Chapter 4
Motion in Two Dimension

Lecture Content

- Position, velocity, and acceleration vectors Acceleration
- Two dimension with constant acceleration
- Projectile motion
- Uniform circular motion
- Radial and tangential acceleration

Position, average velocity and instantaneous velocity

- Position vector is defined by vector **r.**
- The displacement vector is Δ**r.**

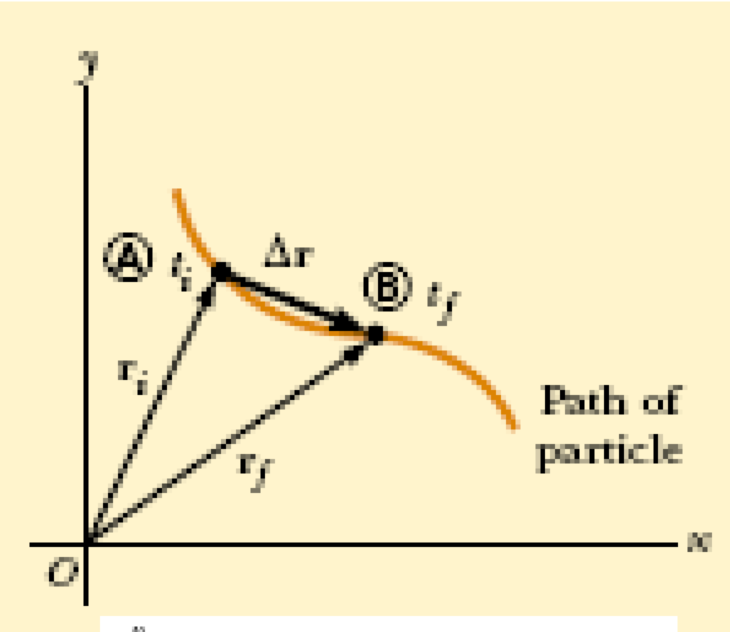
$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

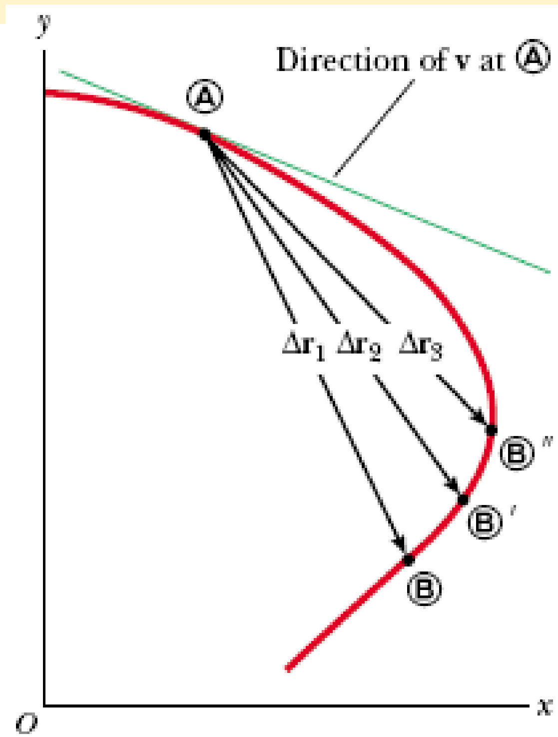
- The average velocity is defined as:

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

- The average acceleration is

$$\bar{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$





Acceleration

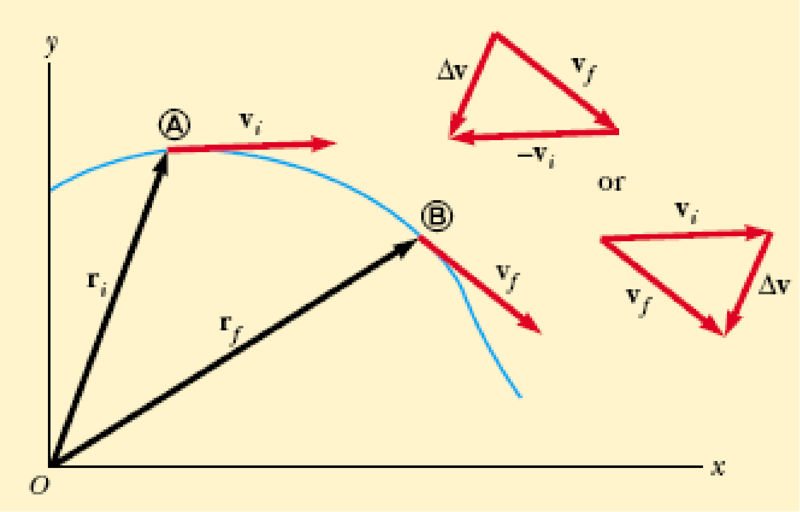
- The average acceleration is

$$\bar{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

- Instantaneous acceleration

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

- ***Caution:*** In vector change calculations, magnitude may change, or direction, or both.



Two dimensional motion with constant acceleration

- Position

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

- Velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$$

$$\begin{aligned}\mathbf{v}_f &= (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}} \\ &= (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}}) + (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t\end{aligned}$$

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$$

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

$$\begin{aligned}\mathbf{r}_f &= (x_i + v_{xi}t + \tfrac{1}{2}a_x t^2)\hat{\mathbf{i}} + (y_i + v_{yi}t + \tfrac{1}{2}a_y t^2)\hat{\mathbf{j}} \\ &= (x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}}) + (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}})t + \tfrac{1}{2}(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t^2\end{aligned}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2$$

A particle starts from the origin at $t = 0$ with an initial velocity having an x component of 20 m/s and a y component of − 15 m/s. The particle moves in the xy plane with an x component of acceleration only, given by $a_x = 4.0\ \text{m/s}^2$.

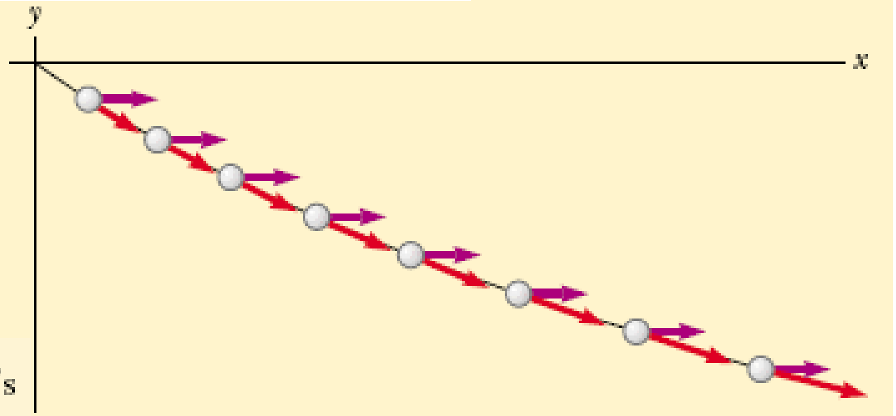
(A) Determine the components of the velocity vector at any time and the total velocity vector at any time.

(B) Calculate the velocity and speed of the particle at $t = 5.0$ s.

$$v_{xf} = v_{xi} + a_x t = (20 + 4.0t)\ \text{m/s}$$

$$v_{yf} = v_{yi} + a_y t = -15\ \text{m/s} + 0 = -15\ \text{m/s}$$

$$\mathbf{v}_f = v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}} = [(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s}$$



$$\mathbf{v}_f = [(20 + 4.0(5.0))\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s} = (40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})\ \text{m/s}$$

$$v_f = |\mathbf{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ \text{m/s} = 43\ \text{m/s}$$

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right)$$

Θ = 360 – 21 = 339°

(C) Determine the x and y coordinates of the particle at any time t and the position vector at this time.

$$x_f = v_{xi}t + \tfrac{1}{2}a_x t^2 = (20t + 2.0t^2)\ \text{m}$$

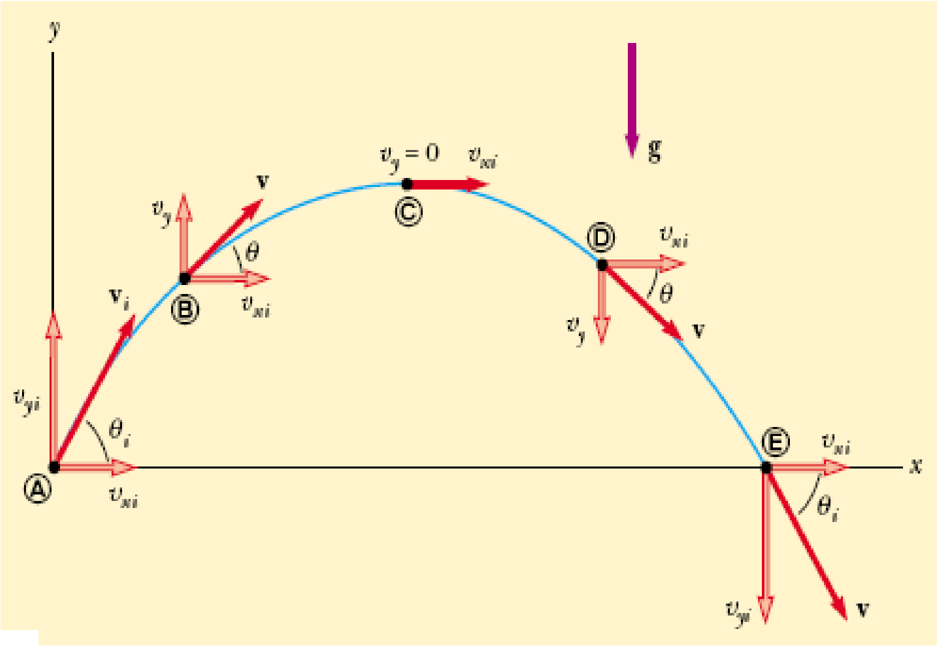
$$y_f = v_{yi}t = (-15t)\ \text{m}$$

$$\mathbf{r}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} = [(20t + 2.0t^2)\hat{\mathbf{i}} - 15t\hat{\mathbf{j}}]\ \text{m}$$

$$r_f = |\mathbf{r}_f| = \sqrt{(150)^2 + (-75)^2}\ \text{m} = 170\ \text{m}$$

Projectile Motion

- Important to separate the two components in the calculations.
- Assumptions:
- g is constant
- Air resistance is neglected



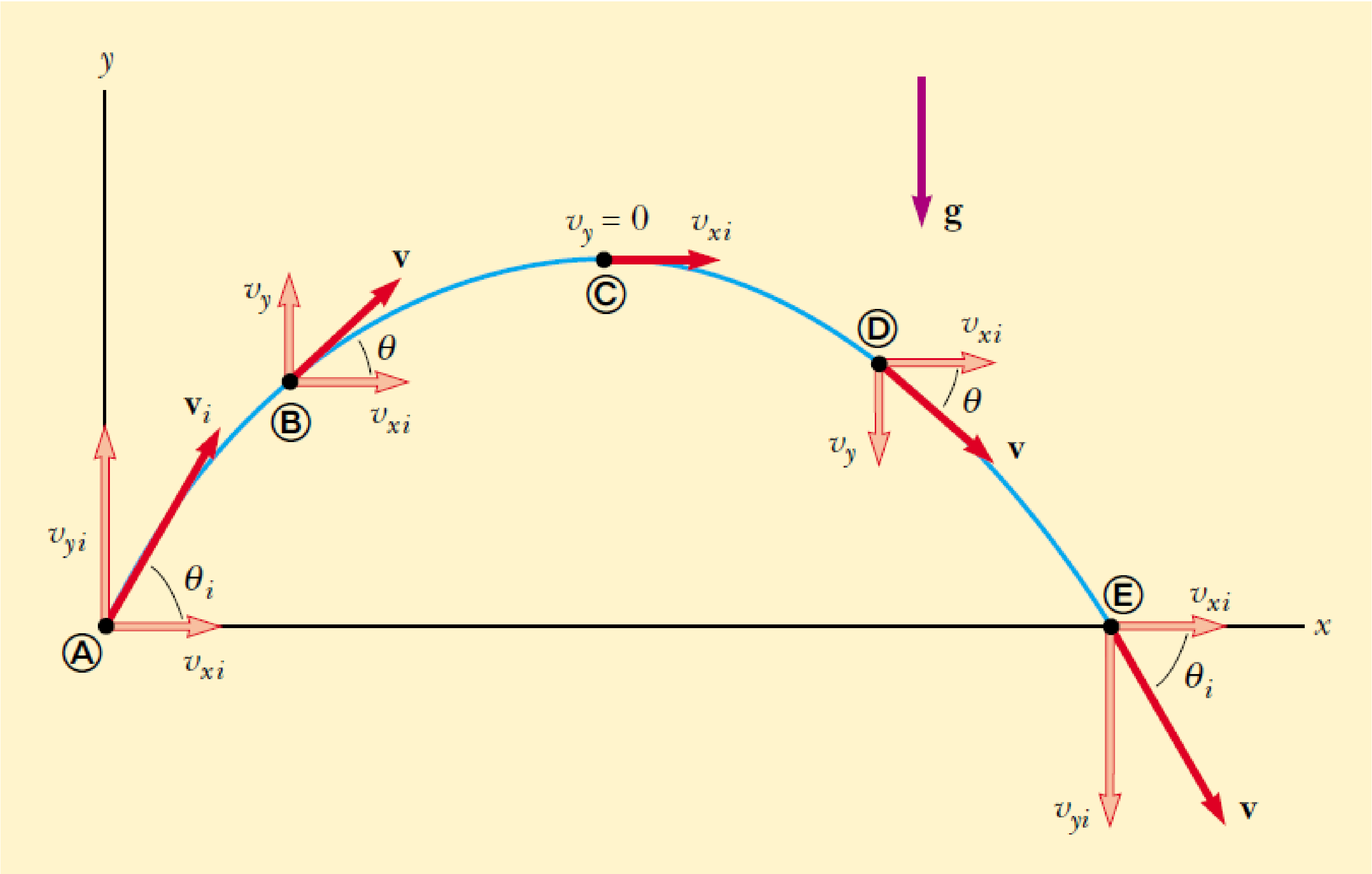
$$\cos\theta_i = v_{xi}/v_i \qquad \sin\theta_i = v_{yi}/v_i$$

$$v_{xi} = v_i\cos\theta_i \qquad v_{yi} = v_i\sin\theta_i$$

$$x_f = v_{xi}t = (v_i\cos\theta_i)t$$

$$y_f = v_{yi}t + \frac{1}{2}a_y t^2 = (v_i\sin\theta_i)t - \frac{1}{2}gt^2$$

$$t = x_f/(v_i\cos\theta_i) \qquad y = (\tan\theta_i)x - \left(\frac{g}{2{v_i}^2\cos^2\theta_i}\right)x^2$$



Active Figure 4.7 The parabolic path of a projectile that leaves the origin with a velocity $\mathbf{v}_i$. The velocity vector $\mathbf{v}$ changes with time in both magnitude and direction. This change is the result of acceleration in the negative y direction. The x component of velocity remains constant in time because there is no acceleration along the horizontal direction. The y component of velocity is zero at the peak of the path.

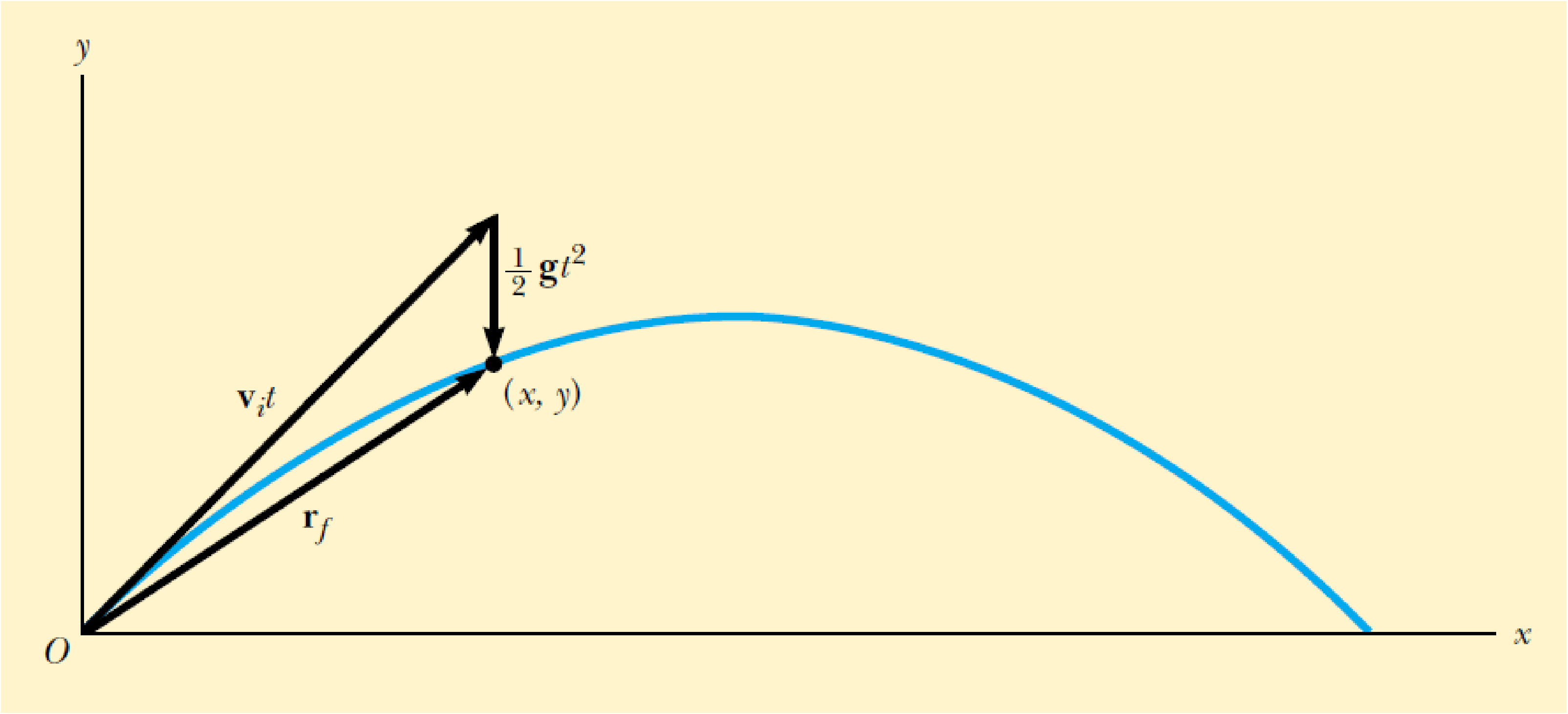


Figure 4.8 The position vector $\mathbf{r}_f$ of a projectile launched from the origin whose initial velocity at the origin is $\mathbf{v}_i$. The vector $\mathbf{v}_i t$ would be the displacement of the projectile if gravity were absent, and the vector $\frac{1}{2}\mathbf{g}t^2$ is its vertical displacement due to its downward gravitational acceleration.

Horizontal Range and Maximum Height of a Projectile

- Let us assume that a projectile is launched from the origin at $t_i = 0$ with a positive v_{yi} component, as shown in Figure 4.10

Two points are especially interesting to analyze:
the peak point A, which has Cartesian coordinates (R/2, h), and the point B which has coordinates (R, 0). The distance R is called the ***horizontal range*** of the projectile, and the distance h is its ***maximum height***. Let us find h and R in terms of v_i, θ_i, and g.
We can determine h by noting that at the peak, v_{yA}, = 0 Therefore, we can use Equation 4.8a to determine the time t_A at which the projectile reaches the peak:

$$v_{yf} = v_{yi} + a_y t$$

$$0 = v_i \sin \theta_i - g t_A$$

$$t_A = \frac{v_i \sin \theta_i}{g}$$

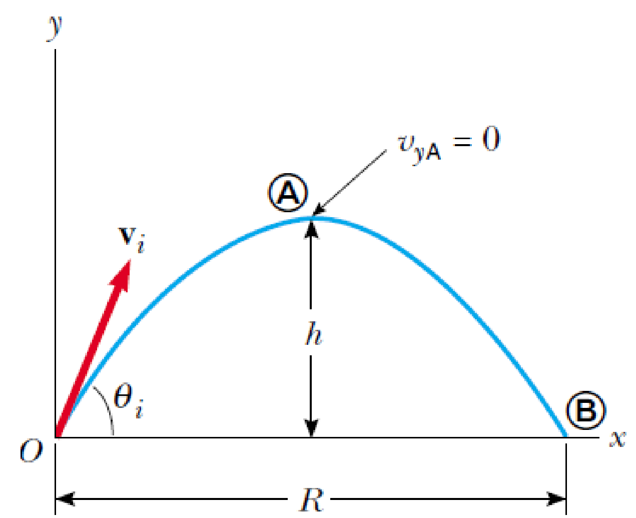


Figure 4.10 A projectile launched from the origin at $t_i = 0$ with an initial velocity $\mathbf{v}_i$. The maximum height of the projectile is h, and the horizontal range is R. At Ⓐ, the peak of the trajectory, the particle has coordinates (R/2, h).

Substituting this expression for t_A into the y part of Equation 4.9a and replacing $y = y_A$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$h = (v_i \sin \theta_i)\frac{v_i \sin \theta_i}{g} - \tfrac{1}{2}g\left(\frac{v_i \sin \theta_i}{g}\right)^2$$

$$h = \frac{{v_i}^2 \sin^2 \theta_i}{2g}$$

The range R is the horizontal position of the projectile at a time that is twice the time at which it reaches its peak, that is, at time $t_B = 2t_A$. Using the x part of Equation 4.9a, noting that $v_{xi} = v_{xB} = v_i \cos \theta_i$ and setting $x_B = R$ at $t = 2t_A$, we find that

$$R = v_{xi} t_B = (v_i \cos \theta_i) 2t_A$$

$$= (v_i \cos \theta_i)\frac{2v_i \sin \theta_i}{g} = \frac{2{v_i}^2 \sin \theta_i \cos \theta_i}{g}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$ (see Appendix B.4), we write R in the more compact form

$$R = \frac{{v_i}^2 \sin 2\theta_i}{g} \tag{4.14}$$

R is a maximum when $\theta_i = 45°$.

A long-jumper (Fig. 4.12) leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction? (Assume his motion is equivalent to that of a particle.)

(B) What is the maximum height reached?

Solution, A

$$x_f = x_B = (v_i \cos \theta_i) t_B = (11.0 \text{ m/s})(\cos 20.0°) t_B$$

$$v_{yf} = v_{yA} = v_i \sin \theta_i - g t_A$$

$$0 = (11.0 \text{ m/s}) \sin 20.0° - (9.80 \text{ m/s}^2) t_A$$

$$t_A = 0.384 \text{ s}$$

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

$$t_B = 2 t_A = 0.768 \text{ s.}$$

$$x_f = x_B = (11.0 \text{ m/s})(\cos 20.0°)(0.768 \text{ s}) = 7.94 \text{ m}$$

Solution, B

$$y_{max} = y_A = (v_i \sin \theta_i) t_A - \tfrac{1}{2} g t_A^2$$

$$= (11.0 \text{ m/s})(\sin 20.0°)(0.384 \text{ s})$$

$$-\tfrac{1}{2}(9.80 \text{ m/s}^2)(0.384 \text{ s})^2 = 0.722 \text{ m}$$

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

Uniform Circular Motion

- Figure 4.17a shows a car moving in a circular path with *constant speed v*. Such motion is called uniform circular motion, and occurs in many situations. It is often surprising to students to find that even though an object moves at a constant speed in a circular path, it still has an acceleration.
- The velocity vector is always tangent to the path of the object and perpendicular to the radius of the circular path. We now show that the acceleration vector in uniform circular motion is always perpendicular to the path and always points toward the center of the circle. An acceleration of this nature is called a centripetal acceleration (*centripetal* means *center-seeking*), and its magnitude is

$$a_c = \frac{v^2}{r}$$

a_c : centripetal acceleration
v : is the velocity
r : is the radius of circular path

$$T = \frac{2\pi r}{v}$$

T: is the period , the time required for one complete revolution.

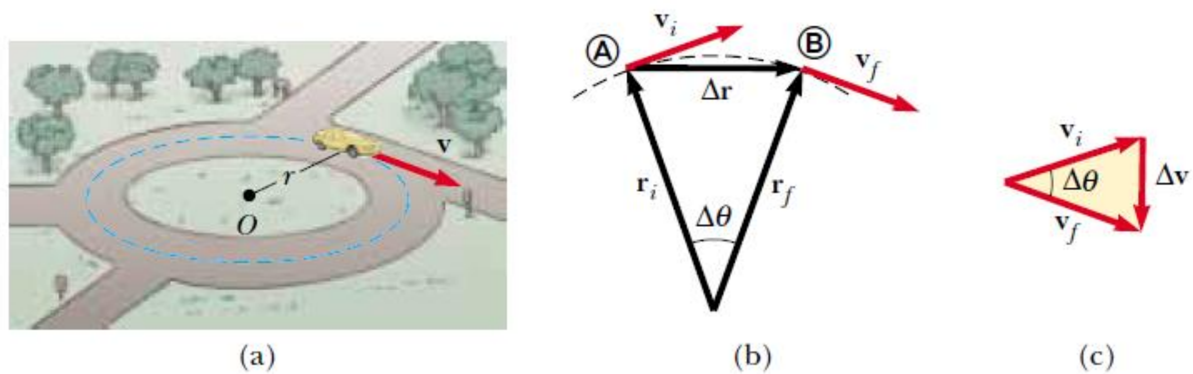


Figure 4.17 (a) A car moving along a circular path at constant speed experiences uniform circular motion. (b) As a particle moves from Ⓐ to Ⓑ, its velocity vector changes from $\mathbf{v}_i$ to $\mathbf{v}_f$. (c) The construction for determining the direction of the change in velocity $\Delta\mathbf{v}$, which is toward the center of the circle for small $\Delta\mathbf{r}$.

Example: a car moves at a constant speed of 10 m/s around a circular path with radius 25 m. Find the following:

A- The centripetal acceleration

B- The period

$$a_c = \frac{v^2}{r}$$

$$T = \frac{2\pi r}{v}$$

Solution:

A- $a_c = v^2 / r = 10^2 / 25 = 4 \text{ m/s}^2$

B- $T = 2*\pi*25/10 = 15.71$ s

Tangential and Radial Acceleration

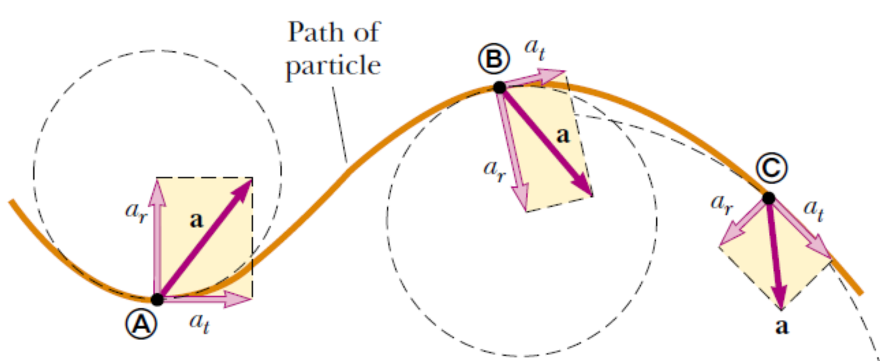
- **Total acceleration** $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t$

- **Tangential acceleration** $a_t = \dfrac{d|\mathbf{v}|}{dt}$

- Radial acceleration

$$a_r = -a_c = -\frac{v^2}{r}$$

Because $\mathbf{a}_r$ and $\mathbf{a}_t$ are perpendicular component vectors of $\mathbf{a}$, it follows that the magnitude of $\mathbf{a}$ is $a = \sqrt{a_r^2 + a_t^2}$. At a given speed, a_r is large when the radius of curvature is small (as at points Ⓐ and Ⓑ in Fig. 4.18) and small when r is large (such as at point Ⓒ). The direction of $\mathbf{a}_t$ is either in the same direction as $\mathbf{v}$ (if v is increasing) or opposite $\mathbf{v}$ (if v is decreasing).



Example: A car exhibits a constant acceleration of 0.300 m/s2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s.

What is the direction of the total acceleration vector for the car at this instant?

$$a_r = -\frac{v^2}{r} = -\frac{(6.00\ \text{m/s})^2}{500\ \text{m}} = -0.0720\ \text{m/s}^2$$

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(-0.0720)^2 + (0.300)^2}\ \text{m/s}^2$$

$$= 0.309\ \text{m/s}^2$$

If ϕ is the angle between **a** and the horizontal, then

$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.0720\ \text{m/s}^2}{0.300\ \text{m/s}^2}\right) = -13.5°$$

This angle is measured downward from the horizontal. (See Figure 4.20b.)