King Saud University

College of Science

Department of Mathematics

# 254 Math Exercises
# Ordinary Differential Equation
# Ch. (6)

***Khaled A Tanash***

**خالد طنش**

**محاضر بقسم الرياضيات**

*e-mail: ktanash@ksu.edu.sa*

*url:http//fac.ksu.edu.sa/ktanash*

# Taylor's Method

1- Taylor's Method of order 1( Euler's Method)

$$y_{i+1} = y_i + hf(x_i, y_i)$$

2- Taylor's Method of order 2

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!} f'(x_i, y_i)$$

Where

$$f'(x\,, y) = f_x + f_y f$$

# Runge-Kutta Method

## (Modified Euler's Method)

$$\mathrm{y}_{\mathrm{i+1}} = y_i + \frac{h}{2}(k_1 + k_2) \text{ where}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_{i+1}, y_i + hk_1)$$

***Example:*** Given the IVP $y' = x + y$ , $y(0) = 1$, the approximate value of $y(0.1)$ using Euler's method with $n = 1$ is

***Solution:***

$y' = f(x,y) = x + y$

$x_0 = 0$ , $y_0 = 1$ , $n = 1$

$\Rightarrow h = \frac{b-a}{n} = \frac{0.1-0}{1} = 0.1$

$y_{i+1} = y_i + h f(x_i, y_i)$

$y_{i+1} = y_i + 0.1(x_i + y_i)$

$i = 0$

$y_1 = y_0 + 0.1(x_0 + y_0)$

$= 1 + 0.1(0 + 1) = 1.1$

mean

$y(x_1) = y(0.1) = 1.1$

***Example:*** For the IVP $y' = 1 + \frac{y}{x}$ , $y(1) = 1$ the approximate value of $y(1.2)$ using Euler's method with $h = 0.1$

***Solution:***

$$y' = f(x, y) = 1 + \frac{y}{x}$$

$x_0 = 1$ , $y_0 = 1$ ; $h = 0.1$

$x_1 = 1.1$ , $x_2 = 1.2$

$$y_{i+1} = y_i + h f(x_i, y_i)$$

$$= y_i + h\left(1 + \frac{y}{x}\right) = y_i + 0.1\left(1 + \frac{y}{x}\right)$$

$i = 0$:

$$y_1 = y_0 + 0.1(1 + y_0/x_0)$$

$$= 1 + 0.1(1 + 1/1) = 1.2$$

mean $y(x_1) = y(1.1) \approx y_1 = 1.2$

$i = 1$

$$y_2 = y_1 + 0.1(1 + y_1/x_1)$$

$$= 1.2 + 0.1(1 + 1.2/1.1) = 1.4091$$

mean

$$y(x_2) = y(1.2) \approx y_2 = 1.4091$$

***Example:*** For IVP $y' + 3y = 4$, $y(0) = 5$ the approximate value of $y(0.1)$ using Taylor's method of order two when $n = 1$ is

***Solution:***

$$y_{i+1} = y_i + h f(x_i, y_i) + \frac{h^2}{2!} f'(x_i, y_i)$$

$$f(x, y) = y' = 4 - 3y$$

$$f'(x, y) = f_x + f f_y$$

$$= 0 - (-3)(4 - 3y) = 9y - 12$$

$$x_0 = 0, \quad y_0 = 5 \quad \text{and } h = \frac{0.1 - 0}{1} = 0.1$$

$$i = 0 \quad ; \quad x_1 = 0.1$$

$$\Rightarrow y(x_1) = y(0.1) \simeq y_1$$

$$= y_0 + h f(x_0, y_0) + h^2/2 \, f'(x_0, y_0)$$

$$= 5 + 0.1(4 - 3(5)) + \frac{(0.1)^2}{2}\left(9(5) - 12\right)$$

$$= 5 + 0.1(-11) + 0.005(33)$$

$$= 4.065$$

**Example:** For the IVP $y' = x(y+1)$ , $y(0) = 0$ the approximate value of $y(0.2)$ using Taylor's method of order 2 with $n = 1$

**Solution:**

$y' = f(x,y) = x(y+1)$

$x_0 = 0$ , $y_0 = 0$ , $n = 1$ $\Rightarrow h = \frac{0.2 - 0}{1} = 0.2$

$\Rightarrow x_1 = 0.2$

$f'(x,y) = f_x + f_y f =$

$= (y+1) + x \cdot x(y+1) = y + 1 + x^2 y + x^2$

$y_{i+1} = y_i + h f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i)$

$= y_i + 0.2(x_i(y_i+1)) + \frac{h^2}{2}(y_i + 1 + x_i^2 y_i + x_i^2)$

$i = 0$

$y_1 = y_0 + 0.2(x_0(y_0+1)) + \frac{(0.2)^2}{2}(y_0 + 1 + x_0^2 y_0 + x_0^2)$

$y_1 = 0 + 0.2(0(0+1)) + \frac{(0.2)^2}{2}(0 + 1 + 0 + 0)$

$= 0.02$

mean $y(x_1) = y(0.2) \simeq y_1 = 0.02$

***Example:*** Use Taylor's method of order 2 for the IVP

$$5y' - y = 0\,, \qquad 0 \le x \le 1, \qquad y\,(0) = 1\; with\; n = 2$$

What are the values of $y_0\,, y_1\,, y_2$. Compare your approximate solution with the exact solution $y(x) = e^{x/5}$

***Solution:***

$y' = \frac{1}{5} y \Rightarrow f(x, y) = \frac{1}{5} y$

$x_0 = 0,\; y_0 = 1,\; n = 2 \Rightarrow h = 0.5$

$x_1 = 0.5,\; x_2 = 1$

$f'(x, y) = f_x + f_y f = 0 + \frac{1}{5}\left(\frac{1}{5} y\right) = \frac{1}{25} y$

$y_{i+1} = y_i + h\, f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i)$

$= y_i + \frac{h}{5} y_i + \frac{h^2}{50} y_i$

$i = 0 \Rightarrow y_1 = y_0 + \frac{0.5}{5} y_0 + \frac{(0.5)^2}{50} y_0$

$= 1 + \frac{0.5}{5} + \frac{(0.5)^2}{50} = 1.105$

mean $y(x_1) = y(0.5) \simeq y_1 = 1.105$

$i = 1 \Rightarrow y_2 = 1.105 + \frac{0.5}{5}(1.105) + \frac{(0.5)^2}{50}(1.105)$

$= 1.3868$

mean $y(x_2) = y(1) \simeq y_2 = 1.3868$

Now

Error $= |\text{exact} - \text{approx}| = |e^{1/5} - 1.3868| = 0.1654$

***Example:*** Show that second order Taylor's method for the given IVP $xy' + xy = x^3$ , $y(0) = 0.5$ , $n = 2$

Is

$$y(x_{i+1}) \approx y_{i+1} = y_i\left[1 - h + \frac{h^2}{2}\right] + x_i\left[h x_i + h^2 - \frac{h^2}{2}x_i\right]$$

$$i = 0,1, \dots , n-1$$

Use it to find approximate of $y(0.4)$. compare your approximate solution with the exact solution $y(x) = -1.5e^{-x} + x^2 - 2x + 2$

***Solution:***

Now $y' = x^2 - y \Rightarrow f(x,y) = x^2 - y$

$x_0 = 0$ , $y_0 = 0.5$ ; $n = 2$

$\Rightarrow h = \frac{b-a}{n} = \frac{0.4}{2} = 0.2$

$\Rightarrow x_1 = 0.2$ and $x_2 = 0.4$

$f'(x,y) = f_x + f_y f = 2x + (-1)(x^2 - y) = 2x - x^2 + y$

Now

$y_{i+1} = y_i + h f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i)$

$= y_i + h(x_i^2 - y_i) + \frac{h^2}{2}(2x_i - x_i^2 + y_i)$

$= y_i + h x_i^2 - h y_i + h^2 x_i - \frac{h^2}{2} x_i^2 + \frac{h^2}{2} y_i$

$$= y_i\left[1-h+\frac{h^2}{2}\right]+x_i\left[h x_i+h^2-\frac{h^2}{2}x_i\right]$$

$i=0$

$$y_1 = y_0\left[1-h+\frac{h^2}{2}\right]+x_0\left[h x_0+h^2-\frac{h^2}{2}x_0\right]$$

$$y_1 = 0.5\left[1-0.2+\frac{(0.2)^2}{2}\right]+0 = 0.41$$

mean $y(0.2) \simeq y_1 = 0.41$

$i=1$

$$y_2 = y_1\left[1-h+\frac{h^2}{2}\right]+x_1\left[h x_1+h^2-\frac{h^2}{2}x_1\right]$$

$$= 0.41\left[1-0.2+\frac{(0.2)^2}{2}\right]+0.2\left[(0.2)(0.2)+(0.2)^2-\frac{(0.2)^2}{2}(0.2)\right]$$

$$= 0.3362 + 0.0152 = 0.3514$$

mean $y(x_2) = y(0.4) \simeq y_2 = 0.3514$

Now

Error $= |\text{exact} - \text{approx}|$

$$= \left|\left(-1.5e^{-0.4}+(0.4)^2-2(0.2)+2\right) - 0.3514\right|$$

$$= 0.003119$$

***Example:*** Show that Taylor's method of order 2 for the IVP

$$e^y y' - e^x = 0\,, \qquad 0 \le x \le 1\,, y(0) = 1, with\ h = 0.5$$

Is

$$y_{i+1} = y_i + he^{(x_i - y_i)}\left[1 + \frac{h}{2}\left(1 - e^{(x_i - y_i)}\right)\right], i \ge 0$$

What are the values of $y_0, y_1, y_2$. Compare your approximate solution with the exact solution $y(x) = \ln(e^x + e - 1)$

***Solution:***

$$y_{i+1} = y_i + h\, f(x_i, y_i) + \frac{h^2}{2!} f'(x_i, y_i)$$

where $f'(x,y) = f_x + f_y f$

Now $y' = \dfrac{e^x}{e^y} = e^{x-y} = f(x,y)$

and

$$f'(x,y) = e^{x-y} + e^{x-y}\cdot(-e^{x-y}) = e^{x-y}(1 - e^{x-y})$$

Now

$$y_{i+1} = y_i + h\, e^{x_i - y_i} + \frac{h^2}{2}\left(e^{x-y}(1 - e^{x-y})\right]$$

$$= y_i + h\,e^{x_i - y_i}\left[1 + h/2\,(1 - e^{x_i - y_i})\right]$$

Now to find $y_0, y_1, y_2$

$i=$ $h = 0.5$ , $x_0 = 0$ , $y_0 = 1$

$i = 0$

$$\Rightarrow y_1 = y_0 + h e^{x_0 - y_0}\left[1 + \frac{h}{2}\left(1 - e^{x_0 - y_0}\right)\right]$$

$$y_1 = 1 + 0.5 e^{0-1}\left[1 + \frac{0.5}{2}\left(1 - e^{0-1}\right)\right]$$

$$= 1.213 \quad \text{this mean } y(0.5) = y_1$$

$i = 1$ and $x_1 = 0.5$

$$\Rightarrow y_2 = y_1 + h e^{x_1 - y_1}\left[1 + \frac{h}{2}\left(1 - e^{x_1 - y_1}\right)\right]$$

$$= 1.213 + 0.5 e^{0.5 - 1.213}\left[1 + \frac{0.5}{2}\left(1 - e^{0.5 - 1.213}\right)\right]$$

$$= 1.4893$$

this mean $y(1) \simeq y_2$

Now

$$|y(1) - y_2| = |\ln(e^1 + e - 1) - 1.4893|$$

$$= 5.5646 \times 10^{-4}$$

***Example:*** Use modified Euler's method to solve the IVP

$$2xy' + y = 2x^2, \qquad y(1) = 0.25$$

With $n = 2$ to get the value of $y$ at $x = 1.5$

***Solution:***

modified Euler's method

$y_{i+1} = y_i + \frac{h}{2}(K_1 + K_2)$

$K_1 = f(x_i, y_i)$ , $K_2 = f(x_{i+1}, y_i + hK_1)$

Now

$f(x,y) = x - \frac{y}{2x}$ ; $x_0 = 1$ , $y_0 = 0.25$

$h = 0.25$

$i = 0 \Rightarrow y_1 = y_0 + \frac{h}{2}(K_1 + K_2)$

$K_1 = f(x_0, y_0) = f(1, 0.25) = 1 - \frac{0.25}{2} = 0.875$

$K_2 = f(x_1, y_0 + hK_1)$ and $x_1 = x_0 + h = 1 + 0.25 = 1.25$

$= f(1.25, 0.25 + 0.25(0.875))$

$= f(1.25, 0.4687)$

$= 1.25 - \frac{0.4687}{2(1.25)} = 1.0625$

$\Rightarrow y_1 = y_0 + \frac{h}{2}(K_1 + K_2)$

$= 0.25 + \frac{0.25}{2}(0.875 + 1.0625) = 0.4922$

means $y(1.25) = y_1 \simeq 0.4922$

Now $i=1$

$y_2 = y_1 + h/2\,(K_1 + K_2)$

where

$K_1 = f(x_1, y_1) = f(1.25, 0.4922)$

$= 1.25 - \dfrac{0.4922}{2(1.25)} = 1.0531$ , $\boxed{x_2 = 1.5}$

$K_2 = f(x_2, y_1 + hK_1) = f(1.5, 0.4922 + 0.25(1.0531))$

$= f(1.5, 0.7555)$

$= 1.5 - \dfrac{0.7555}{2(1.5)} = 1.2482$

$y_2 = 0.4922 + \dfrac{0.25}{2}(1.0531 + 1.2482)$

$= 0.7799$

mean $y(1.5) = y_2 \simeq 0.7799$

**Example:** Consider the IVP

$$y' - y = x^2 \,, \qquad y(0) = 1$$

Use the Runge-kutta method of order 2 (Modified Euler's method) with $n = 2$ to compute approximation for $y(0.4)$

**Solution:**

$$y_{i+1} = y_i + \frac{h}{2}(K_1 + K_2)$$

$$K_1 = f(x_i, y_i) \,, \quad K_2 = f(x_{i+1}, y_i + hK_1)$$

$$y' = x^2 + y = f(x, y)$$

$x_0 = 0$ , $y_0 = 1$ and $n = 2$ $\Rightarrow h = \frac{b-a}{n}$

$\Rightarrow h = \frac{0.4 - 0}{2} = 0.2 \Rightarrow x_1 = 0.2$ , $x_2 = 0.4$

Now $i = 0$

$y_1 = y_0 + h/2(K_1 + K_2)$

$K_1 = f(x_0, y_0) = f(0, 1) = 1$

$K_2 = f(x_1, y_0 + hK_1) = f(0.2, 1 + 0.2(1))$

$= f(0.2, 1.2) = (0.2)^2 + 1.2 = 1.24$

$\Rightarrow y_1 = 1 + 0.1/2(1 + 1.24) = 1.112$

mean $y(x_1) = y(0.2) \simeq y_1 = 1.112$

similarly $i=1$

$y_2 = y_1 + h/2 (k_1 + k_2)$

$k_1 = f(x_1, y_1) = f(0.2, 1.112) = (0.2)^2 + 1.112 = 1.152$

$k_2 = f(x_2, y_1 + hk_1)$

$= f(0.4, 1.112 + 0.2(1.152))$

$= f(0.4, 1.3424) = (0.4)^2 + 1.3424 = 1.7424$

$\Rightarrow y_2 = 1.112 + \frac{0.2}{2}(1.152 + 1.7424)$

$= 1.4014$

mean $y(x_2) = y(0.4) \simeq y_2 = 1.4014$