Manual for SOA Exam MLC.

Chapter 5. Life annuities
Actuarial problems.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

(#24, Exam M, Fall 2005) For a special increasing whole life annuity-due on (40), you are given:
(i) Y is the present-value random variable.
(ii) Payments are made once every 30 years, beginning immediately.
(iii) The payment in year 1 is 10, and payments increase by 10 every 30 years.
(iv) Mortality follows DeMoivre's law, with $\omega = 110$.
(v) $i = 0.04$
Calculate $\mathrm{Var}(Y)$.
(A) 10.5 (B) 11.0 (C) 11.5 (D) 12.0 (E) 12.5

(E) The possible payments to the living are 10, 20, 30, at times 0, 30, 60, respectively. Let T_{40} be the age–at-death of (40). T_{40} has a uniform distribution (0.70). We have that:
if $30 \geq T_{40}$, $Y = 10$,
if $60 \geq T_{40} > 30$, $Y = 10 + (20)(1.04)^{-30} = 16.16637336$,
if $T_{40} > 60$,
$Y = 10 + (20)(1.04)^{-30} + (30)(1.04)^{-60} = 19.01818539$.
We also have that

$$\mathbb{P}\{30 \geq T_{40}\} = \frac{30}{70}, \ \mathbb{P}\{60 \geq T_{40} > 30\} = \frac{30}{70} \text{ and } \mathbb{P}\{T_{40} > 60\} = \frac{10}{70}.$$

Hence,

$$E[Y] = (10)\frac{30}{70} + (16.16637336)\frac{30}{70} + (19.01818539)\frac{10}{70} = 13.93104364,$$

$$E[Y^2] = (10)^2\frac{30}{70} + (16.16637336)^2\frac{30}{70} + (19.01818539)^2\frac{10}{70} = 206.53517$$

$$\mathrm{Var}(Y) = 206.5351798 - (13.93104364)^2 = 12.4612029.$$

(#26, Exam M, Spring 2005) You are given:

(i) $\mu_x(t) = 0.03,\ t \geq 0$

(ii) $\delta = 0.05$

(iii) $T(x)$ is the future lifetime random variable.

(iv) g is the standard deviation of $\overline{a}_{\overline{T(x)}|}$.

Calculate $\mathbb{P}\left(\overline{a}_{\overline{T(x)}|} \geq \overline{a}_x - g\right)$.

(A) 0.53 (B) 0.56 (C) 0.63 (D) 0.68 (E) 0.79

(E) We have that $\overline{a}_{\overline{T(x)}|} = \frac{1-e^{-\delta T(x)}}{\delta}$,

$$\overline{A}_x = \frac{\mu}{\delta+\mu} = \frac{0.03}{0.05+0.03} = \frac{3}{8},$$

$$^2\overline{A}_x = \frac{0.03}{(2)0.05+0.03} = \frac{3}{13},$$

$$\overline{a}_x = \frac{1-\overline{A}_x}{\delta} = \frac{1-\frac{3}{8}}{0.05} = \frac{100}{8} = 12.5,$$

$$\mathrm{Var}(\overline{a}_{\overline{T(x)}|}) = \frac{1}{(0.05)^2}\left(\frac{3}{13} - \left(\frac{3}{8}\right)^2\right) = 36.05769231$$

Hence,

$$\begin{aligned}
&\mathbb{P}\left(\overline{a}_{\overline{T(x)}|} \geq \overline{a}_x - g\right)\\
=&\mathbb{P}\left(\frac{1-e^{-(0.05)T(x)}}{0.05} \geq 12.5 - \sqrt{36.05769231}\right)\\
=&\mathbb{P}\left(1-(0.05)(12.5-\sqrt{36.05769231}) \geq e^{-(0.05)T(x)}\right)\\
=&\mathbb{P}\left(0.6752402884 \geq e^{-(0.05)T(x)}\right)\\
=&\mathbb{P}\left(-(20)\ln(0.6752402884) \leq T(x)\right) = e^{(0.03)(20)\ln(0.6752402884)}\\
=&0.7900871674.
\end{aligned}$$

(#37, Exam M, Spring 2005) Company ABC sets the contract premium for a continuous life annuity of 1 per year on (x) equal to the single benefit premium calculated using:
(i) $\delta = 0.03$
(ii) $\mu_x(t) = 0.02,\ t \geq 0$
However, a revised mortality assumption reflects future mortality improvement and is given by

$$\mu_x(t) = \begin{cases} 0.02 & \text{for } t \leq 10 \\ 0.01 & \text{for } t > 10 \end{cases}$$

Calculate the expected loss at issue for ABC (using the revised mortality assumption) as a percentage of the contract premium.
(A) 2% (B) 8% (C) 15% (D) 20% (E) 23%

(C) Let $\overline{a}_x$ be the contract premium. We have that

$$\overline{a}_x = \frac{1}{0.02 + 0.03} = 20.$$

Under the revised mortality rate

$$\begin{aligned}
&\overline{a}_x^{\text{rev}} = \overline{a}_{x:\overline{10}|}^{\text{rev}} + {}_{10|}\overline{a}_x^{\text{rev}} = \overline{a}_{x:\overline{10}|}^{\text{rev}} + {}_{10}E_x \overline{a}_{x+10}^{\text{rev}} \\
=&(1 - e^{-(10)(0.03+0.02)}) \frac{1}{0.02 + 0.03} + e^{-(10)(0.03+0.02)} \frac{1}{0.01 + 0.03} \\
=&(1 - e^{-0.5})(20) + e^{-0.5}(25) = 20 + (5)e^{-0.5} = 23.0326533.
\end{aligned}$$

The expected loss with the revised mortality rate is

$$\overline{a}_x^{\text{rev}} - \overline{a}_x = 23.0326533 - 20 = 3.0326533.$$

The expected loss as a percentage of the contract premium is

$$\frac{3.0326533}{20} = 0.151632665 = 15.1632665\%.$$

(#11, Exam M, Fall 2005) For a group of 250 individuals age x, you are given:
(i) The future lifetimes are independent.
(ii) Each individual is paid 500 at the beginning of each year, if living.
(iii) $A_x = 0.369131$
(iv) ${}^2A_x = 0.1774113$
(v) $i = 0.06$
Using the normal approximation, calculate the size of the fund needed at inception in order to be 90% certain of having enough money to pay the life annuities.
(A) 1.43 million (B) 1.53 million (C) 1.63 million (D) 1.73 million (E) 1.83 million

(A) Let $\ddot{Y}_1, \dots, \ddot{Y}_{250}$ be the present value random variables made by this insurance to each of the 250 individuals. We have that

$$\ddot{a}_x = \frac{1 - A_x}{d} = \frac{1 - 0.369131}{\frac{0.06}{1.06}} = 11.14535233,$$

$$E\left[\sum_{j=1}^{250} \ddot{Y}_j\right] = (250)(500)(11.14535233) = 1393169.041,$$

$$\mathrm{Var}\left(\sum_{j=1}^{250} \ddot{Y}_j\right) = (250)(500)^2 \frac{{}^2A_x - (A_x)^2}{d^2}$$

$$=(250)(500)^2 \frac{0.1774113 - (0.369131)^2}{\left(\frac{0.06}{1.06}\right)^2} = 802781083.3.$$

Let Q be the size of the funds satisfying the requirement. Hence, Q is a 90% percentile of a normal distribution with mean 1393169.041 and variance 802781083.3. So,

$$Q = 1393169.041 + (1.28)\sqrt{802781083.3} = 1429435.782.$$

(#20, Exam M, Fall 2005) For a group of lives age x, you are given:
 (i) Each member of the group has a constant force of mortality that is drawn from the uniform distribution on [0.01, 0.02].
(ii) $\delta = 0.01$
For a member selected at random from this group, calculate the actuarial present value of a continuous lifetime annuity of 1 per year.
(A) 40.0 (B) 40.5 (C) 41.1 (D) 41.7 (E) 42.3

(B) The APV of a continuous annuity with rate one per year under constant mortality rate μ is $E[\overline{Y}|\mu] = \frac{1}{\mu+0.01}$. The APV for a random member of the group is

$$E[\overline{Y}] = E[E[\overline{Y}|\mu]] = \int_0^\infty \frac{1}{\mu + 0.01} f_\mu(\mu)\, d\mu$$

$$= \int_{0.01}^{0.02} \frac{1}{0.01 + \mu} \frac{1}{0.02 - 0.01}\, d\mu$$

$$= (100) \ln(0.01 + \mu) \Big|_{0.01}^{0.02} = (100) \ln(3/2) = 40.54651081.$$

(#4, Exam M, Fall 2006) For a pension plan portfolio, you are given:

(i) 80 individuals with mutually independent future lifetimes are each to receive a whole life annuity-due.

(ii) $i = 0.06$

(iii)

Age	Number of annuitants	Annual annuity payment	$\ddot{a}_x$	A_x	${}^2A_x$
65	50	2	9.8969	0.43980	0.23603
75	30	1	7.2170	0.59149	0.38681

Using the normal approximation, calculate the 95^{th} percentile of the distribution of the present value random variable of this portfolio.

(A) 1220 (B) 1239 (C) 1258 (D) 1277 (E) 1296

(E) The actuarial present value of this portfolio is

$$Z = \sum_{j=1}^{50} 2U_j + \sum_{j=1}^{30} V_j,$$

where $U_1, \dots, U_{50}, V_1, \dots, V_{30}$ are independent r.v.'s, $U_1, \dots, U_{50}$ have the distribution of $\ddot{Y}_{65}$ and $V_1, \dots, V_{30}$ have the distribution of $\ddot{Y}_{75}$. Using that $E[\ddot{Y}_x] = \ddot{a}_x$, we get that

$$E[Z] = (50)(2)(9.8969) + (30)(7.2170) = 1206.2.$$

Using that $\mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - (A_x)^2}{d^2}$, we get that

$$\mathrm{Var}(Z) = (50)(2)^2 \frac{0.23603 - (0.43980)^2}{(0.06/1.06)^2} + (30)\frac{0.38681 - (0.59149)^2}{(0.06/1.06)^2}$$

$$=3013.464959.$$

Using the normal approximation the 95–th percentile is

$$1206.2 + (1.645)\sqrt{3013.464959} = 1296.502334.$$

(#5, Exam M, Fall 2006) Your company sells a product that pays the cost of nursing home care for the remaining lifetime of the insured.

(i) Insureds who enter a nursing home remain there until death.

(ii) The force of mortality, μ, for each insured who enters a nursing home is constant.

(iii) μ is uniformly distributed on the interval $[0.5, 1]$.

(iv) The cost of nursing home care is 50,000 per year payable continuously.

(v) $\delta = 0.045$

Calculate the actuarial present value of this benefit for a randomly selected insured who has just entered a nursing home.

(A) 60,800 (B) 62,900 (C) 65,100 (D) 67,400 (E) 69,800

(C) For each insured, the APV of the benefit is

$$(50000)\overline{a}_x = \frac{50000}{\mu+\delta} = \frac{50000}{\mu+0.045}.$$

The APV of the benefit for a random selected insured is

$$\int_{0.5}^{1} \frac{50000}{\mu+0.045}\frac{1}{1-0.5}\,d\,\mu = (100000)\ln(\mu+0.045)\Big|_{0.5}^{1}$$
$$=(100000)\ln(1.045/0.545) = 65098.63697.$$

(#33, Exam M, Fall 2006) You are given:
(i) Y is the present value random variable for a continuous whole life annuity of 1 per year on (40).
(ii) Mortality follows DeMoivre's Law with $\omega = 120$.
(iii) $\delta = 0.05$
Calculate the 75^{th} percentile of the distribution of Y.
(A) 12.6 (B) 14.0 (C) 15.3 (D) 17.7 (E) 19.0

(D) We have that $Y = \overline{Y}_x = \frac{1-e^{-0.05T_x}}{0.05}$. Let q be the 75–th quantile of the distribution of $\overline{Y}_x$. Let ξ be the 75–th quantile of the distribution of T_x. Since $y = h(t) = \frac{1-e^{-0.05t}}{0.05}$ is increasing, $q = h(\xi)$. We have that

$$0.75 = \mathbb{P}\{T_x \le \xi\} = \frac{\xi}{80}.$$

and $\xi = 60$. Hence,

$$q = \frac{1 - e^{-(0.05)(60)}}{0.05} = \frac{1 - e^{-3}}{0.05} = 19.00426.$$

(#2, Exam MLC, Spring 2007) You are given:
(i) $\mu_x(t) = c$, $t \geq 0$
(ii) $\delta = 0.08$
(iii) $\overline{A}_x = 0.3443$
(iv) $T(x)$ is the future lifetime random variable for (x).
Calculate $\mathrm{Var}(\overline{a}_{\overline{T(x)|}})$.
(A) 12 (B) 14 (C) 16 (D) 18 (E) 20

(D) We have that $0.3443 = \overline{A}_x = \frac{\mu}{\mu+0.08}$ and $\mu = \frac{(0.3443)(0.08)}{1-0.3443} = 0.04200702$. Therefore,

$$^2\overline{A}_x = \frac{\mu}{\mu + 2\delta} = \frac{0.04200702}{0.04200702 + 2(0.08)} = 0.2079483,$$

$$\mathrm{Var}(\overline{a}_{\overline{T(x)}|}) = \mathrm{Var}(\overline{Y}_x) = \frac{^2\overline{A}_x - (\overline{A}_x)^2}{\delta^2} = \frac{0.2079483 - (0.3443)^2}{(0.08)^2}$$

$=13.96966.$

(#23, Exam MLC, Spring 2007) For a three–year temporary life annuity due of 100 on (75), you are given:
(i) $\int_0^x \mu(t)\,dt = 0.01x^{1.2}$, $x > 0$
(ii) $i = 0.11$
Calculate the actuarial present value of this annuity.
(A) 264 (B) 266 (C) 268 (D) 270 (E) 272

(#23, Exam MLC, Spring 2007) For a three–year temporary life annuity due of 100 on (75), you are given:
(i) $\int_0^x \mu(t)\,dt = 0.01x^{1.2}$, $x > 0$
(ii) $i = 0.11$
Calculate the actuarial present value of this annuity.
(A) 264 (B) 266 (C) 268 (D) 270 (E) 272
Solution: (A) We have that $s(x) = e^{-\int_0^x \mu(t)\,dt} = e^{-0.01x^{1.2}}$. We need to find

$$
\begin{aligned}
&(100)\ddot{a}_{75:\overline{3}|} = 100 + (100)(1+i)^{-1}p_{75} + (100)(1+i)^{-1}p_{75}p_{76} \\
=&100 + (100)(1.11)^{-1}\frac{s(76)}{s(75)} + (100)(1.1)^{-1}\frac{s(77)}{s(75)} \\
=&100 + (100)(1.11)^{-1}e^{-0.01((76)^{1.2}-(75)^{1.2})} \\
+&(100)(1.11)^{-2}e^{-0.01((77)^{1.2}-(75)^{1.2})} = 264.2196.
\end{aligned}
$$

(#29, Exam MLC, Spring 2007) For a special fully discrete, 30–year deferred, annual life annuity-due of 200 on (30), you are given:
(i) The single benefit premium is refunded without interest at the end of the year of death if death occurs during the deferral period.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$
Calculate the single benefit premium for this annuity.
(A) 350 (B) 360 (C) 370 (D) 380 (E) 390

(A) Let π be the single benefit premium for this annuity. We have that

$$\pi = \pi A^1_{30:\overline{30}|} + (200)_{30}E_{30}\ddot{a}_{60} = \pi(A_{30} - {}_{30}E_{30}A_{60}) + (200)_{30}E_{30}\ddot{a}_{60}$$

$$= \pi(0.10248 - (1.06)^{-30}\frac{8188074}{9501381}(0.36913))$$

$$+ (200)(1.06)^{-30}\frac{8188074}{9501381}(11.1454)$$

$$= 0.04709420291\pi + 334.4604139$$

and

$$\pi = \frac{334.4604139}{1 - 0.04709420291} = 350.9900086.$$

(#7, MLC–09–08) For an annuity payable semiannually, you are given:

(i) Deaths are uniformly distributed over each year of age.

(ii) $q_{69} = 0.03$

(iii) $i = 0.06$

(iv) $1000\overline{A}_{70} = 530$

Calculate $\ddot{a}^{(2)}_{69}$.

(A) 8.35 (B) 8.47 (C) 8.59 (D) 8.72 (E) 8.85

(C) We have that

$$A_{70} = \frac{\delta}{i}\overline{A}_{70} = \frac{\ln(1.06)}{0.06}(0.53) = 0.5147086884,$$

$$A_{69} = q_{69}v + p_{69}vA_{70}$$

$$=(0.03)(1.06)^{-1} + (0.03)(1.06)^{-1}(0.5147086884) = 0.4993088941,$$

$$A^{(2)}_{69} = \frac{i}{i^{(2)}}A_{69} = \frac{0.06}{0.0591260282}(0.4993088941) = 0.5066894321,$$

$$\ddot{a}^{(2)}_{69} = \frac{1 - A^{(2)}_{69}}{d^{(2)}} = \frac{1 - 0.5066894321}{0.05742827529} = 8.59002931.$$

(#11, MLC–09–08) For a group of individuals all age x, of which 30% are smokers and 70% are non-smokers,
you are given:
(i) $\delta = 0.10$
(ii) $\bar{A}_x^{\text{smoker}} = 0.444$.
(iii) $\bar{A}_x^{\text{non-smoker}} = 0.286$.
(iv) T is the future lifetime of (x)
(v) $\mathrm{Var}(\bar{a}_{\overline{T}|}^{\text{smoker}}) = 8.818$
(vi)$\mathrm{Var}(\bar{a}_{\overline{T}|}^{\text{non-smoker}}) = 8.503$
Calculate $\mathrm{Var}(\bar{a}_{\overline{T}|})$ for an individual chosen at random from this group.
(A) 8.5 (B) 8.6 (C) 8.8 (D) 9.0 (E) 9.1

(E) **Solution 1:** Let $S = I$(an individual is a smoker). We have that

$$\mathrm{Var}(\overline{Y}_x) = \mathrm{Var}(E[\overline{Y}_x|S]) + E[\mathrm{Var}(\overline{Y}_x|S)].$$

$$E[\overline{Y}_x|S=1] = \frac{1-0.444}{0.1} = 5.56,\ E[\overline{Y}_x|S=0] = \frac{1-0.286}{0.1} = 7.14.$$

So, $E[\overline{Y}_x|S]$ takes only two values and

$$\mathrm{Var}(E[\overline{Y}_x|S]) = (0.3)(0.7)\,(5.56-7.14)^2 = 0.524244.$$

Hence,

$$E[\mathrm{Var}(\overline{Y}_x|S)] = (0.3)(8.818) + (0.7)(8.503) = 8.5975,$$
$$\mathrm{Var}(\overline{Y}_x) = 0.524244 + 8.5975 = 9.121744.$$

(E) **Solution 2:** From $8.818 = \mathrm{Var}(\overline{a}^{\text{smoker}}_{\overline{T}|}) = \frac{{}^2A_x^{\text{smoker}} - (A_x^{\text{smoker}})^2}{\delta^2}$, we get that

$$ {}^2A_x^{\text{smoker}} = (0.444)^2 + (8.818)(0.1)^2 = 0.285316. $$

Similarly,

$$ {}^2A_x^{\text{non-smoker}} = (0.286)^2 + (8.503)(0.1)^2 = 0.166826. $$

$$ \begin{aligned} A_x &= (0.3)(0.444) + (0.7)(0.286) = 0.3334, \\ {}^2A_x &= (0.3)(0.285316) + (0.7)(0.166826) = 0.202373, \\ \mathrm{Var}(\overline{a}_{\overline{T}|}) &= \frac{{}^2A_x - (A_x)^2}{\delta^2} = \frac{0.202373 - (0.3334)^2}{(0.1)^2} = 9.121744 \end{aligned} $$

(#25, MLC–09–08) Your company currently offers a whole life annuity product that pays the annuitant 12,000 at the beginning of each year. A member of your product development team suggests enhancing the product by adding a death benefit that will be paid at the end of the year of death.
Using a discount rate, d, of 8%, calculate the death benefit that minimizes the variance of the present value random variable of the new product.
(A) 0 (B) 50,000 (C) 100,000 (D) 150,000 (E) 200,000

(#25, MLC–09–08) Your company currently offers a whole life annuity product that pays the annuitant 12,000 at the beginning of each year. A member of your product development team suggests enhancing the product by adding a death benefit that will be paid at the end of the year of death.
Using a discount rate, d, of 8%, calculate the death benefit that minimizes the variance of the present value random variable of the new product.
(A) 0 (B) 50,000 (C) 100,000 (D) 150,000 (E) 200,000

Solution: (D) Let $\pi = 12000$ and let P be the death benefit. The present value random variable of the new product is
$Y = \pi \ddot{Y}_x + PZ_x = \left(P - \frac{\pi}{d}\right) Z_x + \frac{\pi}{d}$. The variance of Y is
$\left(P - \frac{\pi}{d}\right)^2 \mathrm{Var}(Z_x)$, which is minimized when
$P = \frac{\pi}{d} = \frac{12000}{0.08} = 150000$.

(#35, MLC–09–08) You are given:
(i) $\mu_x(t) = 0.01$, $0 \leq t < 5$
(ii) $\mu_x(t) = 0.02$, $5 \leq t$
(iii) $\delta = 0.06$
Calculate $\overline{a}_x$.
(A) 12.5 (B) 13.0 (C) 13.4 (D) 13.9 (E) 14.3

(#35, MLC–09–08) You are given:
(i) $\mu_x(t) = 0.01$, $0 \le t < 5$
(ii) $\mu_x(t) = 0.02$, $5 \le t$
(iii) $\delta = 0.06$
Calculate $\overline{a}_x$.
(A) 12.5 (B) 13.0 (C) 13.4 (D) 13.9 (E) 14.3
Solution: (B)

$$\overline{a}_x = \overline{a}_{x:\overline{5}|} + {}_5E_x \overline{a}_{x+5} = \frac{1 - e^{-5(0.01+0.06)}}{0.01 + 0.06} + e^{-5(0.01+0.06)} \frac{1}{0.02 + 0.06}$$
$$= \frac{1 - e^{-0.35}}{0.07} + e^{-0.35} \frac{1}{0.08} = 13.0273427.$$

(#55, MLC–09–08) For a 20-year deferred whole life annuity-due of 1 per year on (45), you are given:
(i) Mortality follows De Moivre's law with $\omega = 105$.
(ii) $i = 0$
Calculate the probability that the sum of the annuity payments actually made will exceed the actuarial present value at issue of the annuity.
(A) 0.425 (B) 0.450 (C) 0.475 (D) 0.500 (E) 0.525

(B) The present value of the sum of the annuity payments is $\max(K_{45} - 20, 0)$. So,

$$ {}_{20|}\ddot{a}_{45} = \sum_{j=1}^{40} j\mathbb{P}\{K_{45} = 20+j\} = \sum_{j=1}^{40} j\frac{1}{60} = \frac{\frac{(40)(41)}{2}}{60} = \frac{41}{3} = 13.66666667. $$

The probability that the sum of the annuity payments actually made will exceed the actuarial present value at issue of the annuity is

$$ \mathbb{P}\{K_{45} - 20 > 13.66666667\} = \mathbb{P}\{K_{45} > 33\} = \mathbb{P}\{T_{45} > 33\} $$
$$ = \frac{60 - 33}{60} = 0.45. $$

(#63, MLC–09–08) For a whole life insurance of 1 on (x), you are given:
(i) The force of mortality is $\mu_x(t)$.
(ii) The benefits are payable at the moment of death.
(iii) $\delta = 0.06$
(iv) $\overline{A}_x = 0.60$
Calculate the revised actuarial present value of this insurance assuming $\mu_x(t)$ is increased by 0.03 for all t and δ is decreased by 0.03.
(A) 0.5 (B) 0.6 (C) 0.7 (D) 0.8 (E) 0.9

(D) **Solution 1:** Let ${}_tp_x^*$ and $\overline{A}_x^*$ be the values for the revised table. We have that

$$
\begin{aligned}
{}_tp_x^* &= e^{-\int_0^t(\mu_x(s)+0.03)\,ds} = e^{-0.03t}\,{}_tp_x,\\
\overline{A}_x^* &= \int_0^\infty e^{-0.03t}\,{}_tp_x^*(\mu_x(t)+0.03)\,dt\\
&= \int_0^\infty e^{-0.03t}e^{-0.03t}\,{}_tp_x(\mu_x(t)+0.03)\,dt\\
&= \int_0^\infty e^{-0.03t}e^{-0.03t}\,{}_tp_x\mu_x(t)\,dt + \int_0^\infty e^{-0.03t}e^{-0.03t}\,{}_tp_x 0.03\,dt\\
&= \overline{A}_x + (0.03)\overline{a}_x\\
&= (0.60) + (0.03)\frac{1-0.60}{0.06} = 0.8.
\end{aligned}
$$

Solution 2: Let ${}_tp_x^*$, $\overline{A}_x^*$ and $\overline{a}_x^*$ be the values for the revised table. We have that

$$
\begin{aligned}
{}_tp_x^* &= e^{-\int_0^t (\mu_x(s)+0.03)\,ds} = e^{-0.03t}\,{}_tp_x,\\
\overline{a}_x^* &= \int_0^\infty e^{-0.03t}\,{}_tp_x^*\,dt = \int_0^\infty e^{-0.03t} e^{-0.03t}\,{}_tp_x\,dt\\
&= \int_0^\infty e^{-0.06t}\,{}_tp_x\,dt = \overline{a}_x = \frac{1-0.60}{0.06} = \frac{20}{3},\\
\overline{A}_x^* &= 1 - \frac{20}{3}(0.03) = 0.8.
\end{aligned}
$$

(#67, MLC–09–08) For a continuous whole life annuity of 1 on (x):
(i) $T(x)$ is the future lifetime random variable for (x).
(ii) The force of interest and force of mortality are constant and equal.
(iii) $\overline{a}_x = 12.50$
Calculate the standard deviation of $\overline{a}_{\overline{T(x)}|}$.
(A) 1.67 (B) 2.50 (C) 2.89 (D) 6.25 (E) 7.22

(E) If $\mu = \delta$, then

$$\overline{A}_x = \frac{\mu}{\mu+\delta} = \frac{1}{2}, \ {}^2\overline{A}_x = \frac{\mu}{\mu+2\delta} = \frac{1}{3}, \ \mathrm{Var}(\overline{Z}_x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12},$$
$$12.50 = \overline{a}_x = \frac{1}{\mu+\delta} = \frac{1}{2\delta}.$$

So, $\delta = \frac{1}{25} = 0.04$ and

$$\sqrt{\mathrm{Var}(\overline{Y}_x)} = \frac{\sqrt{\mathrm{Var}(\overline{Z}_x)}}{\delta} = \frac{25}{\sqrt{12}} = 7.216878365.$$

(#79, MLC–09–08) For a group of individuals all age x, you are given:
(i) 30% are smokers and 70% are non-smokers.
(ii) The constant force of mortality for smokers is 0.06.
(iii) The constant force of mortality for non-smokers is 0.03.
(iv) $\delta = 0.08$
Calculate $\mathrm{Var}(\overline{a}_{\overline{T(x)}|})$ for an individual chosen at random from this group.
(A) 13.0 (B) 13.3 (C) 13.8 (D) 14.1 (E) 14.6

(D) We have that

$$\overline{A}_x = (0.3)\frac{0.06}{0.06+0.08} + (0.7)\frac{0.03}{0.03+0.08} = 0.3194805195,$$

$$^2\overline{A}_x = (0.3)\frac{0.06}{0.06+(2)0.08} + (0.7)\frac{0.03}{0.03+(2)0.08} = 0.1923444976,$$

$$\mathrm{Var}(\overline{a}_{\overline{T(x)}|}) = \frac{\mathrm{Var}(\overline{Z}_x)}{\delta^2} = \frac{0.1923444976-(0.3194805195)^2}{(0.08)^2} = 14.10573$$

(#86, MLC–09–08) You are given:
(i) $A_x = 0.28$
(ii) $A_{x+20} = 0.40$
(iii) $A_{x:\overline{20|}}^{\;\,1} = 0.25$
(iv) $i = 0.05$
Calculate $a_{x:\overline{20|}}$.
(A) 11.0 (B) 11.2 (C) 11.7 (D) 12.0 (E) 12.3

(B) We have that

$$
\begin{aligned}
{}_{20|}A_x &= {}_{20}E_x A_{x+20} = (0.4)(0.25) = 0.1, \\
A^{1}_{x:\overline{20|}} &= A_x - {}_{20|}A_x = 0.28 - 0.1 = 0.18, \\
A_{x:\overline{20|}} &= A^{1}_{x:\overline{20|}} + {}_{20}E_x = 0.18 + 0.25 = 0.43, \\
\ddot{a}_{x:\overline{20|}} &= \frac{1 - A_{x:\overline{20|}}}{d} = (21)(1 - 0.43) = 11.97, \\
a_{x:\overline{20|}} &= \ddot{a}_{x:\overline{20|}} - 1 + {}_{20}E_x = 11.97 - 1 + 0.25 = 11.22.
\end{aligned}
$$

(#88, MLC–09–08) At interest rate i:
(i) $\ddot{a}_x = 5.6$
(ii) The actuarial present value of a 2-year certain and life annuity-due of 1 on (x) is $\ddot{a}_{\overline{x:\overline{2}|}} = 5.6459$.
(iii) $e_x = 8.83$
(iv) $e_{x+1} = 8.29$
Calculate i.
(A) 0.077 (B) 0.079 (C) 0.081 (D) 0.083 (E) 0.084

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Calculate i.
(A) 0.077 (B) 0.079 (C) 0.081 (D) 0.083 (E) 0.084

Solution: (B) From $8.83 = e_x = p_x(1 + e_{x+1}) = p_x 9.29$, we get that $p_x = \frac{8.83}{9.29} = 0.9504843918$. We have that

$$5.6459 = \ddot{a}_{\overline{\overline{x:\overline{2}|}}} = 1 + v + v^2 p_x p_{x+1} \ddot{a}_{x+2},$$

$$5.6 = \ddot{a}_x = 1 + vp_x + v^2 p_x p_{x+1} \ddot{a}_{x+2}$$

Hence, $5.6459 - 5.6 = 0.0459 = vq_x$ and
$1 + i = \frac{1-0.9504843918}{0.0459} = 1.07877142$.

(#113, MLC–09–08) For a disability insurance claim:
(i) The claimant will receive payments at the rate of 20,000 per year, payable continuously as long as she remains disabled.
(ii) The length of the payment period in years is a random variable with the gamma distribution with parameters $\alpha = 2$ and $\theta = 1$. That is, $f(t) = te^{-t}$, $t > 0$
(iii) Payments begin immediately.
(iv) $\delta = 0.05$
Calculate the actuarial present value of the disability payments at the time of disability.
(A) 36,400 (B) 37,200 (C) 38,100 (D) 39,200 (E) 40,000

(B) We have that

$$E\left[(20000)\bar{a}_{\overline{T}|}\right] = \int_0^\infty (20000)\bar{a}_{\overline{T}|} te^{-t}\,dt$$

$$= \int_0^\infty (20000)\frac{1-e^{-0.05t}}{0.05} te^{-t}\,dt = (20000)(20)\int_0^\infty (te^{-t} - te^{-1.05t})$$

$$=(400000)\left(1-\frac{1}{(1.05)^2}\right) = 37188.20862.$$

(#114, MLC–09–08) For a special 3-year temporary life annuity-due on (x), you are given: (i)

t	Annuity Payment	p_{x+t}
0	15	0.95
1	20	0.90
2	25	0.85

(ii) $i = 0.06$
Calculate the variance of the present value random variable for this annuity.
(A) 91 (B) 102 (C) 114 (D) 127 (E) 139

(C) We have that:
if $T_x \le 1$, $\ddot{Y}_{x:\overline{3}|} = 15$,
if $1 < T_x \le 2$, $\ddot{Y}_{x:\overline{3}|} = 15 + 20(1.06)^{-1} = 33.8679245283019$,
if $2 < T_x$,
$\ddot{Y}_{x:\overline{3}|} = 15 + 20(1.06)^{-1} + 25(1.06)^{-2} = 56.1178355286579$.
Hence,

$$
\begin{aligned}
&E[\ddot{Y}_{x:\overline{3}|}] = (15)(0.05) + (33.8679245283019)(0.95)(0.1) \\
&\quad + (56.1178355286579)(0.95)(0.90) \\
&= 51.94820221, \\
&E[(\ddot{Y}_{x:\overline{3}|})^2] = ((15)^2(0.05) + (33.8679245283019)^2(0.95)(0.1) \\
&\quad + (56.1178355286579)^2(0.95)(0.90) \\
&= 2812.794252, \\
&\mathrm{Var}(\ddot{Y}_{x:\overline{3}|}) = 2812.794252 - (51.94820221)^2 = 114.1785391.
\end{aligned}
$$

(#126, MLC–09–08) A government creates a fund to pay this year's lottery winners.

You are given:

(i) There are 100 winners each age 40.

(ii) Each winner receives payments of 10 per year for life, payable annually, beginning immediately.

(iii) Mortality follows the Illustrative Life Table.

(iv) The lifetimes are independent.

(v) $i = 0.06$

(vi) The amount of the fund is determined, using the normal approximation, such that the probability that the fund is sufficient to make all payments is 95%.

Calculate the initial amount of the fund.

(A) 14,800 (B) 14,900 (C) 15,050 (D) 15,150 (E) 15,250

(E) Let $Y = \sum_{j=1}^{100} 10\ddot{Y}_{40,j}$ be the present value of the payments. We have that

$$E[Y] = (100)(10)\ddot{a}_{40} = (100)(10)(14.8166) = 14816.6,$$

$$\mathrm{Var}(Y) = (100)(10)^2\mathrm{Var}(\ddot{Y}_{40}) = (100)(10)^2\frac{0.04863 - 0.16132^2}{(6/106)^2}$$

$=70555.39333.$

The initial amount of the fund is $14816.6 + (1.645)\sqrt{70555.39333} = 15253.54926.$

(#130, MLC–09–08) A person age 40 wins 10,000 in the actuarial lottery. Rather than receiving the money at once, the winner is offered the actuarially equivalent option of receiving an annual payment of K (at the beginning of each year) guaranteed for 10 years and continuing thereafter for life.

You are given:

(i) $i = 0.04$

(ii) $A_{40} = 0.30$

(iii) $A_{50} = 0.35$

(iv) $A^{1}_{40:\overline{10}|} = 0.09$

Calculate K.

(A) 538 (B) 541 (C) 545 (D) 548 (E) 551

(A) We have that

$$0.30 = A_{40} = A^{1}_{40:\overline{10}|} + {}_{10}E_{40} \cdot A_{50} = 0.09 + {}_{10}E_{40} \cdot 0.35,$$

$${}_{10}E_{40} = \frac{0.30 - 0.09}{0.35} = 0.6,$$

$$\ddot{a}_{\overline{10}|} + {}_{10}E_{40}\ddot{a}_{50} = \ddot{a}_{\overline{10}|} + (0.6)\frac{1 - 0.35}{d} = 18.57533161,$$

$$K = \frac{10000}{18.57533161} = 538.3483972.$$

(#140, MLC–09–08) Y is the present-value random variable for a special 3-year temporary life annuity-due on (x). You are given:

(i) ${}_tp_x = 0.9^t$, $t \geq 0$

(ii) K is the curtate-future-lifetime random variable for (x).

(iii) $Y = \begin{cases} 1.00, & K = 0 \\ 1.87, & K = 1 \\ 2.72, & K = 2, 3, \ldots \end{cases}$

Calculate $\mathrm{Var}(Y)$.

(A) 0.19 (B) 0.30 (C) 0.37 (D) 0.46 (E) 0.55

(B) We have that

$$\mathbb{P}\{K=0\}=0.1,\ \mathbb{P}\{K=1\}=(0.9)(0.1)=0.09,$$
$$\mathbb{P}\{K\geq 2\}=(0.9)(0.9)=0.81,$$
$$E[Y]=(1)(0.1)+(1.87)(0.09)+(2.72)(0.81)=2.4715,$$
$$E[Y^2]=(1)^2(0.1)+(187)^2(0.09)+(2.72)^2(0.81)=6.407425,$$
$$\mathrm{Var}(Y)=6.407425-(2.4715)^2=0.29911275.$$

(#146, MLC–09–08) A fund is established to pay annuities to 100 independent lives age x. Each annuitant will receive 10,000 per year continuously until death. You are given:
(i) $\delta = 0.06$
(ii) $\overline{A}_x = 0.40$
(iii) ${}^2\overline{A}_x = 0.25$
Calculate the amount (in millions) needed in the fund so that the probability, using the normal approximation, is 0.90 that the fund will be sufficient to provide the payments.
(A) 9.74 (B) 9.96 (C) 10.30 (D) 10.64 (E) 11.10

(D) The present value of all annuities is $\overline{Y} = \sum_{j=1}^{100} (10)^4 \overline{Y}_{x,j}$.
Hence,

$$E[\overline{Y}] = (100)(10)^4 \overline{a}_x = (10)^6 \frac{1 - 0.40}{0.06} = (10)^7,$$

$$\mathrm{Var}(\overline{Y}) = (100)(10)^8 \mathrm{Var}(\overline{Y}_x) = (10)^{10} \frac{0.25 - (0.40)^2}{(0.06)^2} = (10)^{10} \frac{0.09}{(0.06)^2}.$$

The amount needed is

$$(10)^7 + (1.28)(10)^5 \frac{\sqrt{0.09}}{0.06}.$$

The amount (in millions) needed is

$$(10) + (0.128) \frac{0.3}{0.06} = 10.64.$$

(#154, MLC–09–08) For a special 30-year deferred annual whole life annuity-due of 1 on (35):
(i) If death occurs during the deferral period, the single benefit premium is refunded without interest at the end of the year of death.
(ii) $\ddot{a}_{65} = 9.90$
(iii) $A_{35:\overline{30|}} = 0.21$
(iv) $A^{1}_{35:\overline{30|}} = 0.07$
Calculate the single benefit premium for this special deferred annuity.
(A) 1.3 (B) 1.4 (C) 1.5 (D) 1.6 (E) 1.7

(C) Let π be the single benefit premium. We have that $\pi = \pi A^{1}_{35:\overline{30}|} + {}_{35}E_{30}\ddot{a}_{65} = 0.7\pi + {}_{35}E_{30}(9.9)$. So,

$$
\begin{aligned}
{}_{35}E_{30} &= A_{35:\overline{30}|} - A^{1}_{35:\overline{30}|} = 0.21 - 0.07 = 0.14, \\
\pi &= \frac{(0.14)(9.9)}{1 - 0.07} = 1.490322581.
\end{aligned}
$$

