Chapter 5: Bayesian Estimation

. **5.1: Let $X_1, X_2, \ldots, X_n$ be a random sample from Bernoulli with parameter θ, and the prior distribution of Θ is a uniform distribution, where 0 < θ < 1. Find the posterior distribution and the Bayes' point estimator of Θ when the loss function be the <u>squared error</u> loss function**

given that $X_i \sim Bernoulli(\theta) \Rightarrow f(x|\theta) = \theta^x(1-\theta)^{1-x}, \quad x = 0, 1$

$\theta \sim Uniform(0,1) \qquad \Rightarrow h(\theta) = 1, \quad 0 < \theta < 1$

$L(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$

(a) <u>The posterior distribution:</u>

$$k(\theta|x) \propto L(x|\theta)\mathrm{h}(\theta)$$

$$K(\theta|x) \propto \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i} = \theta^{\sum_{i=1}^{n} x_i+1-1}(1-\theta)^{n-\sum_{i=1}^{n} x_i+1-1}$$

Then $\theta|X \sim Beta\left(\sum_{i=1}^{n} x_i + 1\ ,\ n - \sum_{i=1}^{n} x_i + 1\right)$

(b) <u>Then the Bayes Point estimator of θ is</u>

$$E(\theta|X) = \frac{\sum_{i=1}^{n} x_i + 1}{\sum_{i=1}^{n} x_i + 1 + n - \sum_{i=1}^{n} x_i + 1} = \frac{\sum_{i=1}^{n} x_i + 1}{n+2}$$

- The pdf of Beta Distribution

  $X \sim Beta\ (a, b)$

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}[1-x]^{b-1}$$

$$E(X) = \frac{a}{a+b}$$

. **5.2:** Let Y have a binominal distribution in which $n = 20$ and $p = \theta$. The prior probability on Θ is $Beta(a, b)$, where $a, b > 0$ are known constants. Find the following:

. (a) Posterior distribution.

(b) Bayes' point estimate of Θ, when $\mathrm{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$.

given that $Y \sim Binomial(20, \theta) \Rightarrow f(y|\theta) = \binom{20}{y}\theta^y(1-\theta)^{20-y}, \quad y = 0, 1, \ldots, 20$

$\theta \sim Beta(a, b), a, b > 0 \qquad \Rightarrow h(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}[1-\theta]^{b-1}, \quad 0 < \theta < 1$

$L(y|\theta) = \prod_{i=1}^{20} f(y_i|\theta) = \prod_{i=1}^{20} \binom{20}{y} \theta^{\sum_{i=1}^{20} y_i}(1-\theta)^{400-\sum_{i=1}^{20} y_i}$

(a) The posterior distribution

$$K(\theta|y) \propto \theta^{\sum_{i=1}^{20} y_i}(1-\theta)^{400-\sum_{i=1}^{20} y_i}\boldsymbol{\theta}^{\boldsymbol{a-1}}[\boldsymbol{1-\theta}]^{b-1}$$

$$\propto \theta^{\sum_{i=1}^{20} y_i+a-1}(1-\theta)^{400-\sum_{i=1}^{20} y_i+b-1}$$

Then $\theta|Y \sim Beta\,(\sum_{i=1}^{20} y_i + a\,, 400 - \sum_{i=1}^{20} y_i + b)$

(b) the Bayes Point estimator of $\boldsymbol{\theta}$

$$E(\theta|X\,) = \frac{\sum_{i=1}^{20} y_i + a}{400+a+b}$$

5.3: Let $X_1, X_2, \dots, X_{10}$ denote a random sample from a Poisson distribution with mean θ, $0 < \theta < \infty$. Let $Y = \sum_{i=1}^{10} X_i$. Use the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If Θ has the pdf $h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$, for $0 < \theta < \infty$. Find:

(a) The posterior distribution.

(b) The Bayes' solution $\delta(y)$ for a point estimate for θ, when $Y = 22$.

given that $X_1, X_2, \dots, X_{10} \sim \boldsymbol{poisson(\theta)}, \boldsymbol{0} < \boldsymbol{x} < \infty \Rightarrow \boldsymbol{f(x|\theta)} = \frac{\boldsymbol{e^{-\theta}\theta^x}}{\boldsymbol{x!}}$

$Y = \sum_{i=1}^{10} X_i \quad h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$ for $\boldsymbol{0} < \boldsymbol{\theta} < \infty$

- The pdf of Gamma Distribution

$$X \sim Gamma\left(n, \frac{1}{a}\right)$$

$$f(x) = \frac{a^n}{\Gamma(n)} x^{n-1} e^{-ax}$$

(a) The posterior distribution

$$L(x|\theta) = \prod_{i=1}^{n} \boldsymbol{f(x_i|\theta)} = \prod_{i=1}^{10} \frac{e^{-\theta}\theta^x}{x!} = \frac{e^{-10\theta}\ \theta^{\sum_{i=1}^{10} x_i}}{\prod_{i=1}^{10} x!}$$

$$k(\theta|x) \propto L(x|\theta)\mathrm{h}(\theta)$$

$$K(\theta|X) \propto \boldsymbol{e^{-10\theta}}\ \boldsymbol{\theta^{\sum_{i=1}^{n} x_i}}\ \theta^2 e^{-\frac{1}{2}\theta} = \boldsymbol{\theta^{\sum_{i=1}^{n} x_i+2}}\ e^{-10.5\theta} = \boldsymbol{\theta^{\sum_{i=1}^{n} x_i+2+1-1}}\ e^{-10.5\theta}$$

Then $\theta|X \sim Gamma\,(\sum_{i=1}^{n} x_i + 3, \frac{1}{10.5})$

(b) $\boldsymbol{Y=22}$ the Bayes Point estimator of θ is

$$E(\theta|X) = \frac{\sum_{i=1}^{10} x_i + 3}{10.5} = \frac{22+3}{10.5} = 2.38 \quad ,(Y = \sum_{i=1}^{10} X_i = 22)$$

(C) **90% CI for** θ $\boldsymbol{Y=22}$

Note: $\boldsymbol{Gamma(\alpha, \theta = 2) \equiv \chi^2_v} \quad \boldsymbol{\Rightarrow v = 2\alpha}$

X~$\boldsymbol{Gamma(n,\theta)}$ $\Rightarrow$ **aX~**$\boldsymbol{Gamma(n, a\theta)}$

We have from (a) $\theta|X \sim Gamma\left(\sum_{i=1}^{n} x_i + 3, \frac{1}{10.5}\right)$

$$\Rightarrow \quad 2(10.5)\theta|X \sim Gamma\left(\sum_{i=1}^{n} x_i + 3, 2\right)$$

$$2(10.5)\,\theta|X \sim Gamma\left(\sum_{i=1}^{n} x_i + 3, 2\right) \equiv 2(10.5)\ \ \theta|X \sim \chi^2_{2(\sum_{i=1}^{n} x_i + 3)}$$

$$\Rightarrow\ 21\,\theta|X \sim \chi^2_{2(22+3)}$$

$$\Rightarrow\ 21\,\theta|X \sim \chi^2_{50}$$

Then, $\chi^2_{1-\frac{\alpha}{2},50} < 21\,\theta < \chi^2_{\frac{\alpha}{2},50}$

$$\alpha = 0.1 \ \Rightarrow \frac{\alpha}{2} = 0.05 \ \Rightarrow\ 1 - \frac{\alpha}{2} = 0.95 \Rightarrow \chi^2_{0.95,50} = 34.76 \quad \chi^2_{0.05,50} = 67.5$$

$$34.76 < 21\,\theta < 67.5$$

$$1.66 < \theta < 3.21$$

. **5.4:** Let Y be the nth order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = \frac{1}{\theta}$, $0 < x \le \theta$, zero elsewhere. Take the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. Let θ be an observed value of the random variable Θ, which has pdf $\boldsymbol{h(\theta)} = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}$, $\alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0, \beta > 0$. Find the Bayes' solution $\delta(y)$ for a point estimate of θ.

given that $\boldsymbol{f(x) = \frac{1}{\theta}}$, $\boldsymbol{0 < x < \theta}$

$$\boldsymbol{h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \quad \alpha < \theta < \infty, \quad \alpha > 0, \quad \beta > 0}$$

$$\boldsymbol{Y_n = max}(X_1, X_2, \dots, X_n)$$

$$f_{Y_n}(y_n) = n f_X(y_n)[F_X(y_n)]^{n-1}, \quad 0 \le y_n < \theta$$

$$= n\frac{1}{\theta}\left[\frac{y_n}{\theta}\right]^{n-1} = \frac{n{y_n}^{n-1}}{\theta^n}$$

$$K(\theta|y_n) \propto \frac{1}{\theta^n}\frac{1}{\theta^{\beta+1}}$$

$$K(\theta|y_n) = c\frac{1}{\theta^{n+\beta+1}}, \quad \alpha < \theta < \infty$$

We have proved in example (3.5) page:55

that $Y = \text{Maximum}(X_1, X_2, \dots, X_n)$is a sufficient statistic

then

$k(\theta|x) \propto f_{Y_n}(y_n|\theta)\, h(\theta)$ page: 112

To find the constant c :

Since $K(\theta|y_n)$ is a pdf , then

$$\int_\alpha^\infty K(\theta|y_n)\, d\theta = 1$$

$$\Rightarrow \int_\alpha^\infty c\frac{1}{\theta^{n+\beta+1}}\, d\theta = 1$$

$$\Rightarrow \int_\alpha^\infty c\theta^{-n-\beta-1}\, d\theta = 1$$

$$c\frac{\theta^{-n-\beta-1+1}}{-n-\beta-1+1}\Big|_\alpha^\infty = 1 \Rightarrow c\frac{\theta^{-n-\beta}}{-n-\beta}\Big|_\alpha^\infty = 1$$

$$\Rightarrow \frac{c}{-n-\beta}\left(0 - \alpha^{-n-\beta}\right) = 1$$

$$\Rightarrow \frac{c}{n+\beta}\alpha^{-n-\beta} = 1$$

$$\Rightarrow C = \frac{n+\beta}{\alpha^{-n-\beta}} = (n+\beta)\alpha^{n+\beta}$$

Then, $K(\theta|y_n) = c\frac{1}{\theta^{n+\beta+1}} = \frac{(n+\beta)\alpha^{n+\beta}}{\theta^{n+\beta+1}}$

The Bayes point estimate of θ such that the loss function is the squared error loss is

$$E(\theta|Y_n) = \int_\alpha^\infty \theta K(\theta|y_n)\, d\theta$$

$$= \int_\alpha^\infty \theta\frac{(n+\beta)\alpha^{n+\beta}}{\theta^{n+\beta+1}}\, d\theta$$

$$= (n+\beta)\alpha^{n+\beta}\int_\alpha^\infty \theta^{-n-\beta}\, d\theta$$

$$= (n+\beta)\alpha^{n+\beta}\frac{\theta^{-n-\beta+1}}{-n-\beta+1}\Big|_\alpha^\infty$$

$$= \frac{n+\beta}{-n-\beta+1}\alpha^{n+\beta}(0-\alpha^{-n-\beta+1})$$

$$= \frac{n+\beta}{-n-\beta+1}\alpha^{n+\beta}(-\alpha^{-n-\beta+1})$$

$$= \frac{n+\beta}{-n-\beta+1}(-\alpha^{n+\beta-n-\beta+1})$$

$$= \frac{n+\beta}{-n-\beta+1}(-\alpha) = \frac{n+\beta}{-(n+\beta-1)}(-\alpha) = \frac{n+\beta}{(n+\beta-1)}\alpha \quad \text{.... (1)}$$

. **5.5:** In Exercise 5.4, let n=4 from the uniform pdf $f(x,\theta)=\frac{1}{\theta}$.$0<x<\theta$,and the prior θ pdf be $g(\theta) = \frac{2}{\theta^3}$,$1 < \theta < \infty$, zero elsewhere. Find:

(a) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4 , using the loss function $[\theta - \delta(Y_4)]^2$.

(b) The Bayesian estimator $\delta(Y_4)$ofθ,based upon the sufficient statistic Y_4, using the loss function $|\delta(Y_4) - \theta|$.

in 5.4 if n=4 $f(x) = \frac{1}{\theta}$, $0 < x < \theta$

$$g(\theta) = \frac{2}{\theta^3},\quad 1 < \theta < \infty \Rightarrow g(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}} = \frac{2(1)^2}{\theta^{2+1}} = \frac{2}{\theta^3}$$

(a) Y_4 **is the maximum order statistic in the sample** X_1, X_2, X_3, X_4

To find the Bayesian estimate of θ, we substitute in (1) by n=4 ,$\beta = 2$ and $\alpha = 1$

$$E(\theta|Y_4) = \frac{n+\beta}{(n+\beta-1)}\alpha = \frac{4+2}{(4+2-1)}(1) = \frac{6}{5}$$

(b) **The Bayesian estimator (Y4) of θ, based upon the sufficient statistic Y4, using the loss function $|\delta(y_4)-\theta|$.**

it Is the median m of $K(\theta|y_n)$ $\alpha < \theta < \infty$ which is the solution of

$$\int_\alpha^m K(\theta|y_n)d\theta = \frac{1}{2}$$

$$\Rightarrow \int_\alpha^m \frac{(n+\beta)\alpha^{n+\beta}}{\theta^{n+\beta+1}}d\theta = \frac{1}{2} \quad (n=4, \beta = 2 \text{ and } \alpha = 1)$$

$$\Rightarrow \int_1^m \frac{6}{\theta^7}d\theta = \frac{1}{2}$$

$\Rightarrow \int_1^m 6\theta^{-7} d\theta = \frac{1}{2}$

$\Rightarrow \frac{6\theta^{-6}}{-6}\Big|_1^m = \frac{1}{2}$

$\Rightarrow -(m^{-6} - 1) = \frac{1}{2} \Rightarrow 1 - m^{-6} = \frac{1}{2} \Rightarrow m^{-6} = \frac{1}{2} \Rightarrow m^6 = 2 \Rightarrow m = 1.1225$

5.6: $f(x|\theta) = \theta e^{-\theta x}$

$h(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta\beta}, \theta > 0$

(a) The posterior distribution of θ

$L(x|\theta) = \prod_{i=1}^n f(x_i|\theta) = \prod_{i=1}^n \theta e^{-\theta x} = \theta^n e^{-\theta \sum_{i=1}^n x_i}$

$K(\theta|X) \propto \theta^n e^{-\theta \sum_{i=1}^n x_i} \theta^{\alpha-1} e^{-\theta\beta} = \theta^{n+\alpha-1} e^{-\theta(\beta + \sum_{i=1}^n x_i)}$

Then $\theta|X \sim Gamma\left(n+\alpha, \frac{1}{\beta + \sum_{i=1}^n x_i}\right)$

(b) The Bayes Point estimate of θ **use** $\mathcal{L}[\theta,\delta(y)]=[\theta - \delta(y)]^2$

$E(\theta|X) = \frac{n+\alpha}{\beta + \sum_{i=1}^n x_i}$

(c) If $X_1=2.5$ $X_2=3.61$ $X_3=4.8$ $X_4=2.74$ $X_5=3.95$ and $\alpha=2$ $\beta=4$. Calculate (b).

$\sum_{i=1}^5 x_i=17.6$, n=5, $\alpha=2$ and $\beta=4$

$E(\theta|X) = \frac{n+\alpha}{\beta + \sum_{i=1}^n x_i} = \frac{5+2}{17.6+4} = 0.3241$