Chapter 2

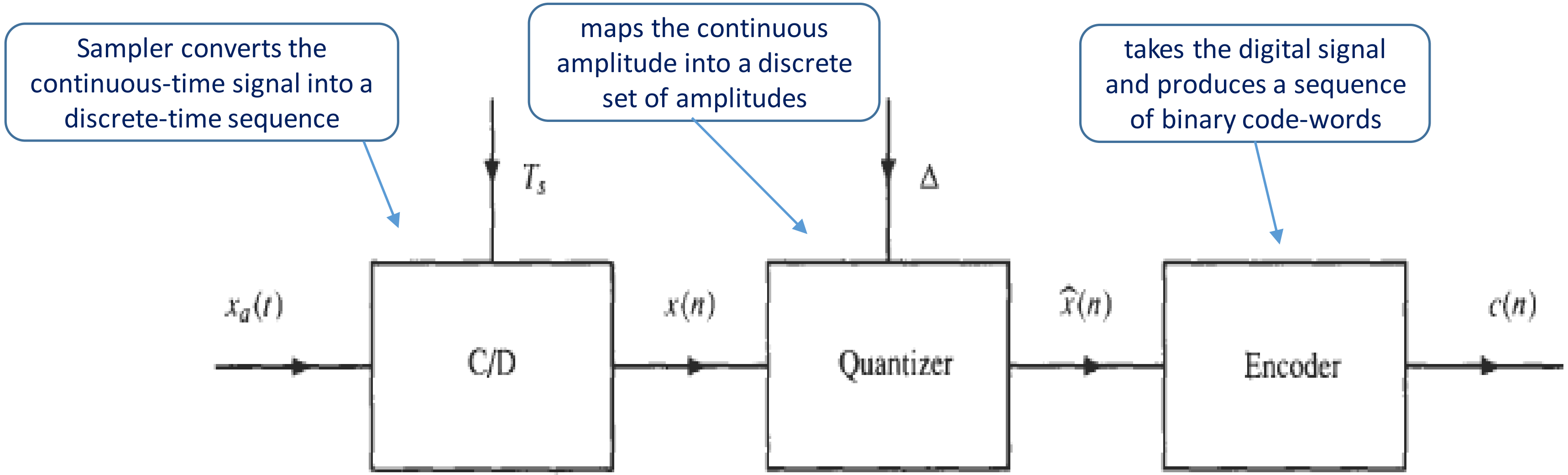
Signal Sampling and Quantization

Introduction$_1$

- Even most of signals are in continuous-time domain, they should be converted to a number at different discrete time to be processed by a microprocessor.

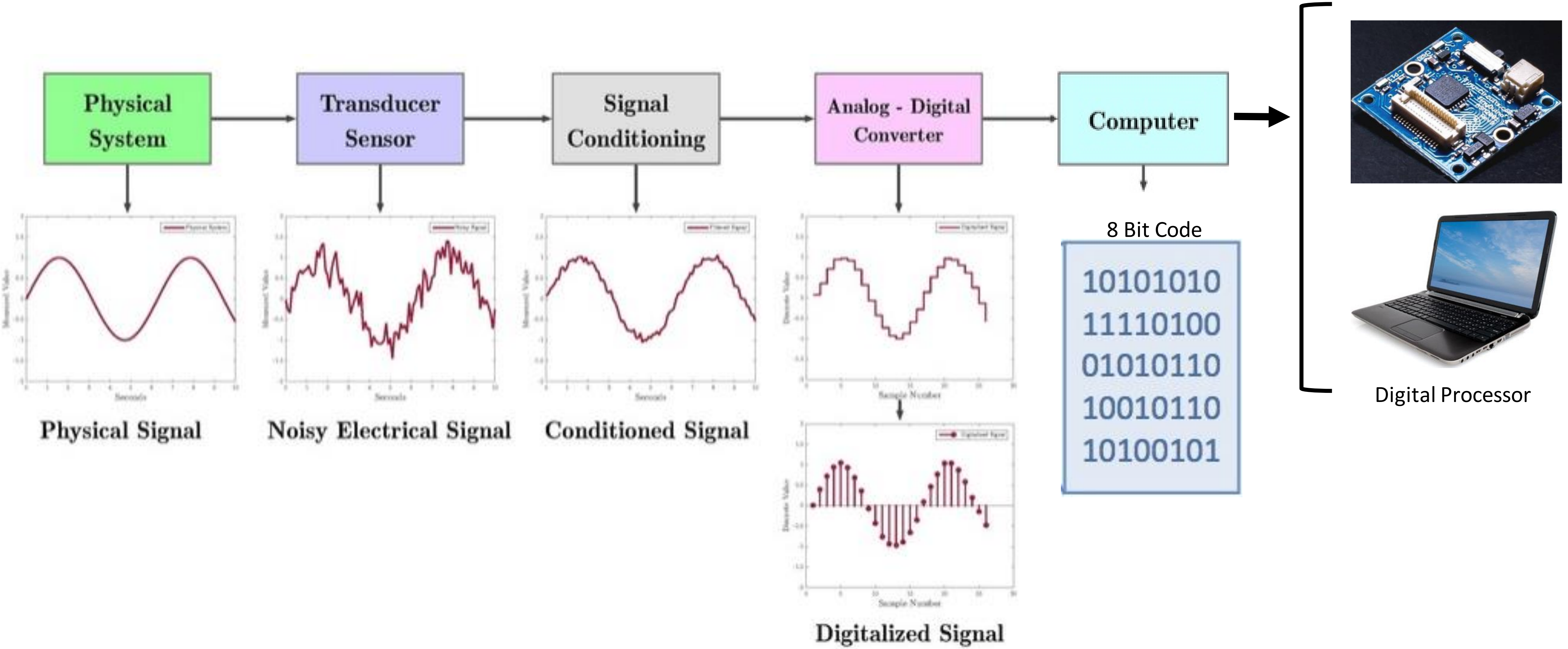
- The process of converting these signals into digital form is called analog-to-digital (A/D) conversion.

- The reverse process of reconstructing an analog signal from its samples is known as digital-to-analog (D/A) conversion.



Components of an analog-to-digital converter (ADC)

Introduction$_2$



Sampling

- *Periodic or uniform sampling*, a sequence of samples $x[n]$ is obtained from a continuous-time signal $x_c\ [t]$ by taking values at equally spaced points in time. *T* is the fixed time interval between samples, is known as the *the sampling period.*

$$x[n] \triangleq x_c(t)|_{t=nT} = x_c(nT), \quad -\infty < n < \infty$$

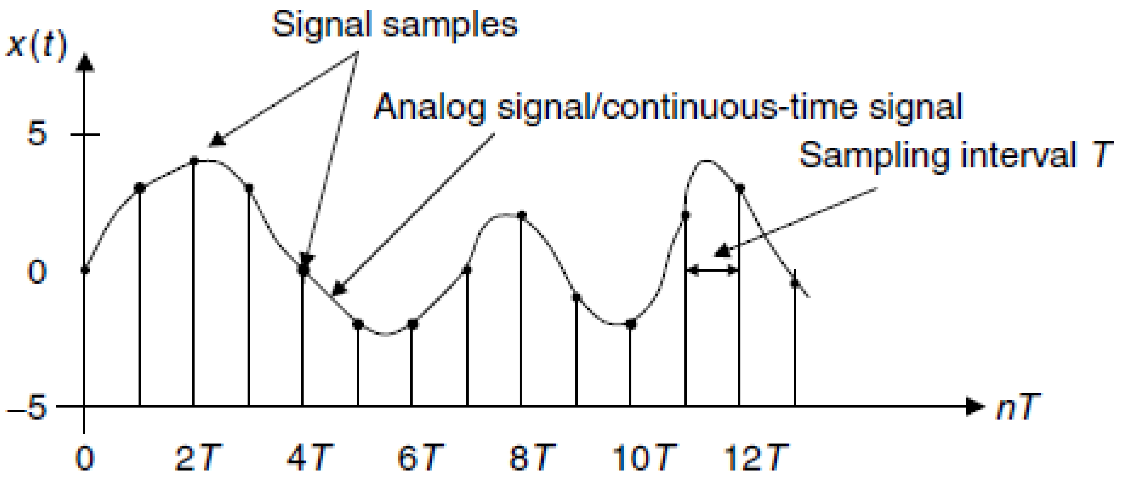
- The reciprocal F_s is called *sampling frequency* (cycles per second or Hz) or *sampling rate* (samples per second).

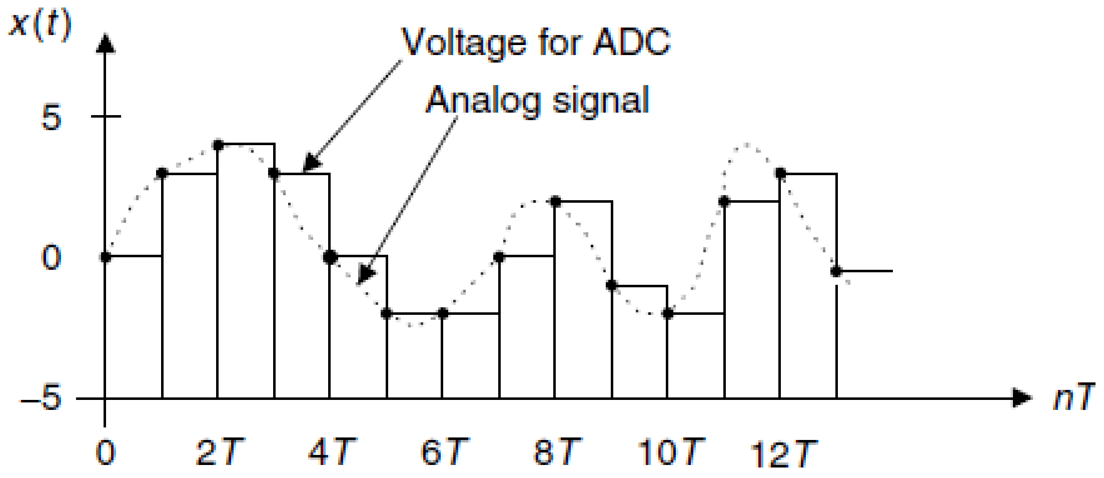
Sampling rate
Sample per second (Hz) → $F_s = 1/T$ ← Sampling period (second)

Example ⇒

sampling period: T = 125 µs.

sampling rate: F_s =1/125µs = 8,000 samples per second (Hz).

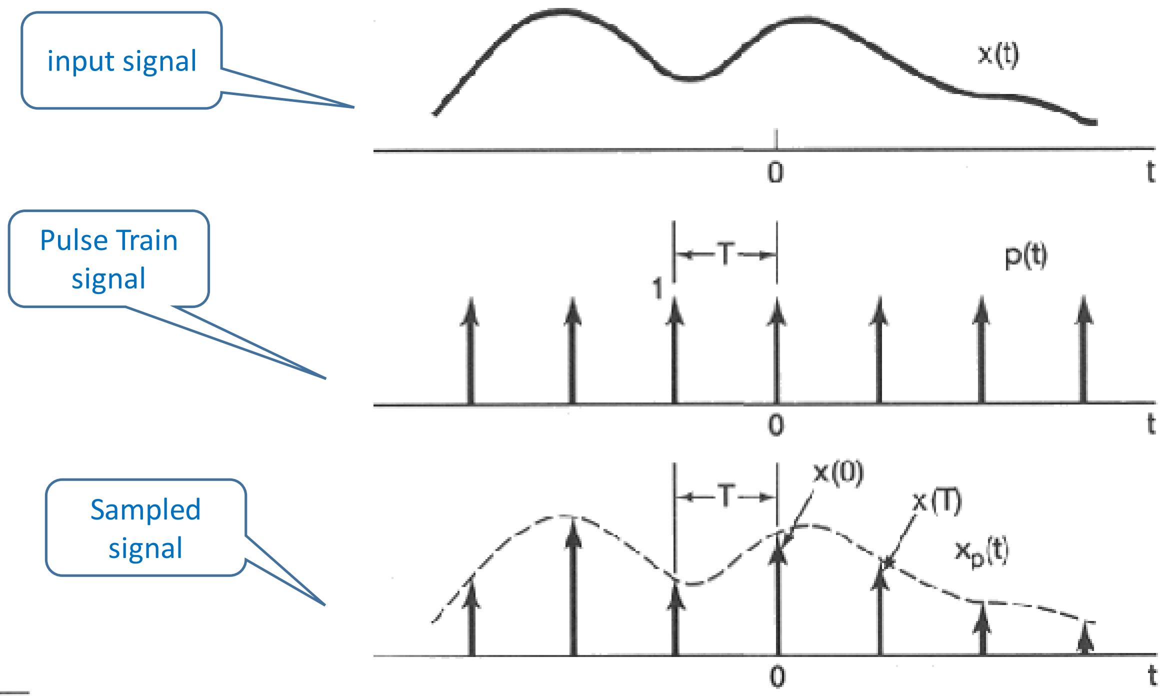


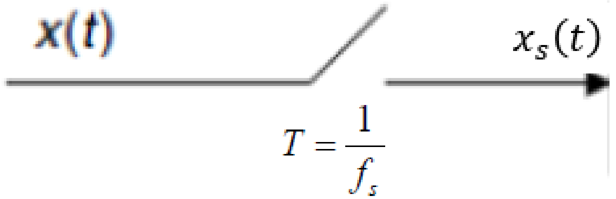


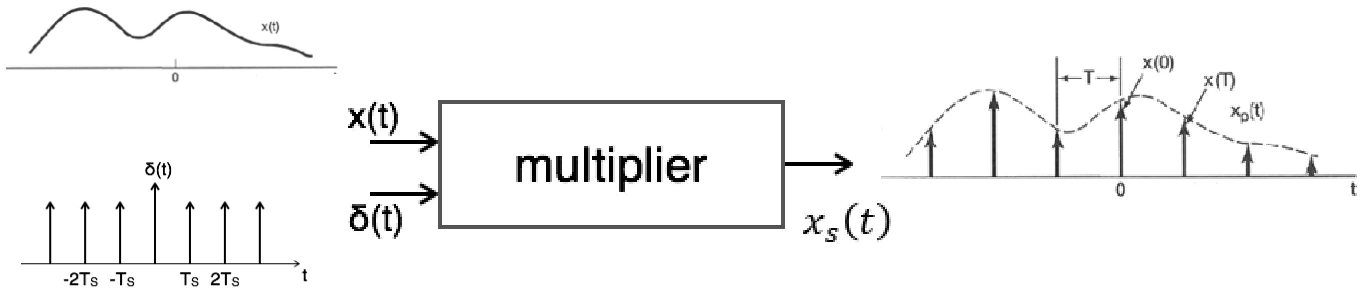
Sample and Hold

Sampling Process

- The sampling of a continuous-time signal $x[n]$ is equivalent to multiply the signal with pulse train signal $S(t)$.







$x(t)$ Input analog signal

$S(t)$ Pulse train

Sampled signal

$$x_s(t) = x(t) \cdot S(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\, \delta(t - nT_s)$$

Sampling Process - frequency domain 1

- The signal $x_c\ [t]$ and its spectrum $X_c\ (j\Omega)$

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt,$$ Fourier Transform

$$x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)\, e^{j\Omega t}d\Omega. \quad (1)$$ Inverse Fourier Transform

- The sequence $x[n]$ and its *periodic* spectrum $X\ (j\omega)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \quad (2)$$

- Since $x[n]$ is related to $x_c\ [t]$ $with\ t = nT = \frac{n}{F_s}$ (1)(2) ⇒

$$\omega = \Omega T = 2\pi FT = 2\pi \frac{F}{F_s} = 2\pi f.$$

Analog Frequency (Hz)

Normalized Frequency (Cycles/ Sample)

Sampling Frequency (Hz)

- The desired relationship between sampled signal spectrum $X_s\ (F)$ and the continuous signal spectrum $X_c\ (F)$

From spectral analysis , and after some mathematical operations ⇒

$$X_s(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(F - kF_s)$$

$X_s(F)$: Sampled signal spectrum
$X_c(F)$: Original signal spectrum
$X(F \pm kF_s)$: Replica spectrum

Sampling Process - frequency domain 2

$$X_s(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(F - kF_s) \quad \Rightarrow \quad X_s(f) = \cdots + \frac{1}{T}X_c(F + F_s) + \frac{1}{T}X_c(F) + \frac{1}{T}X_c(F - F_s) + \cdots$$

- Spectrum of $x[n]$ is obtained by scaling the spectrum of $x_c[t]$, putting copies of the scaled spectrum $\left(\frac{1}{T}\right) X_c(F)$, at all integer multiples of the sampling frequency $F_s = \frac{1}{T}$.

- The spectrum of $x[n]$ can be readily sketched if $x_c(t)$ is assumed to be band-limited. $X_c(F) = 0 \; for \; |F| > F_H$

- Two conditions obviously are necessary to prevent *overlapping spectral bands*:

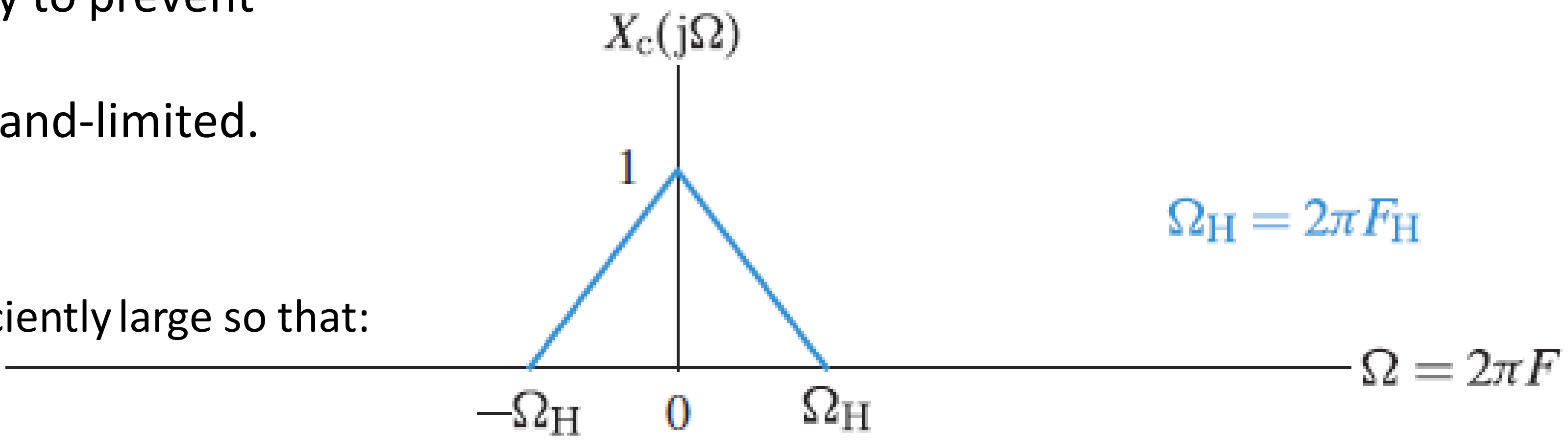
1. The continuous-time signal must be band-limited.

$$X_c(j\Omega) = 0, \quad |\Omega| > \Omega_H$$

2. The sampling frequency Ω_s must be sufficiently large so that:

$$\Omega_s - \Omega_H > \Omega_H$$

$$\Rightarrow \quad \Omega_s \geq 2\Omega_H \quad \text{or} \quad T \leq \frac{1}{2F_H}$$

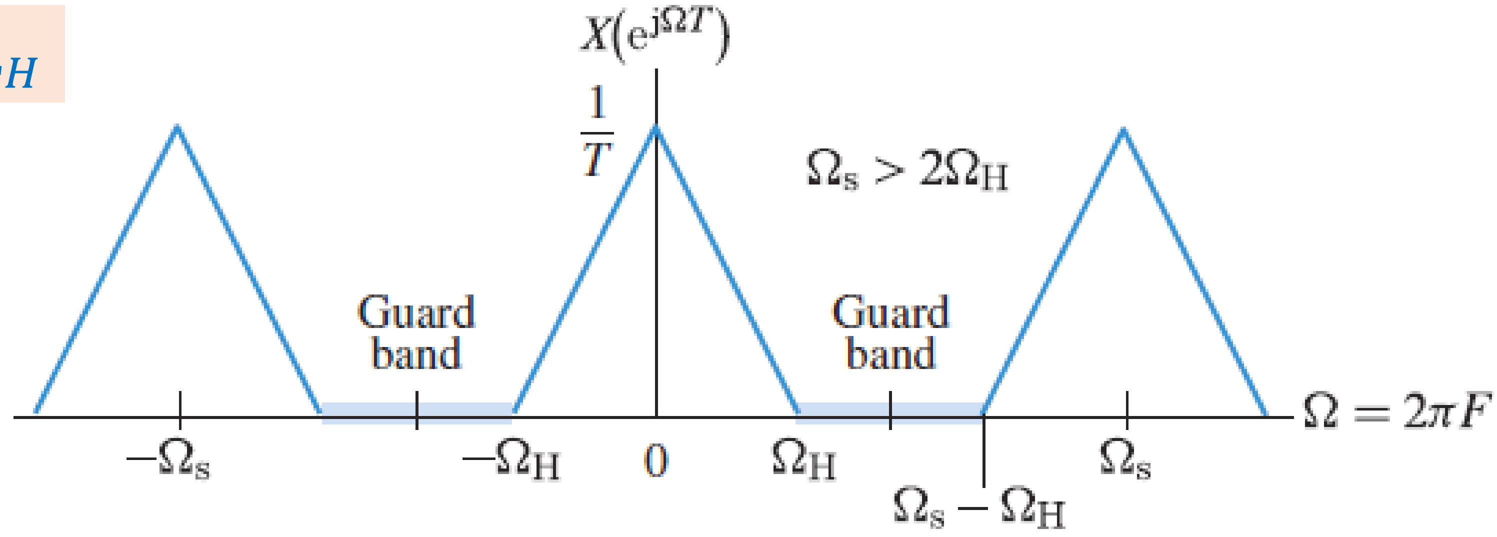


$\Omega_H = 2\pi F_H$

Spectrum of continuous-time band-limited signal $x_c(t)$

Sampling Process – frequency domain 3

Case 1: $\Omega_s > 2\Omega_H$



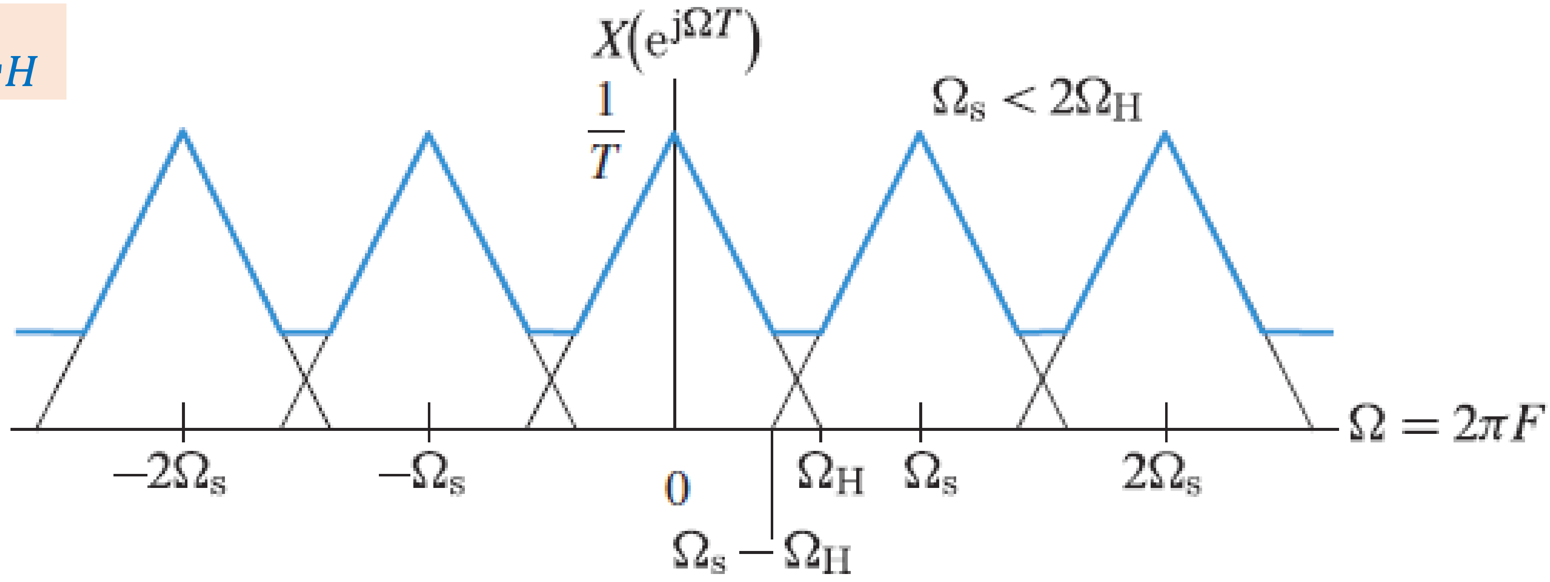
spectrum of discrete-time signal $x[n] = x_c[nT]$ with $\Omega_s > 2\Omega_H$

- The sampling operation leaves the input spectrum $X_c(\Omega)$ *intact* when $\Omega_s > 2\Omega_H$, therefore, it should be possible to *recover* or *reconstruct* $x_c(t)$ from the sequence $x[n]$.

- Sampling at $\Omega_s > 2\Omega_H$ creates a *guard band* which simplifies the reconstruction process in practical applications.

Sampling Process – frequency domain 4

Case2: $\Omega_s < 2\Omega_H$



spectrum of x[n], showing aliasing distortion, when s $\Omega_s < 2\Omega_H$

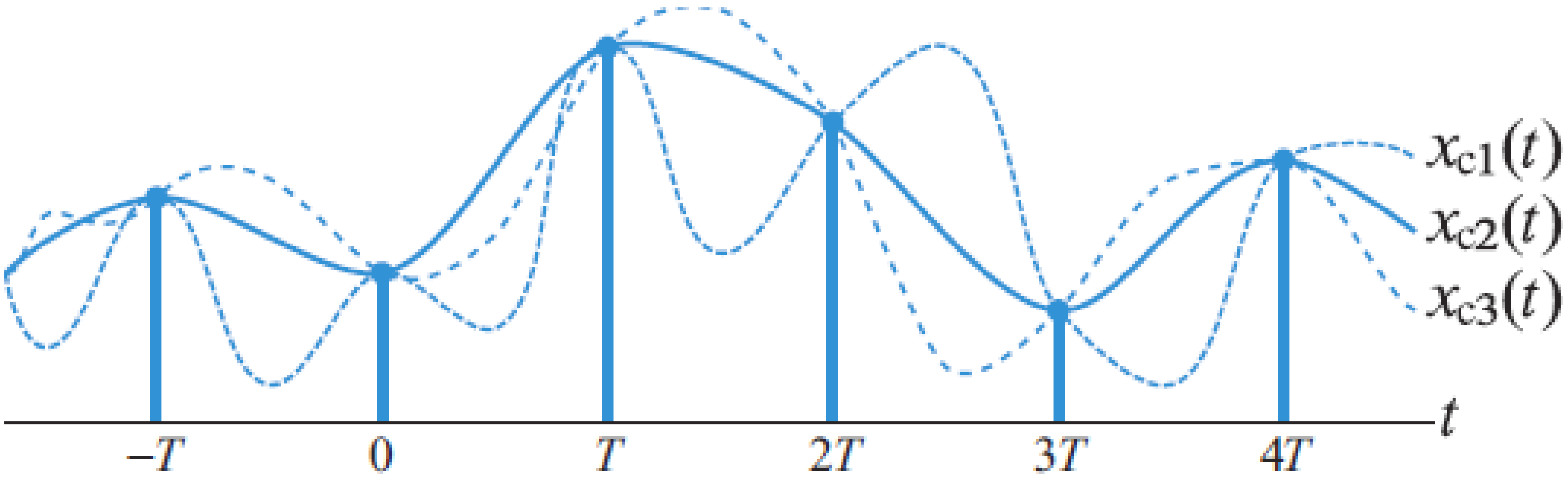
- If $\Omega_s < 2\Omega_H$, the scaled copies of $X_c(\Omega)$ overlap, so that when they are added together, $X_c(\Omega)$ *cannot be recovered* from $X(\Omega)$.

- This effect, in which individual terms overlap is known as *aliasing distortion* or simply *aliasing*.

Sampling Theorem 1

- ***Question***: Are the samples $x[n]$ sufficient to describe uniquely the original continuous-time signal and, if so, how can $x_c\ [t]$ be reconstructed from $x[n]$? An infinite number of signals can generate the same set of samples.

- ***Answer:*** The response lies in the frequency domain, in the relation between the *spectra* of $x_c\ [t]$ and $x[n]$.



different continuous-time signals with the same set of sample values

Sampling Theorem 2

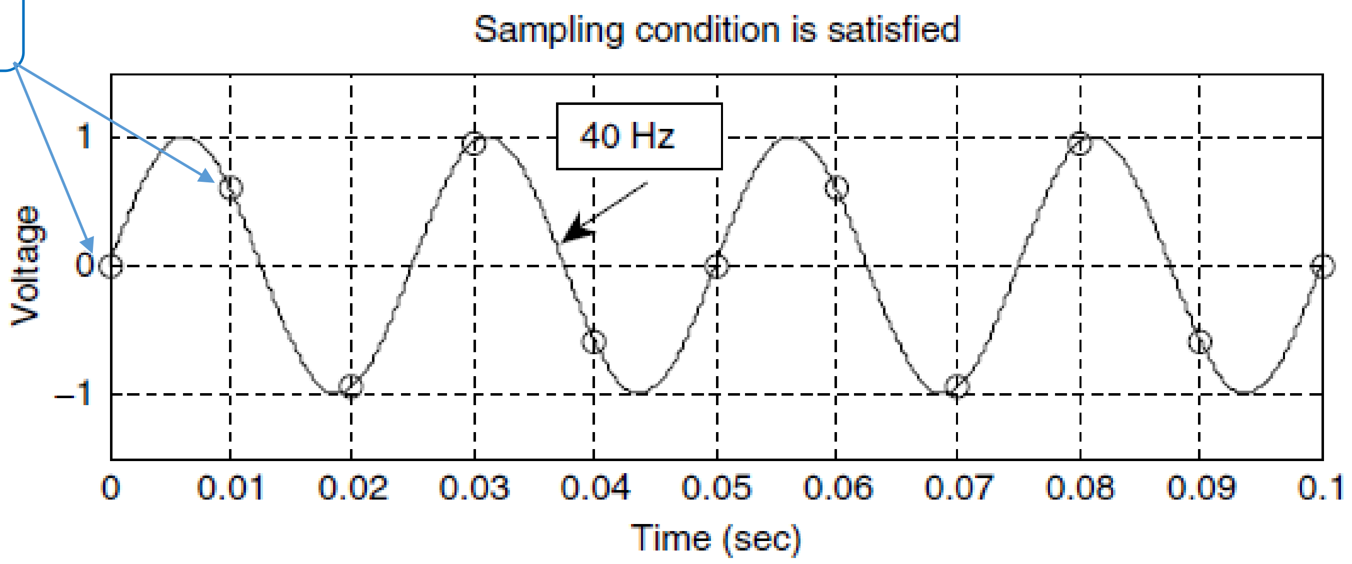
$$T = 0.01\ \text{sec} \rightarrow F_s = \frac{1}{T} = \frac{1}{0.01} = 100\ Hz$$

One sample each 0.01 s

Sampling interval T= 0.01 s
Sampling rate f_s= 100 Hz
Sinusoid freq. = 4 cycles / 0.1
= 40 Hz

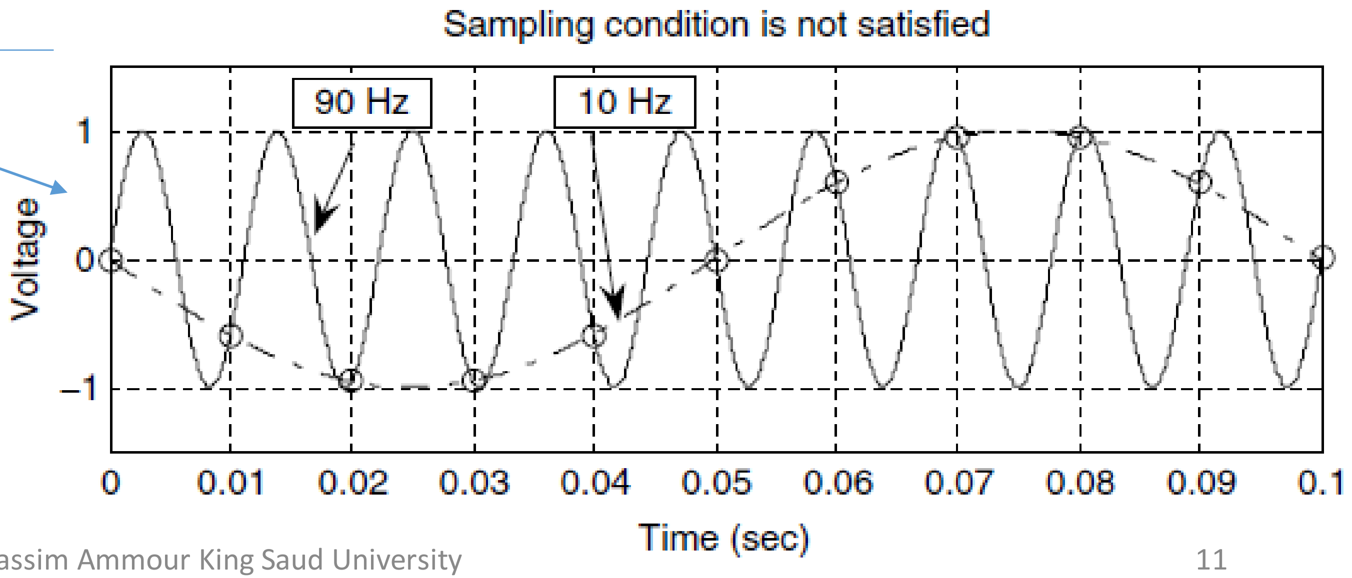
$$2f_{\max} = 80\ \text{Hz} < f_s.$$

Sampling condition is satisfied, so reconstruction from digital to analog is possible.



The signal is under-sampled 2 fmax=180 > fs

Do this by yourself!



Sampling Theorem 3

- An analog signal can be perfectly recovered (*reconstruction filter*) as long as the sampling rate is at *least twice* as large as the highest-frequency component of the analog signal to be sampled (**Shannon sampling theorem**).

- Let $x_c(t)$ be a continuous-time band-limited signal with Fourier transform: $X_c(j\Omega) = 0 \quad \text{for} |\Omega| > \Omega_H.$

Then $x_c(t)$ can be uniquely determined by its samples $x[n] = x_c(nT)$, where $n = 0, \pm1, \pm2, \ldots$ if the sampling frequency Ω_s satisfies the condition:

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_H. \quad \Rightarrow \quad F_s = \frac{2\pi}{T_s} \geq 2\, F_{max}$$

- Half of the sampling frequency $\frac{F_s}{2}$ is usually called the *Nyquist frequency* (Nyquist limit), or *folding frequency*.

Example: To sample a speech signal containing frequencies up to 4 kHz, the minimum sampling rate is chosen to be at least 8 kHz, or 8,000 samples per second.

Example 1

Problem:

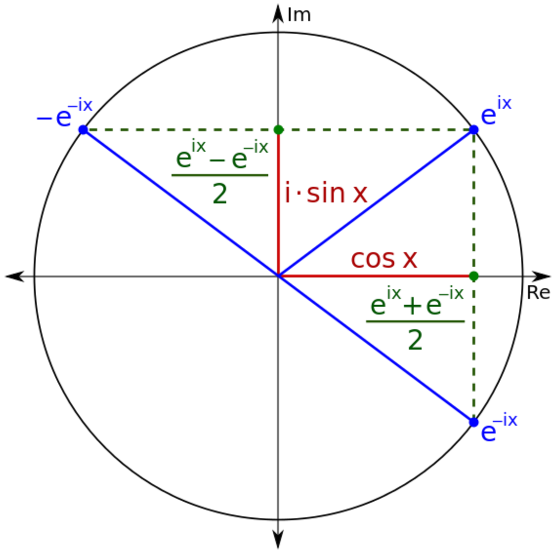
Suppose that an analog signal is given as

$$x(t) = 5\cos(2\pi \cdot 1000t), \text{ for } t \geq 0$$

and is sampled at the rate of 8,000 Hz.

a. Sketch the spectrum for the original signal.

b. Sketch the spectrum for the sampled signal from 0 to 20 kHz.



Euler's identity

$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Solution:

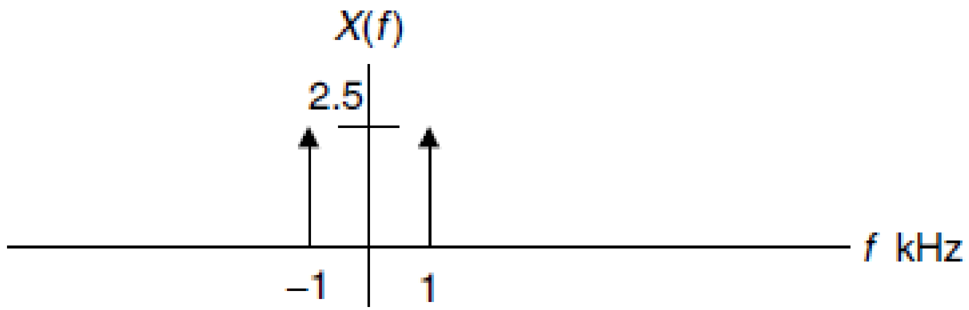
Using Euler's identity,

$$5\cos(2\pi \times 1000t) = 5 \cdot \left(\frac{e^{j2\pi\times1000t} + e^{-j2\pi\times1000t}}{2}\right) = 2.5e^{j2\pi\times1000t} + 2.5e^{-j2\pi\times1000t}$$

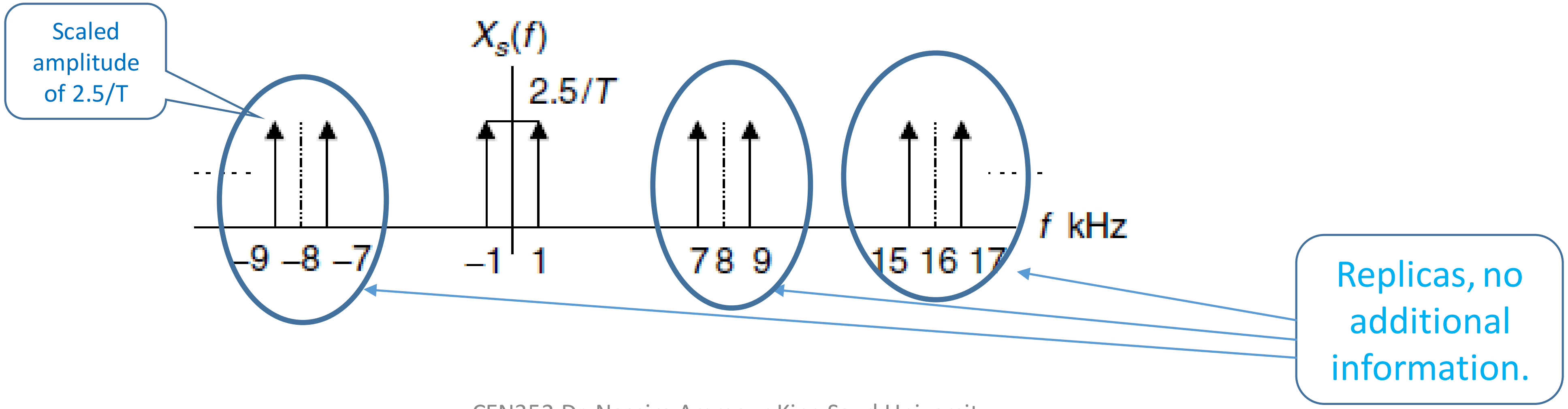
Hence, the Fourier series coefficients are: $c_1 = 2.5$, and $c_{-1} = 2.5$.

Example 1 - Contd.

a.



b. After the analog signal is sampled at the rate of 8,000Hz, the sampled signal spectrum and its replicas centered at the frequencies $\pm nf_s$, each with the scaled amplitude being 2.5/T .



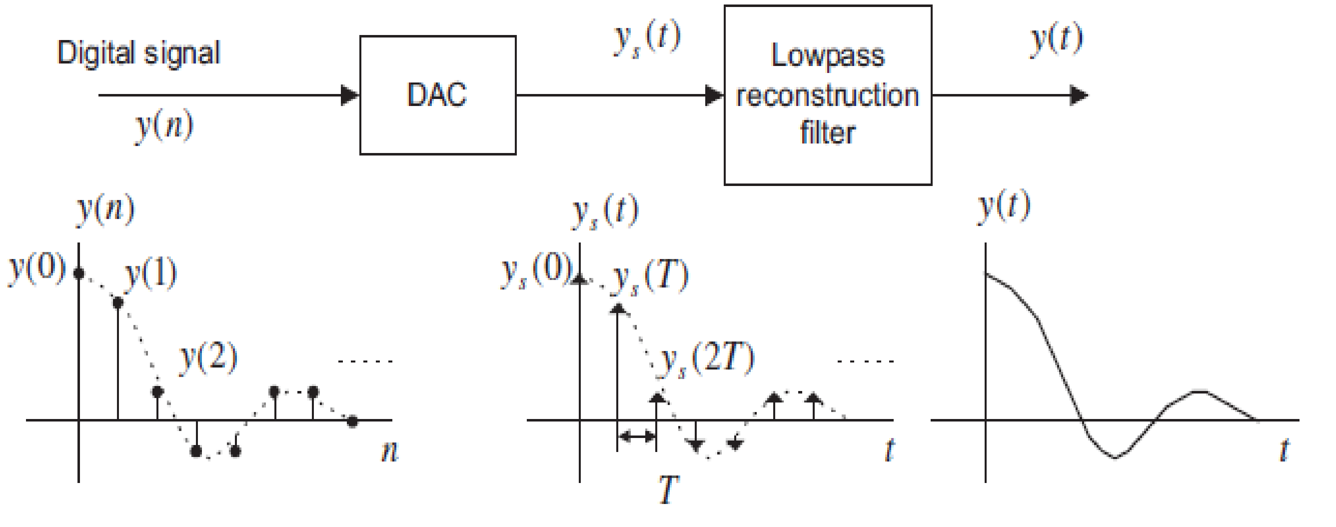
Signal Reconstruction

(Digital-to-Analog Conversion)

- The reconstruction process (recovering the analog signal from its sampled signal) involves two steps.

- **First:** the samples $x(n)$ (digital signal)are converted into a sequence of ideal impulses $x_s(t)$, in which each impulse has its amplitude proportional to digital output $x(n)$, and two consecutive impulses are separated by a sampling period of *T.*

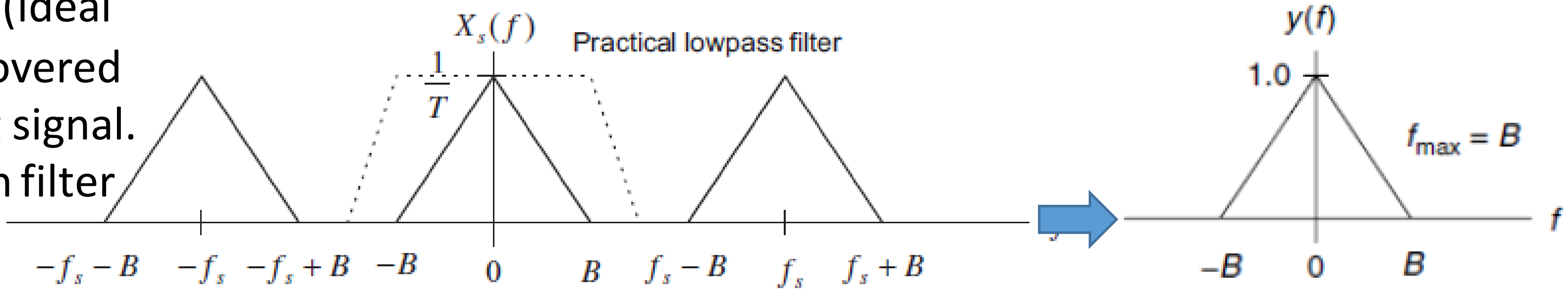
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT_s)$$



a.Digital signal processed b.Sampled signal recovered c.Analog signal recovered.

- **Second:** The analog reconstruction filter (ideal low-pass filter) is applied to the ideally recovered signal $x_s(t)$ to obtain the recovered analog signal. The impulse response of the reconstruction filter is

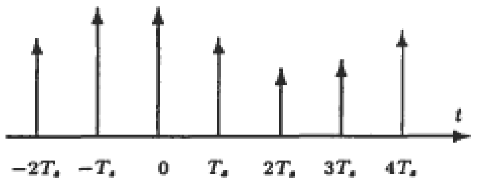
$$h_r(t) = \frac{\sin(\frac{\pi t}{T_s})}{\frac{\pi t}{T_s}}$$

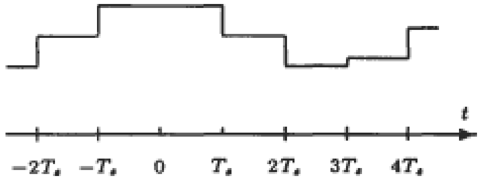


D Recovered signal spectrum

Signal Reconstruction – Contd.

- Before applying the reconstruction filter, a zero-order hold is used to interpolate between the samples in $x_s(t)$.

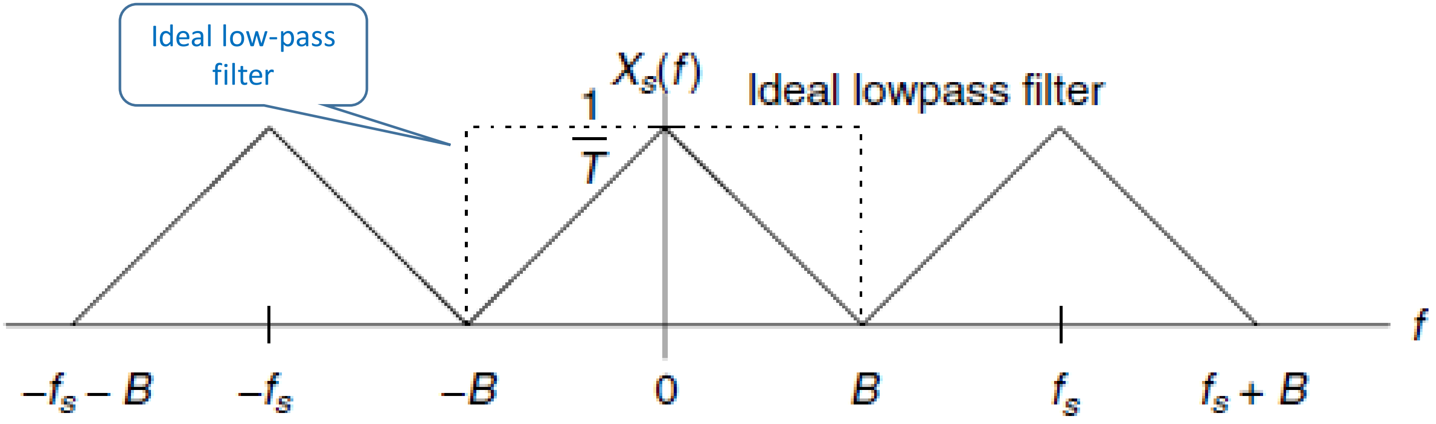




- Reconstruction filter

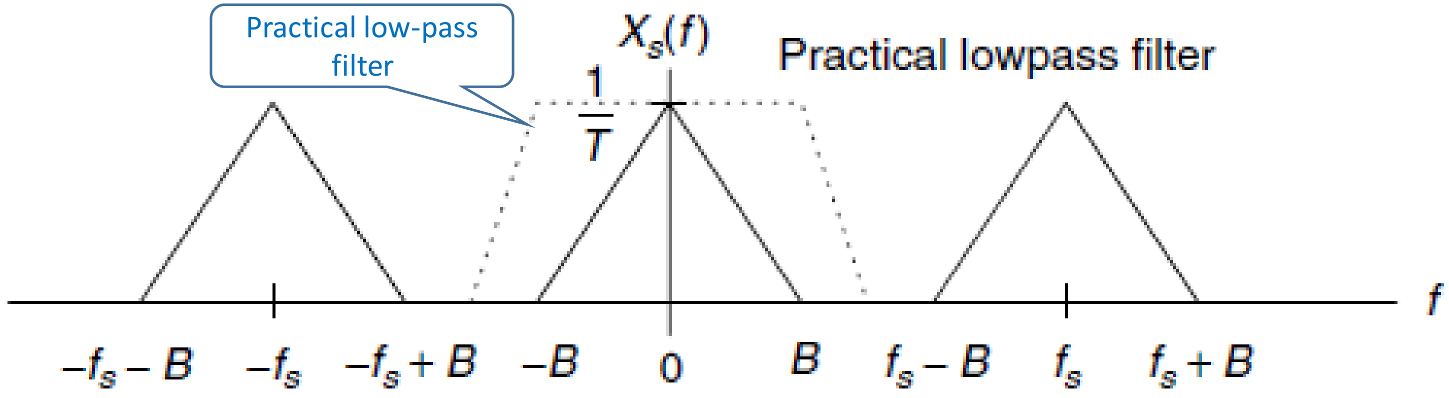
Case 1: $f_s = 2f_{\max}$

An ideal low-pass reconstruction filter is required to recover the analog signal Spectrum (an impractical case).



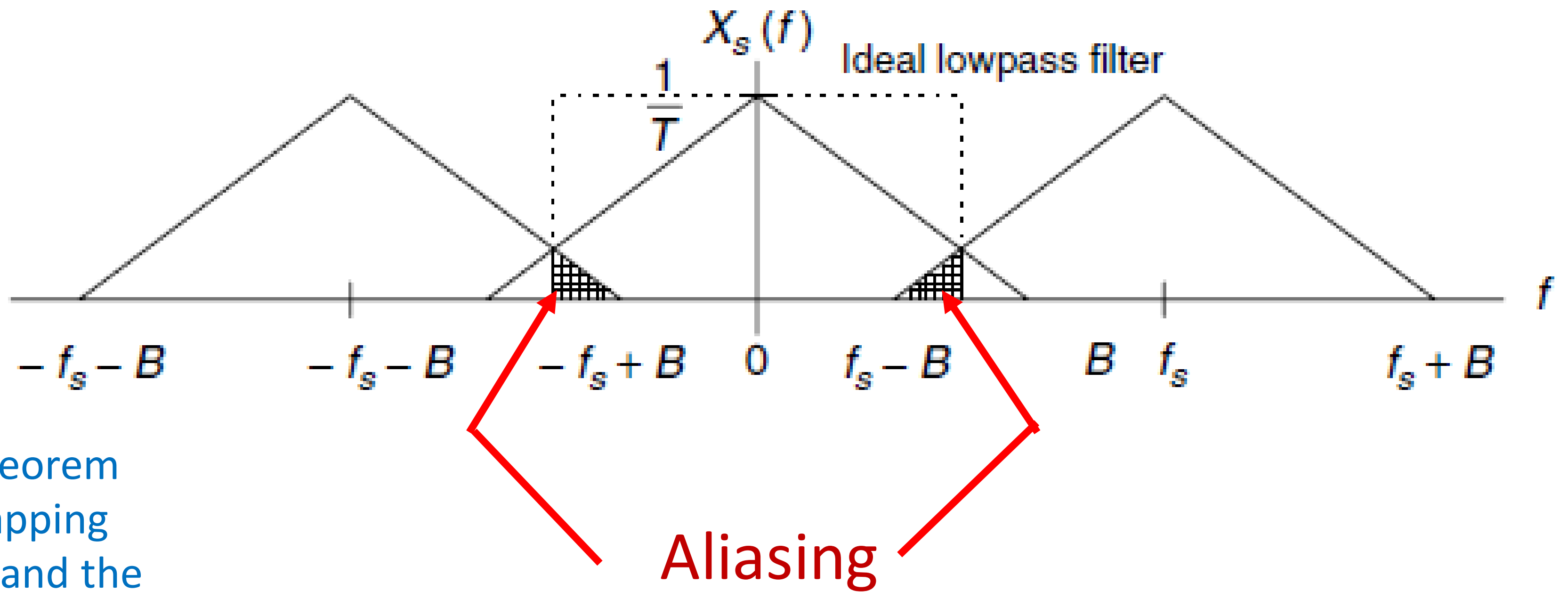
Case 2: $f_s > 2f_{\max}$

A practical low-pass reconstruction (anti-image) filter can be designed to reject all the images and achieve the original signal spectrum.



Signal Reconstruction – Contd.

Case 3: $f_s < 2f_{max}$



the condition of the Shannon sampling theorem is violated. We can see the spectral overlapping between the original baseband spectrum and the spectrum of the replica (add aliasing noise)

- Perfect reconstruction is not possible, even if we use ideal low pass filter.
- if an analog signal with a frequency f is under-sampled, the aliasing frequency component f_{alias} in the baseband is simply given by: $f_{alias} = f_s - f$

Example 2

Problem:

Assuming that an analog signal is given by

$$x(t) = 5\cos(2\pi \cdot 2000t) + 3\cos(2\pi \cdot 3000t), \text{ for } t \geq 0$$

and it is sampled at the rate of 8,000 Hz,

a. Sketch the spectrum of the sampled signal up to 20 kHz.

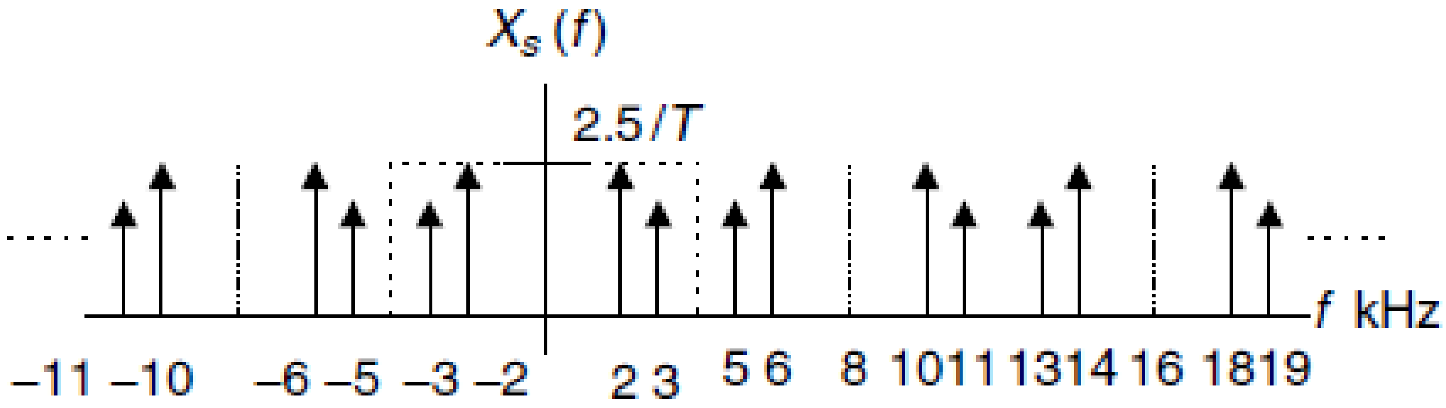
b. Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal ($y(n) = x(n)$ in this case) to recover the original signal.

Solution:

Using the Euler's identity:

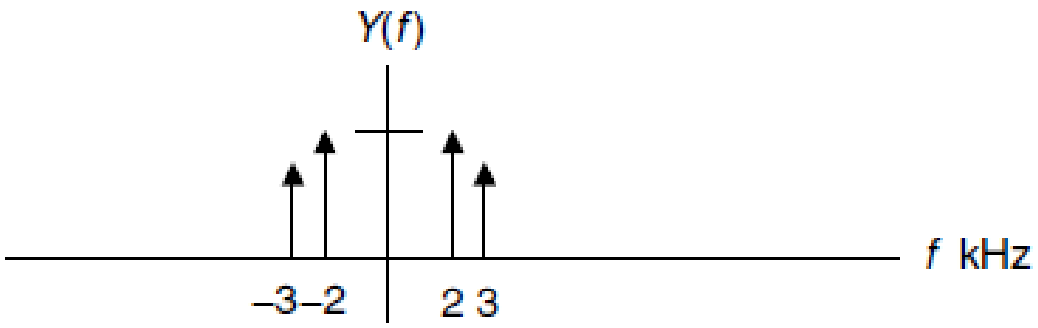
$$x(t) = \frac{3}{2}e^{-j2\pi \cdot 3000t} + \frac{5}{2}e^{-j2\pi \cdot 2000t} + \frac{5}{2}e^{j2\pi \cdot 2000t} + \frac{3}{2}e^{j2\pi \cdot 3000t}$$

a.



b.

The Shannon sampling theory condition is satisfied



Example 3

Problem:

Given an analog signal

$$x(t) = 5\cos(2\pi \times 2000t) + 1\cos(2\pi \times 5000t), \text{ for } t \geq 0,$$

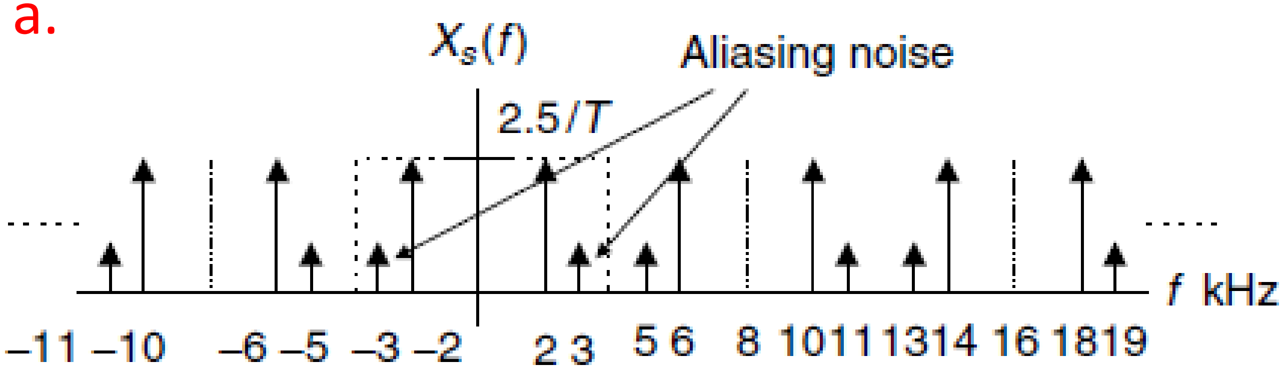
which is sampled at a rate of 8,000 Hz,

a. Sketch the spectrum of the sampled signal up to 20 kHz.

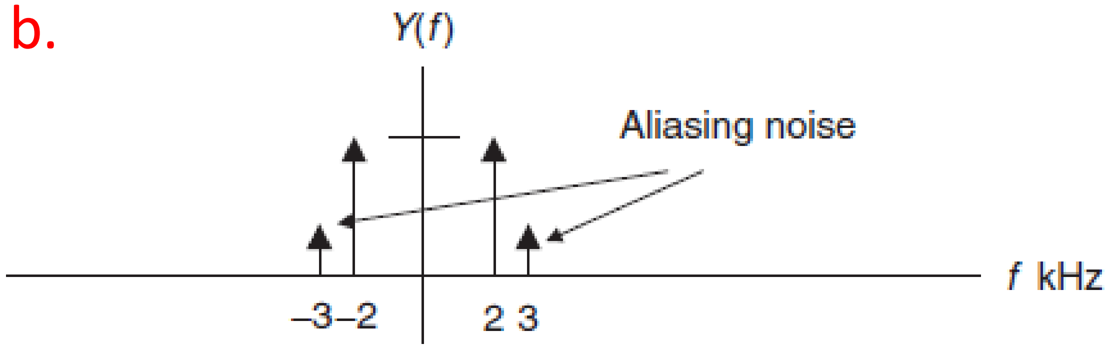
b. Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to recover the original signal ($y(n) = x(n)$ in this case).

Solution:

a.



b.

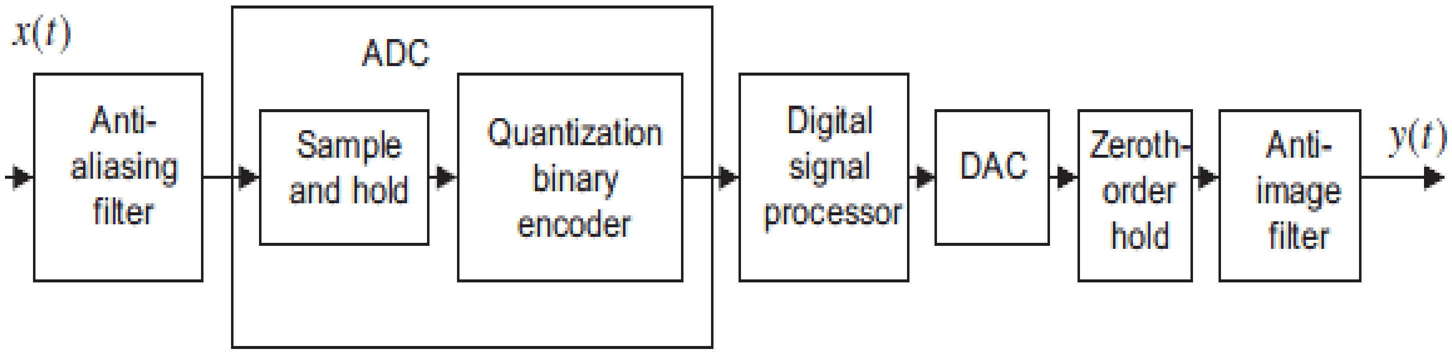


8 Quantization

- During the ADC process, amplitudes of the analog signal to be converted have infinite precision.

- *Quantization :* The quantizer converts the continuous amplitude signal discrete amplitude signal.

- *Encoding:* After quantization, each quantization level is assigned a unique binary code.



A block diagram for a DSP system

8 Quantization – Contd.

- A ***unipolar*** quantizer deals with analog signals ranging from 0 volt to a positive reference voltage

Δ : Step size of quantizer (ADC resolution)

x_{max} : Max value of analog signal

x_{min} : Min value of analog signal

$$\Delta = \frac{(x_{max} - x_{min})}{L}$$

L: Number of quantization level

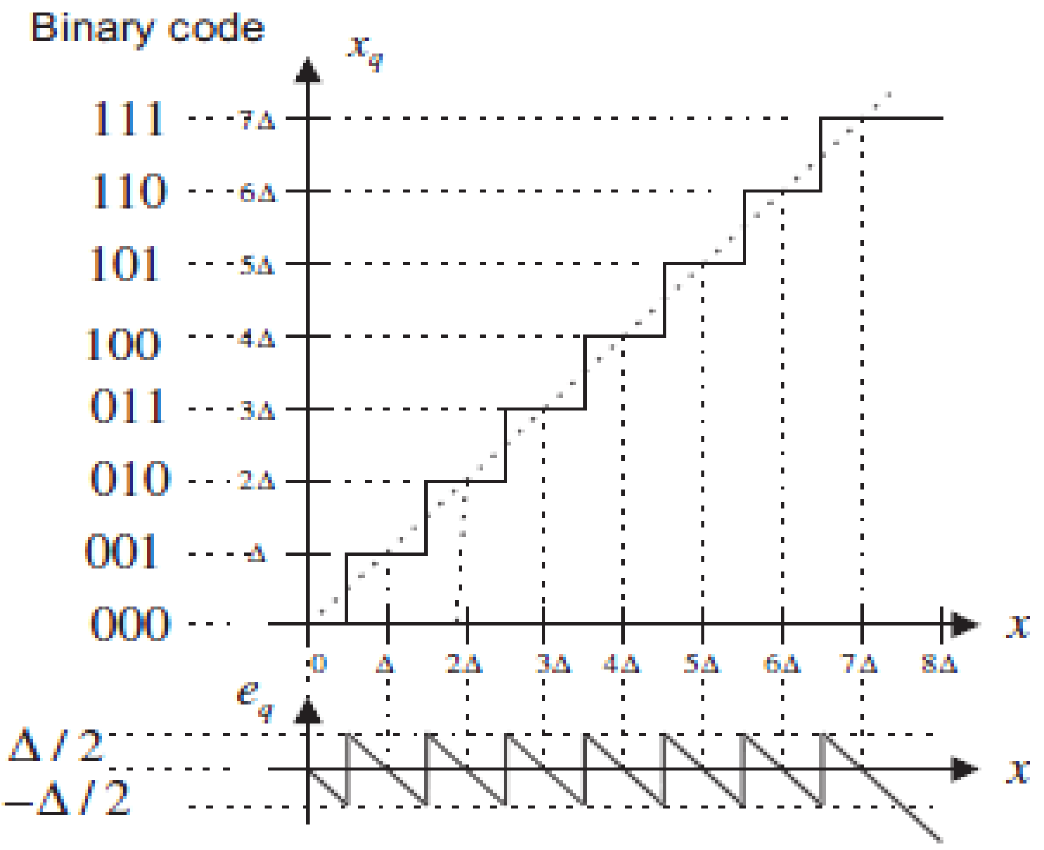
m : Number of bits in ADC

$$L = 2^m$$

i : Index corresponding to binary code $\quad i = round\left(\frac{x - x_{min}}{\Delta}\right)$

x_q: Quantization level $\quad x_q = x_{min} + i \cdot \Delta \qquad i = 0,1,\ldots,L-1$

e_q: Quantization error $\quad e_q = x_q - x \quad with \quad -\frac{\Delta}{2} \le e_q \le \frac{\Delta}{2}$



8 Quantization – Contd.

Example: 3-bit ADC channel accepts analog input ranging from 0 to 5 volts,

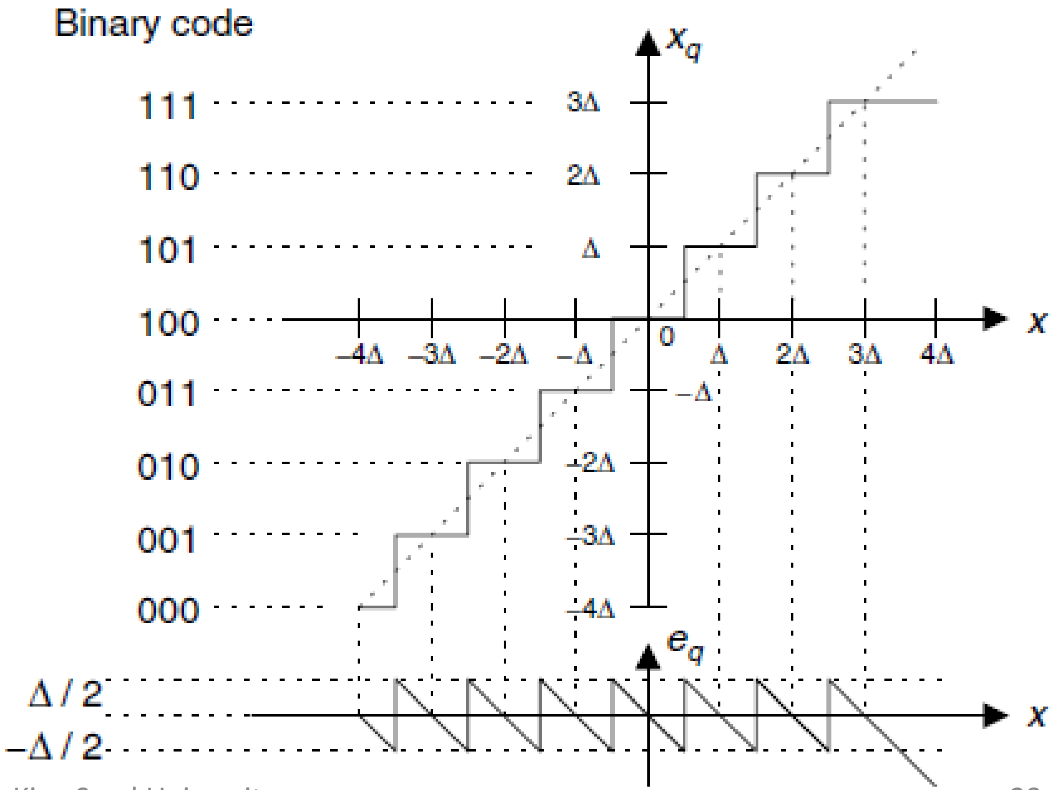
$x_{min} = 0$ volt, $x_{max} = 5$ volts, and $m = 3$ bits

$$\Delta = \frac{5-0}{8} = 0.625 \text{ volt} \qquad i = round\left(\frac{x - x_{min}}{\Delta}\right) \qquad x_q = 0 + i\Delta, i = 0, 1, \ldots, L-1,$$

$L = 2^m = 2^3 = 8$

- ***bipolar*** quantizer deals with analog signals ranging from a negative reference to a positive reference.

$$x_{min} = -4\Delta, \; x_{max} = 4\Delta, \text{ and } m = 3.$$



9 Periodicity

- In discrete-time case, a periodic sequence is a sequence for which

$x[n] = x[n+N], \quad \text{for all } n$ where the period N is necessarily an integer.

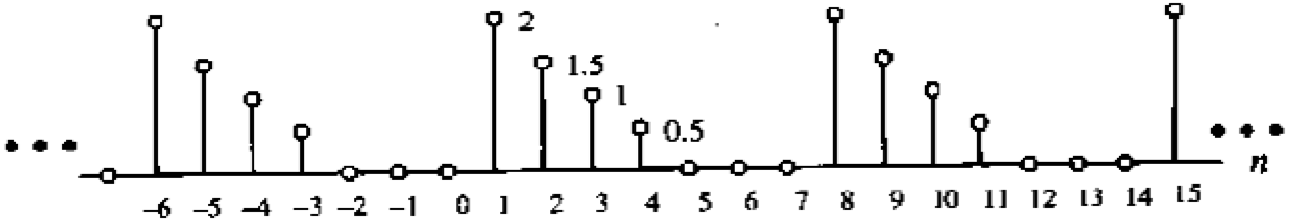
Discrete sinusoidal: $x[n] = A\,cos(\omega n + \theta)$

Pulsation: ω (rad/sample)

Phase shift: θ

Frequency: $f = \frac{\omega}{2\pi}$ cycle/sample

Period: $N = \frac{1}{f}$



periodic sequence with N = 7 samples

Example 1: $x[n] = \cos(\pi n / 4)$

$$\omega = \frac{\pi}{4} \rightarrow f = \frac{\omega}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8} \rightarrow N = \frac{1}{f} = 8$$

Period of N = 8

$$x[n+8] = \cos(\pi(n+8)/4) = \cos(\pi n/4 + 2\pi) = \cos(\pi n/4) = x[n]$$

9 Periodicity - Contd.

Example 2: $x[n] = \cos(3\pi n/8)$ Not periodic with *N* = 8

$$x[n+8] = \cos(3\pi(n+8)/8) = \cos(3\pi n/8 + 3\pi) = -\cos(3\pi n/8) = -x[n]$$

$$\omega = \frac{3\pi}{8} \rightarrow f = \frac{\omega}{2\pi} = \frac{3\,{}^{\pi}/_{8}}{2\pi} = \frac{3}{16} \rightarrow N = \frac{1}{f} = 3 \times \frac{16}{3}$$

But periodic with N = 16 (*Must be integer*)

$$x[n+16] = \cos(3\pi(n+16)/8) = \cos(3\pi n/8 + 6\pi) = \cos(3\pi n/8) = x[n]$$

However, $x[n] = \cos(n)$ is not periodic, because there is no such *N*.