Ordinary Differential Equation Systems with Constant Coefficients

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Introduction

- The D operator

Elimination Method

The D operator

Definition

A **differential operator** is an operator defined as a function of the differentiation operator. It is helpful, as a matter of notation first, to consider differentiation as an abstract operation, accepting a function and returning another (in the style of a higher-order function in computer science).

Here we will use the following notations:

$$D = d/dt, D^2 = d^2/d^2t, \ldots, D^n = d^n,$$

where t is the independent variable.

Simple Equivalents:

- D_u means $D_u \equiv \frac{du}{dt}$ but $uD \equiv u\frac{d}{dt}$.
- $D_y^2 \equiv D \times D_y \equiv \frac{d}{dt}\left(\frac{dy}{dt}\right)$.

- Similarly $D^2 \equiv \frac{d^2}{dt^2}$ and $D^3 \equiv \frac{d^3}{dt^3}$.

The following differential equation:

$$3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 2y = 0,$$

may be expressed as:

$$(3D^2 + 7D + 2)y = 0$$

or

$$3D^2 + 7D + 2 = 0$$

Elimination Method

Definition

The elimination method consists in bringing the system of n differential equations into a single differential equation of order n.

In the elimination method you either add or subtract the system of differential equations into a single differential equation.

Example (1)

Find the general solution of the system

$$\begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} - 3\frac{dy}{dt} - x + 2y = 0, \\ \frac{dx}{dt} + 2\frac{dy}{dt} + 2x - 4y = 0. \end{cases} \tag{1}$$

Solution We first write the system (1) in its operator form

$$\begin{cases} (D^2 + D - 1)[x] + (D^2 - 3D + 2)[y] = 0, \\ (D + 2)[x] + (2D - 4)[y] = 0. \end{cases} \tag{2}$$

To eliminate x, we apply the operator $D^2 + D - 1$ to the second equation in (2) and $D + 2$ to the first one and substrate the first from the second, we get

$$\left((D^2 + D - 1)(2D - 4) - (D + 2)(D^2 - 3D + 2)\right)[y] = 0,$$

or

$$\left(D^3 - D^2 - 2D\right)[y] = 0 \Leftrightarrow y''' - y'' - 2y' = 0. \tag{3}$$

The characteristic equation for (3) is

$$m^3 - m^2 - 2m = 0$$

whose roots are $0, 2, -1$. Thus

$$y(t) = c_1 + c_2 e^{2t} + c_3 e^{-t}.$$

Substitution of this last expression in the second equation of (1) gives

$$x' + 2x = 4c_1 + 6c_3 e^{-t}. \tag{4}$$

Equation (4) is a linear equation, we solve it to obtain

$$x(t) = 2c_1 + 6c_3 e^{-t} + c_4 e^{-2t}.$$

To eliminate the constant c_4 from the solution $x(t)$, we replace $x(t)$ and $y(t)$ in the first equation in (2) and we find $c_4 = 0$. Consequently

$$x(t) = 2c_1 + 6c_3e^{-t}.$$

Example (2)

Solve the system

$$\begin{cases} x'' + y' - 3x' + 2x - y = 0, \\ x' + y' - 2x + y = 0. \end{cases} \tag{5}$$

Solution

We first write the system (5) in its operator form

$$\begin{cases} (D^2 - 3D + 2)[x] + (D - 1)[y] = 0, \\ (D - 2)[x] + (D + 1)[y] = 0. \end{cases} \tag{6}$$

To eliminate y,we apply $(D+1)$ to the first equation in (6) and $(D-1)$ to the second and substrate the first from the second , we get

$$\left(D^3 - 3D^2 + 2D\right)[x] = 0 \Leftrightarrow x''' - 3x'' + 2x' = 0. \tag{7}$$

The general solution of (7) is

$$x(t) = c_1 + c_2e^{2t} + c_3e^t.$$

From the second equation in (5), we have

$$y' + y = c_3 e^t + 2c_1. \tag{8}$$

We solve the linear equation (8) to obtain

$$y(t) = 2c_1 + \frac{c_3}{2} e^t + c_4 e^{-t}.$$

To eliminate the extraneous constant c_4, we substitute for $x(t)$ and $y(t)$ in the first equation in (5) and find that $c_4 = 0$. Hence

$$y(t) = 2c_1 + \frac{c_3}{2} e^t.$$

Example (3)

Find the general solution of the system

$$\begin{cases} \frac{1}{2}x''' - y'' = \cos t, \\ \frac{1}{2}x'' + x + y' = -\cos t \end{cases} \tag{9}$$

Solution We write the system (9) in the operator form

$$\begin{cases} \frac{1}{2}D^3[x] - D^2[y] = \cos t, \\ (\frac{1}{2}D^2 + 1)[x] + D[y] = -\cos t. \end{cases}$$

To eliminate y, we apply the operator D to the second equation and then sum both equations

$$(D^3 + D)[x] = \sin t + \cos t \Leftrightarrow x''' + x' = \sin t + \cos t. \tag{10}$$

By using the method of undetermined coefficients method, we find that the general solution of equation (10) is given by

$$x(t) = c_1 + c_2 \cos t + c_3 \sin t - \frac{t}{2}(\cos t + \sin t). \tag{11}$$

Substitution of the expression (11) in the second equation of (9) gives

$$y(t) = \left(\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}\right)\cos t + \left(-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}\right)\sin t - c_1 t + c_4.$$

To eliminate the extraneous constant c_4, we substitute for $x(t)$ and $y(t)$ in the second equation in (9) and find that $c_4 = 0$. Hence

$$y(t) = \left(\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}\right)\cos t + \left(-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}\right)\sin t - c_1 t.$$

Exercises

Find the general solution of the following systems of differential equations

1. $$\begin{cases} x' + y - x = -t^2, \\ y' - x - 3y = 2t \end{cases}$$

2. $$\begin{cases} x'' - 3x' + y' + 2x - y = 0, \\ y' + x' - 2x + y = 0 \end{cases}$$

3. $$\begin{cases} y'' + x' + x = 0, \\ y' - y + x = \sin t \end{cases}$$