Chapter 6

Numerical Solution of Ordinary Differential Equations

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Introduction

The differential equations are of fundamental importance in engineering mathematics because many physical laws of biology, chemistry, ecology, economics, business, etc., and relations appear mathematically in the form of such equations. We know that many differential equations can be solved explicitly in terms of elementary functions of calculus. For example, the explicit solution of the differential equation

$$\frac{dy}{dx} = e^{x} \quad \Leftrightarrow \quad y(x) == \ln(e^{x} + C),$$

and using initial condition $y(0) = 1$, we get $y(x) == \ln(e^{x} + e - 1)$.

But there are many differential equations which cannot be solved explicitly in terms of the functions of calculus. For example, the solutions of the differential equation of the form

$$\frac{dy}{dx} = e^{-x^2}, \quad \Leftrightarrow \quad y(x) = \int e^{-x^2}\, dx + C,$$

but it is known that these integrals cannot be expressed in terms of the functions of calculus.

Definition (Differential Equation)

An equation which involving functions and their derivatives. are differential equations.

$$(a)\quad \frac{dy}{dx} = 3x, \qquad (b)\quad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0,$$

$$(c)\quad \frac{dy}{dx} = x^2 + y^2, \qquad (d)\quad \left(\frac{d^3y}{dx^3}\right)^2 - 5\frac{d^2y}{dx^2} + 2y = 5.$$

The order of the differential equation is the order of the highest derivative involved. For example, the differential equations (a) and (c) are of first-order since the highest derivatives that appear is of first-order, whereas the differential equations (b) and (d) are respectively, the second-order and the third-order. •

Definition (Order of Differential Equation)

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Definition (Initial Conditions)

When all of the conditions are given at starting value of independent variable x to solve a given differential equation, is called a initial condition.

Classification of Differential Equations

There are two major types of differential equations, called, *ordinary differential equations (ODE)* and *partial differential equations (PDE)*. If an equation contains only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, it is then said to be an *ordinary differential equation*. For example, all the differential equations *(a)-(d)* are ordinary differential equations because there is only one independent variable, called x.

The more general equation is $y' = f(x, y(x))$.

Since the general solution of differential equation is depends on an arbitrary constant C, so this constant can be calculated by specifying the value of function $y(x)$ at a particular point x_0

$$y(x_0) = y_0.$$

The point x_0 is called initial point, and the number y_0 is called the initial value. We call the problem of solving

$$y' = \frac{dy}{dx} = f(x, y); \quad x_0 \leq x \leq x_n, \quad y(x_0) = y_0, \tag{6.6}$$

the initial-value problem (IVP).

Theorem 6.1 (Existence and Uniqueness Theorem)

Let $f(x, y)$ *and* $\frac{\partial f}{\partial y}$ *be continuous functions of* x *and* y *at all points* (x, y) *in some neighborhood of the initial point* (x_0, y_0). *Then there is a unique function* $y(x)$ *defined on some interval* $[x_0 - \epsilon, x_0 + \epsilon]$ *and satisfying*

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad x \in [x_0 - \epsilon, x_0 + \epsilon], \quad \epsilon > 0 \tag{6.7}$$

Numerical Methods for Solving IVP

Euler's Method

In principle, the Euler's method uses the forward difference formula approximation of $y'(x)$

$$y' = \frac{dy}{dx} \approx \frac{y(x_{i+1}) - y(x_i)}{h},$$

where h is the stepsize and it is equal to $x_{i+1} - x_i$. Given that $\dfrac{dy}{dx} = f(x, y)$ and the initial conditions $x = x_0,\ y(x) = y(x_0)$, we have

$$\frac{y(x_1) - y(x_0)}{h} \approx f(x_0, y(x_0)), \quad \text{or} \quad y(x_1) \approx y(x_0) + hf(x_0, y(x_0)),$$

which shows that $y(x_1)$ is approximately given by $y(x_0) + hf(x_0, y(x_0))$. We can now use this approximation for $y(x_1)$ to estimate $y(x_2)$, that is

$$y(x_2) \approx y(x_1) + hf(x_1, y(x_1)),$$

and so on. In general, $\quad y(x_{i+1}) \approx y(x_i) + hf(x_i, y(x_i)), \quad i = 0, 1, \ldots, n-1.$

Taking $y_i \approx y(x_i)$, for each $i = 1, 2, \ldots, n$, we have

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, \ldots, n-1.$$

This simple integration strategy is known as the Euler's method, or the *Euler-Cauchy method*.

Example 6.1 *Use the Euler's method to find the approximate value of $y(1)$ for the given initial-value problem*

$$y' = xy + x, \qquad 0 \le x \le 1, \quad y(0) = 0, \quad \textit{with} \quad h = 0.1,\ 0.2.$$

Compare your approximate solutions with the exact solution $y(x) = -1 + e^{x^2/2}$.

Solution. *Since $f(x,y) = xy + x$, and $x_0 = 0,\ y_0 = 0$, then*

$$y_{i+1} = y_i + hf(x_i, y_i), \quad \textit{for} \quad i = 0, 1, \ldots, 9$$

Then for $h = 0.1$ and taking $i = 0$, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0y_0 + x_0) = 0 + (0.1)[(0)(0) + (0)] = 0.0000.$$

Similar way, we have other approximations by taking $x_i = x_{i-1} + h, i = 1, 2, \ldots, 9$, as follows

$$y_2 = 0.0100, \quad y_3 = 0.0302, \quad y_4 = 0.0611, \quad y_5 = 0.1036, \quad y_6 = 0.1587,$$

$$y_7 = 0.2283, \quad y_8 = 0.3142, \quad y_9 = 0.4194, \quad y_{10} = 0.5471,$$

with possible absolute error

$$|y(1) - y_{10}| = |0.6487 - 0.5471| = 0.1016.$$

Similarly, the approximations for $h = 0.2$, give

$$y_1 = 0.0000, \quad y_2 = 0.0400, \quad y_3 = 0.1232, \quad y_4 = 0.2580, \quad y_5 = 0.4592,$$

with possible absolute error $\quad |y(1) - y_5| = |0.6487 - 0.4592| = 0.1895.$

Analysis of the Euler's Method

The Euler's formula uses the recurrence relation

$$y_{i+1} = y(x_i) + hf(x_i, y_i),$$

to estimate y_{i+1} assuming that $y(x_i)$ is the true solution. The error in y_{i+1} is given by $y_{i+1} - y(x_{i+1})$ which can be written as

$$y_{i+1} - y(x_{i+1}) = -\frac{h^2}{2!} y''(\eta(x_i)),$$

for $i = 0, 1, \ldots, n-1$. We call the term $-\frac{h^2}{2} y''(\eta(x_i))$, the *local* truncation error for the Euler's method.

Theorem 6.2 *For the differential equations* $\frac{dy}{dx} = f(x, y)$, *if the leading term in the local truncation error involves* h^{p+1}, *for some integer* p, *then the global error, for small* h, *is of order* h^p, *that is*

$$y_{i+1} - y(x_{i+1}) \approx ch^p,$$

where c *does not depend on stepsize* h. •

Higher-Order Taylor Methods

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\eta(x_i)),$$

$$\eta(x_i) \in (x_i, x_{i+1}).$$

$$y' = f(x,y) = f$$
$$y'' = f' = f_x + f_y f$$

$$y''' = \frac{\partial y''}{\partial x} = \frac{\partial}{\partial x}\left(f_x(x,y) + f_y(x,y).f(x,y)\right)$$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f_y}{\partial x}.f + f_y\frac{\partial f}{\partial x}$$

$$= f_{xx} + f_{xy}f + \left(\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial x}\right).f + f_y(f_x + f_y f)$$

$$= f_{xx} + f_{xy}f + f_{xy}f + f_{yy}f^2 + f_y f_x + {f_y}^2 f$$

$$= f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_y f_x + {f_y}^2 f$$

Higher-Order Taylor Methods

$$
\begin{aligned}
y' &= f(x,y) = f \\
y'' &= f' = f_x + f_y f \\
y''' &= f'' = f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
y(x_{i+1}) &= y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2!}f'(x_i, y(x_i)) + \cdots \\
&+ \frac{h^n}{n!}f^{(n-1)}(x_i, y(x_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\eta(x_i), y(\eta(x_i))).
\end{aligned}
$$

By taking $y_i \approx y(x_i)$, that the approximation to the exact solution at x_i, for each $i = 0, 1, \ldots, n-1$,

$$
y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!}f'(x_i, y_i) + \cdots + \frac{h^n}{n!}f^{(n-1)}(x_i, y_i).
$$

Then this formula is called the Taylor's method of order n.

the local error of Taylor's method of order n is

$$
E = \frac{h^{n+1}}{(n+1)!}f^{(n)}(\eta_i, y(\eta(x_i))) = \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\eta(x_i)), \qquad x_i < \eta(x_i) < x_{i+1}.
$$

Example 6.3 *Use the Taylor's method of order 2 to find the approximate value of $y(1)$ for the given initial-value problem.*

$$y' = xy + x, \quad 0 \le x \le 1, \quad y(0) = 0, \quad \text{with} \quad h = 0.2$$

Compare your approximate solution with the exact solution $y(x) = -1 + e^{x^2/2}$.

Solution. *Since $f(x,y) = xy + x$, and $x_0 = 0$, $y_0 = 0$, then*

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i), \quad \text{for} \quad i = 0, 1, 2, 3, 4$$

where $f'(x_i, y_i) = y_i + x_i^2 y_i + x_i^2 + 1$. Then for $i = 0$, we have

$$y_1 = y_0 + h(x_0 y_0 + x_0) + \frac{h^2}{2}(y_0 + x_0^2 y_0 + x_0^2 + 1) = 0 + (0.2)(0) + (0.02)(1) = 0.0200,$$

and similar way, we have for $i = 1, 2, 3, 4$, as follows

$$y_2 = 0.0820, \quad y_3 = 0.1937, \quad y_4 = 0.3694, \quad y_5 = 0.6334,$$

with absolute possible error

$$|y(1) - y_5| = |0.6487 - 0.6334| = 0.0153.$$

It showed that the result is entirely correct to 1 decimal place. Clearly, the result using this method is better than the Euler's method and it could be considerable improved by using smaller value of h than 0.2. •

Runge-Kutta Method of Order Two (Modified Euler's Method)

The equation (6.24) can be written in a standard form as

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2], \qquad \text{for each } i = 0, 1, \ldots n-1.$$

where

$$k_1 = f(x_i, y_i) \qquad \text{and} \qquad k_2 = f(x_{i+1}, y_i + hk_1),$$

Example 6.7 *Use Runge-Kutta method of order two (Modified Euler's method) to find the approximate value of $y(1)$ for the given initial-value problem*

$$y' = xy + x, \quad 0 \le x \le 1, \quad y(0) = 0, \quad \textit{with} \quad h = 0.2.$$

Compare your approximate solution with the exact solution $y(x) = -1 + e^{x^2/2}$.

Solution. *Since* $f(x, y) = xy + x$, *and* $x_0 = 0$, $y_0 = 0$, *then for* $i = 0$, *we have*

$$\begin{aligned} k_1 &= f(x_0, y_0) = (x_0 y_0 + x_0) = 0.0000 \\ k_2 &= f(x_1, y_0 + hk_1) = (x_1(y_0 + hk_1) + x_1) = (0 + 0.2) = 0.2000, \end{aligned}$$

and using these values, we have

$$y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 0 + 0.1(0 + 0.2000) = 0.0200.$$

Continuing in this manner, we have

$$\begin{array}{lllll} k_1 = 0.204, & k_2 = 0.4243, & then & y_2 = 0.0828, \\ k_1 = 0.4331, & k_2 = 0.7017, & then & y_3 = 0.1963, \\ k_1 = 0.7178, & k_2 = 1.0719, & then & y_4 = 0.3753, \\ k_1 = 1.1002, & k_2 = 1.5953, & then & y_5 = 0.6449, \end{array}$$

with possible error

$$|y(1) - y_5| = |0.6487 - 0.6449| = 0.0039$$

Example 6.8 *Use the Runge-Kutta method of order two (the Modified Euler's method) to find the approximate value of $y(1.4)$ for the given initial-value problem*

$$xy' + y' - 2y = 0, \quad y(1) = 4, \quad with \quad n = 2.$$

and compare your approximate solution with the exact solution $y(x) = (x+1)^2$.

Solution. *Since $f(x,y) = \dfrac{2y}{x+1}$, and $x_0 = 1$, $y_0 = 4$, $h = (1.4-1)/2 = 0.2$, then for $i = 0$, we have*

$$k_1 = f(x_0, y_0) = f(1,2) = (1)^{-2}(2) = 2,$$

$$k_2 = f(x_1, y_0 + hk_1) = f(1.1, 2.2) = (1.1)^{-2}(2.2) = 1.8182,$$

$$y(1.1) \approx y_1 = y_0 + \frac{h}{2}[k_1 + k_2] = 2 + 0.05(2 + 1.8182) = 2.1909.$$

$k_1 = 1.8107$ *and* $k_2 = 1.6035$,

and by using these values, $y(1.2) \approx y_2 = y_1 + \dfrac{h}{2}[k_1 + k_2] = 2.1909 + 0.05(1.8107 + 1.6035) = 2.3616$,

$|y(1.2) - y_2| = |2.3627 - 2.3616| = 0.0011$, *is the possible absolute error.*