

\[ u_2 = \int \frac{e^{-2x}}{1 + e^{-x}} \, dx = -(1 + e^{-x}) + \ln(1 + e^{-x}), \]
so we have

\[ y = y_c + y_p = (c_1 - 1)e^x + (c_2 - 1)e^{2x} + (e^x + e^{2x}) \ln(1 + e^{-x}), \]
\[ = c_3e^x + c_4e^{2x} + (e^x + e^{2x}) \ln(1 + e^{-x}). \]
Example (4)

Find the general solution of the differential equation

\[ y''' + y' = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}. \]

Solution

1) The general solution of

\[ y''' + y' = 0, \]

is

\[ y_c = c_1 + c_2 \cos x + c_3 \sin x. \]

2) Let \( y_1 = 1, \ y_2 = \cos x \) and \( y_3 = \sin x \). The particular solution of the differential equation has the form

\[ y_p = u_1 y_1 + u_2 y_2 + u_3 y_3. \]
We have

\[
W(x, y_1, y_2, y_3) = \begin{vmatrix}
1 & \cos x & \sin x \\
0 & -\sin x & \cos x \\
0 & -\cos x & -\sin x
\end{vmatrix} = 1,
\]

\[
W_1(x, y_1, y_2, y_3) = \begin{vmatrix}
0 & \cos x & \sin x \\
0 & -\sin x & \cos x \\
\tan x & -\cos x & -\sin x
\end{vmatrix} = \tan x,
\]

\[
W_2(x, y_1, y_2, y_3) = \begin{vmatrix}
1 & 0 & \sin x \\
0 & 0 & \cos x \\
0 & \tan x & -\sin x
\end{vmatrix} = -\sin x,
\]

\[
W_3(x, y_1, y_2, y_3) = \begin{vmatrix}
1 & \cos x & 0 \\
0 & -\sin x & 0 \\
0 & -\cos x & \tan x
\end{vmatrix} = \frac{-\sin^2(x)}{\cos x}.
\]
Then we have,

\[ u'_1 = \frac{W_1}{W} = \tan x, \]

and

\[ u_1 = \int \tan x \, dx = -\ln(\cos x). \]

But

\[ u'_2 = \frac{W_2}{W} = -\sin x, \]

then

\[ u_2 = -\int \sin x \, dx = \cos x. \]

Also

\[ u'_3 = \frac{W_3}{W} = \frac{-\sin^2(x)}{\cos x}, \]
hence,
\[ u_3 = - \int \frac{\sin^2(x)}{\cos x} \, dx = - \int \frac{1 - \cos^2(x)}{\cos x} \, dx = - \ln(\sec x + \tan x) + \sin x. \]

Thus,
\[
y_p = u_1 y_1 + u_2 y_2 + u_3 y_3,
= - \ln(\cos x) + \cos^2(x) - \sin x \ln(\sec x + \tan x) + \sin^2(x),
= 1 - \ln(\cos x) - \sin x \ln(\sec x + \tan x).
\]

So the general solution of the differential equation is
\[
y = y_c + y_p = (c_1 + 1) + c_2 \cos x + c_3 \sin x - \ln(\cos x)
- \sin x \ln(\sec x + \tan x)
\]
\[y = c_4 + c_2 \cos x + c_3 \sin x - \ln(\cos x) - \sin x \ln(\sec x + \tan x).\]
Example (5)

Find the solution of the initial value problem (IVP)

\[
\begin{cases}
2x^2y'' + xy' - 3y = x^{-3} & ; & x > 0 \\
y(1) = 1, & y'(1) = -1.
\end{cases}
\]

Solution

1) We have to find the general solution of

\[2x^2y'' + xy' - 3y = 0.\]

By substituting \( y = x^m \), we have

\[m(m - 1) + m - 3 = (2m - 3)(m + 1) = 0,\]

hence the general solution of the homogeneous differential equation is

\[y_c = c_1 x^{-1} + c_2 x^\frac{3}{2}.\]
2) Let \( y_1 = x^{-1} \), \( y_2 = x^{\frac{3}{2}} \), then

\[ y_p = u_1 y_1 + u_2 y_2. \]

We have

\[ W(x, y_1, y_2) = \begin{vmatrix} x^{-1} & x^{\frac{3}{2}} \\ -x^{-2} & 3x^{\frac{1}{2}} \end{vmatrix} = \frac{5}{2} x^{-\frac{1}{2}}, \]

\[ W_1(x, y_1, y_2) = \begin{vmatrix} 0 & x^{\frac{3}{2}} \\ 1/2 x^{-5} & 3x^{\frac{1}{2}} \end{vmatrix} = -\frac{1}{2} x^{-\frac{7}{2}}, \]

\[ W_2(x, y_1, y_2) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & 1/2 x^{-5} \end{vmatrix} = \frac{1}{2} x^{-6}. \]

Then we have

\[ u'_1 = \frac{W_1}{W} = -\frac{1}{5} x^{-3}, \]

and

\[ u_1 = \frac{1}{10} x^{-2}. \]
Also we have

\[ u'_2 = \frac{W_2}{W} = \frac{1}{5}x^{-\frac{11}{2}}, \]

hence

\[ u_2 = -\frac{2}{45}x^{-\frac{9}{2}}. \]

So

\[ y_p = u_1y_1 + u_2y_2 = \frac{1}{10}x^{-3} - \frac{2}{45}x^{-3} = \frac{1}{18}x^{-3}. \]

Then the general solution of the differential equation is

\[ y = y_c + y_p = c_1x^{-1} + c_2x^{\frac{3}{2}} + \frac{1}{18}x^{-3}. \]

We can obtain \( y_p \) by substituting \( y_p = Ax^{-3} \), which implies \( A = \frac{1}{18} \).
3) \[ y'(x) = -c_1 x^{-2} + \frac{3}{2}c_2 x^{\frac{1}{2}} - \frac{1}{6} x^{-4}. \]

From the conditions \( y(1) = 1 \) and \( y'(1) = -1 \), we deduce

\[ c_1 + c_2 = \frac{17}{18}, \]

and

\[ -c_1 + \frac{3}{2}c_2 = -\frac{5}{6}, \]

which implies \( c_1 = \frac{9}{10} \) and \( c_2 = \frac{2}{45} \). Thus the solution of the IVP is

\[ y = \frac{9}{10} x^{-1} + \frac{2}{45} x^{\frac{3}{2}} + \frac{1}{18} x^{-3}. \]
Exercises

Use the variation of parameters method to find the general solution or initial value problems of the following differential equations.

- \( y'' + y = \sec x; \quad 0 < x < \frac{\pi}{2} \)
- \( y'' - 2y' + y = \frac{e^x}{x}; \quad x > 0 \)
- \( y'' - 12y' + 36y = e^{6x} \ln x; \quad x > 0 \)
- \( y'' - 2y' + y = \frac{e^x}{(e^x + 1)^2} \)
- \( y'' - y = \frac{2}{\sqrt{1 - e^{-2x}}} \)
- \( y''' + 4y' = \sec 2x; \quad 0 < x < \frac{\pi}{4} \)
- \( 2y''' - 6y'' = x^2 \)
- $y'' + y = \tan x; \quad y\left(\frac{\pi}{3}\right) = 1, \quad y'\left(\frac{\pi}{3}\right) = 0$
- $y'' + y = \sec^3(x); \quad y(0) = 1, \quad y'(0) = 1$
- $y'' - 2y' + y = \frac{e^x}{x}; \quad y(1) = e, \quad y'(1) = 0$