Chapter 4
One and Two-Sample Estimation Problems

Department of Statistics and Operations Research

Edited by: Reem Alghamdi

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Statistical Inference: is to obtain information about some unknown aspects in a certain population or to access general judgements on it by taking a random sample from it and observing its behavior. It has two methods:

1. Estimation
2. Hypothesis testing

Suppose we have a population with some unknown parameter.
Example: $Normal(\mu, \sigma)$. μ and σ are parameters.
We need to draw conclusions (make inferences) about the unknown parameters.
Estimation of the parameters.

1. **Point Estimation**.
2. **Interval Estimation (Confidence Interval)**.

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Point Estimation: A point estimate of some population parameter θ is a single value $\widehat{\theta}$ of a statistic $\widehat{\Theta}$.
Example: the value $\overline{x}$ computed from a sample of size n, is a point estimate of the population parameter μ.
Example:$\widehat{p} = x/n$ is a point estimate of the population proportion p for a binomial experiment.
Example:s^2 is a point estimate of the population variance σ^2.

Interval Estimation (Confidence Interval): An interval estimate of some population parameter Θ is an interval of the form $(\hat{\Theta}_L, \hat{\Theta}_U)$, ie:

$$\hat{\Theta}_L < \Theta < \hat{\Theta}_U.$$

Where $\hat{\Theta}_L$ and $\hat{\Theta}_U$ depend on the value of the statistic $\hat{\Theta}$ for a particular sample and also on the sampling distribution of $\hat{\Theta}$. This interval contains the true value of Θ with probability $1 - \alpha$:

$$P(\hat{\Theta}_L < \Theta < \hat{\Theta}_U) = 1 - \alpha.$$

$(\hat{\Theta}_L, \hat{\Theta}_U)$ is called a $(1 - \alpha)100\%$ confidence interval (C.I.) for Θ.

- $1 - \alpha$ is called the confidence coefficient
- $\hat{\Theta}_L$ lower confidence limit
- $\hat{\Theta}_U$ upper confidence limit
- $0 < \alpha < 1$, e.g. $\alpha = 0.1$, $\alpha = 0.05$, $\alpha = 0.01$

Properties of a good estimator:

1. **Unbiased Estimator**

Definition

A statistic $\widehat{\Theta}$ is said to be an unbiased estimator of the parameter θ if $\mu_{\widehat{\Theta}} = E(\widehat{\Theta}) = \theta$.

Example

S^2 is an unbiased estimator of the parameter σ^2 (i.e. $E(S^2) = \sigma^2$).

2. **Minimized Variance (Efficiency)**

If $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$ are two unbiased estimators of the same population parameter θ, we want to choose the estimator whose sampling distribution has the smaller variance.
Hence, if $\sigma^2_{\hat{\theta}_1} < \sigma^2_{\hat{\theta}_2}$, we say that $\widehat{\Theta}_1$ is a more efficient estimator of θ than $\hat{\Theta}_2$.

Definition

If we consider all possible unbiased estimators of some parameter θ, the one with the smallest variance is called the most efficient estimator of θ.

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Single Sample: Estimating the Mean (μ)

Definition (Case I: Confidence Interval on μ, normal population and σ^2 known)

If $\overline{X}$ is the mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

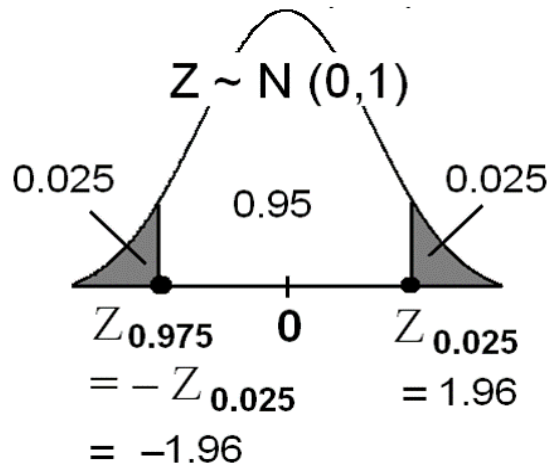
where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example

The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume the population is approximately normally distributed and the standard deviation is 0.3 gram per milliliter.

Solution

The point estimate of μ is $\overline{x} = 2.6$. The z-value leaving an area of 0.025 to the right, and therefore an area of 0.975 to the left, is $z_{0.025} = 1.96$ (Table A.3).



Hence, the 95% confidence interval is

$$2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right)$$

which reduces to $2.50 < \mu < 2.70$. To find a 99% confidence interval, we find the z-value leaving an area of 0.005 to the right and 0.995 to the left. From Table A.3 again, $z_{0.005} = 2.575$, and the 99% confidence interval is

$$2.6 - (2.575)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (2.575)\left(\frac{0.3}{\sqrt{36}}\right)$$

or simply

$$2.47 < \mu < 2.73.$$

The error in estimating μ by $\overline{x}$ is the absolute value of the difference between μ and $\overline{x}$, and we can be $100(1-\alpha)\%$ confident that this difference will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Theorem

If $\overline{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ (i.e. $e = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$).

Theorem

If $\overline{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed a specified amount e when the sample size is

$$n = \left(\frac{z_{\frac{\alpha}{2}}\sigma}{e}\right)^2$$

Example

How large a sample is required if we want to be 95% confident that our estimate of μ in the last Example is off by less than 0.05?

Solution

The population standard deviation is $\sigma = 0.3$. Then,

$$n = \left[\frac{(1.96)(0.3)}{0.05}\right]^2 = 138.3.$$

Therefore, we can be 95% confident that a random sample of size 139 will provide an estimate $\overline{x}$ differing from μ by an amount less than 0.05.

Definition (Case II: Confidence Interval on μ, normal population and σ^2 unknown)

If $\overline{x}$ and s are the mean and standard deviation of a random sample of size n from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is

$$\overline{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $\nu = n-1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Example

The contents of seven similar containers of sulfuric acid are $9.8, 10.2, 10.4, 9.8, 10.0, 10.2, 9.6$ liters.
Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

Solution

The sample mean and standard deviation for the given data are

$$\overline{x} = 10.0 \text{ and } s = 0.283.$$

Using Table A.4, we find $t_{0.025} = 2.447$ for $\nu = 6$ degrees of freedom. Hence, the 95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right)$$

which reduces to $9.74 < \mu < 10.26$.

Concept of a Large-Sample Confidence Interval

Definition (Case III: Confidence Interval on μ, non-normal population and n is large)

When normality cannot be assumed, and $n \to \infty$ (i.e. $n \geq 30$), the confidence interval can be written as:

1. If σ is known, then:
$$\overline{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$
2. If σ is unknown, s can replace σ, then:
$$\overline{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$$

This is often referred to as a large-sample confidence interval.

Example

Scholastic Aptitude Test (SAT) mathematics scores of a random sample of 500 high school seniors in the state of Texas are collected, and the sample mean and standard deviation are found to be 501 and 112, respectively.
Find a 99% confidence interval on the mean SAT mathematics score for seniors in the state of Texas.

Solution

The sample mean and standard deviation are

$$\bar{x} = 501 \text{ and } s = 112.$$

Using Table A.4, we find $z_{0.005} = 2.575$. Hence, the 99% confidence interval for μ is

$$501 - (2.575)\left(\frac{112}{\sqrt{500}}\right) < \mu < 501 + (2.575)\left(\frac{112}{\sqrt{500}}\right)$$

which reduces to $488.1 < \mu < 513.9$.

Standard Error of a Point Estimate

We indicated earlier that a measure of the quality of an unbiased estimator is its variance. The variance of $\overline{X}$ is

$$\sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$$

Thus, the standard deviation of $\overline{X}$, or standard error of $\overline{X}$, is $\sigma/\sqrt{n}$. Simply put, the standard error of an estimator is its standard deviation. For $\overline{X}$, the computed confidence limit

$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ is written } \overline{x} \pm z_{\alpha/2} \text{ s.e.}(\overline{X})$$

In the case where σ is unknown and sampling is from a normal distribution, s replaces σ and the estimated standard error $s/\sqrt{n}$ is involved. Thus, the confidence limits on μ are

$$\overline{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}} \text{ is written } \overline{x} \pm t_{\alpha/2} \text{ s.e.}(\overline{X})$$

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Definition (**Case I: Confidence Interval for $\mu_1 - \mu_2$,** normal population , and σ_1^2 **and** σ_2^2 **known**)

If $\overline{x}_1$ and $\overline{x}_2$ are means of independent random samples of sizes n_1 and n_2 from normal populations with known variances σ_1^2 and σ_2^2, respectively, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\overline{x}_1 - \overline{x}_2) - z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example

A study was conducted in which two types of engines, A and B, were compared. Gas mileage, in miles per gallon, was measured. Fifty experiments were conducted using engine type A and 75 experiments were done with engine type B. The gasoline used and other conditions were held constant. The average gas mileage was 36 miles per gallon for engine A and 42 miles per gallon for engine B. Find a 96% confidence interval on $\mu_B - \mu_A$, where μ_A and μ_B are population mean gas mileages for engines A and B, respectively. Assume that the population standard deviations are 6 and 8 for engines A and B, respectively.

Solution

The point estimate of $\mu_B - \mu_A$ is $\overline{x}_B - \overline{x}_A = 42 - 36 = 6$.
Using $\alpha = 0.04$, we find $z_{0.02} = 2.055$ from Table A.3. Hence, with substitution in the formula above, the 96% confidence interval is

$$6 - 2.055\sqrt{\frac{64}{75} + \frac{36}{50}} < \mu_B - \mu_A < 6 + 2.055\sqrt{\frac{64}{75} + \frac{36}{50}}$$

or simply $3.42 < \mu_B - \mu_A < 8.58$.

Definition (**Case II: Confidence Interval for $\mu_1 - \mu_2$,** normal population and unknown variances **but** $\sigma_1^2 = \sigma_2^2$)

If $\overline{x}_1$ and $\overline{x}_2$ are means of independent random samples of sizes n_1 and n_2, respectively, from approximately normal populations with unknown but equal variances, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\overline{x}_1 - \overline{x}_2) - t_{\frac{\alpha}{2}} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + t_{\frac{\alpha}{2}} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where s_p is the pooled estimate of the population standard deviation and $t_{\alpha/2}$ is the t-value with $\nu = n_1 + n_2 - 2$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Definition (Pooled Variance)

Denoting the pooled estimator by s_p^2, we have the following:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{(n_1 + n_2 - 2)}$$

Example

Two independent sampling stations, station 1 and station 2, were chosen for a study on pollution. For 12 monthly samples collected at station 1, the species diversity index had a mean value $\overline{x}_1 = 3.11$ and a standard deviation $s_1 = 0.771$, while 10 monthly samples collected at the station 2 had a mean index value $\overline{x}_2 = 2.04$ and a standard deviation $s_2 = 0.448$. Find a 90% confidence interval for the difference between the population means for the two locations, assuming that the populations are approximately normally distributed with equal variances.

Solution

Let μ_1 and μ_2 represent the population means, respectively, for the species diversity indices at the downstream and upstream stations. We wish to find a 90% confidence interval for $\mu_1 - \mu_2$. Our point estimate of $\mu_1 - \mu_2$ is

$$\overline{x}_1 - \overline{x}_2 = 3.11 - 2.04 = 1.07.$$

The pooled estimate, s_p^2, of the common variance, σ^2, is

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{(n_1 + n_2 - 2)} = \frac{(11)(0.771)^2 + (9)(0.448)^2}{12 + 10 - 2} = 0.417.$$

Taking the square root, we obtain $s_p = 0.646$. Using $\alpha = 0.1$, we find in Table A.4 that $t_{0.05} = 1.725$ for $\nu = n_1 + n_2 - 2 = 20$ degrees of freedom. Therefore, the 90% confidence interval for $\mu_1 - \mu_2$ is

$$\begin{aligned} 1.07 - 1.725(0.646)\sqrt{\frac{1}{12} + \frac{1}{10}} &< \mu_1 - \mu_2 \\ &< 1.07 + 1.725(0.646)\sqrt{\frac{1}{12} + \frac{1}{10}} \end{aligned}$$

which simplifies to $0.593 < \mu_1 - \mu_2 < 1.547$.

Definition (**Case III: Confidence Interval for $\mu_1 - \mu_2$,** Non- normal population and n_1, n_2 are large)

If $\overline{x}_1$ and $\overline{x}_2$ are means of independent random samples of sizes n_1 and n_2 from non-normal populations with $n_1 \geq 30$ and $n_2 \geq 30$, respectively, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by:

1. If σ_1^2 and σ_2^2 are to be known, then

$$(\overline{x}_1 - \overline{x}_2) - z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

2. If σ_1^2 and σ_2^2 are to be unknown, then

$$(\overline{x}_1 - \overline{x}_2) - z_{\frac{\alpha}{2}} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\frac{\alpha}{2}} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Paired Observations

We consider constructing a confidence interval for the difference between the means of two related (non-independent) normal populations. As before, let us define the difference between the two means as follows:

$$\mu_D = \mu_1 - \mu_2$$

where μ_1 is the mean of the first population and μ_2 where is the mean of the second population. We assume that the two normal populations are not independent.

- 1−st population: $X_1, X_2, X_3, \ldots, X_n$ and with mean μ_1.
- 2−st population: $Y_1, Y_2, Y_3, \ldots, Y_n$ and with mean μ_2.

We define the followings quantities:

- The differences (D-observations)

$$D_i = X_i - Y_i, i = 1, 2 \ldots, n$$

- Sample mean of the D-observations (differences)

$$\overline{D} = \frac{\sum_{i=1}^{n} D_i}{n} = \frac{D_1 + D_2 + \cdots + D_n}{n}$$

- Sample variance of the D-observations (differences)

$$S_D^2 = \frac{\sum_{i=1}^{n} (D_i - \overline{D})^2}{n-1}$$

- Sample standard deviation of the D-observations

$$S_D = \sqrt{S_D^2}$$

Definition (Special Case: Confidence Interval for $\mu_D = \mu_1 - \mu_2$, for Paired Observations)

If $\overline{D}$ and S_D are the mean and standard deviation, respectively, of the normally distributed differences of n random pairs of measurements, a $100(1-\alpha)\%$ confidence interval for $\mu_D = \mu_1 - \mu_2$ is

$$\overline{D} - t_{\alpha/2}\frac{S_D}{\sqrt{n}} < \mu < \overline{D} + t_{\alpha/2}\frac{S_D}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $\nu = n - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right..

Example

A study published in Chemosphere reported the levels of the dioxin TCDD of 10 Massachusetts Vietnam veterans who were possibly exposed to Agent Orange. The TCDD levels in plasma and in fat tissue are listed in the following Table.
Find a 95% confidence interval for $\mu_1 - \mu_2$, where μ_1 and μ_2 represent the true mean TCDD levels in plasma and in fat tissue, respectively. Assume the distribution of the differences to be approximately normal.

Veteran	TCDD levels in Plasma	TCDD levels in Fat Tissue	D_i
1	2.5	4.9	-2.4
2	3.1	5.9	-2.8
3	2.1	4.4	-2.3
4	3.5	6.9	-3.4
5	3.1	7.0	-3.9
6	1.8	4.2	-2.4
7	6.0	10.0	-4.0
8	3.0	5.5	-2.5
9	36.0	41.0	-5.0
10	4.7	4.4	0.3

Solution

The point estimate of μ_D is $\overline{D} = -2.84$. The standard deviation, S_D, of the sample differences is 1.42. Using $\alpha = 0.05$, we find in Table A.4 that $t_{0.025} = 2.262$ for $\nu = n - 1 = 9$ degrees of freedom. Therefore, the 95% confidence interval is

$$-2.84 - (2.262)\left(\frac{1.42}{\sqrt{10}}\right) < \mu_D < -2.84 + (2.262)\left(\frac{1.42}{\sqrt{10}}\right)$$

or simply $-3.86 < \mu_D < -1.82$.

Plan

Single Sample: Estimating a Proportion

A point estimator of the proportion p in a binomial experiment is given by the statistic $\widehat{p} = X/n$, where X represents the number of successes in n trials.

Definition

The sample proportion $\widehat{p} = X/n$ will be used as the point estimate of the parameter p.

Theorem(Large-Sample Confidence Intervals for p)

If $\widehat{p}$ is the proportion of successes in a random sample of large size n and $\widehat{q} = 1 - \widehat{p}$, an approximate $100(1-\alpha)\%$ confidence interval, for the binomial parameter p is given by

$$\widehat{p} - z_{\alpha/2}\sqrt{\frac{\widehat{p}\widehat{q}}{n}} < p < \widehat{p} + z_{\alpha/2}\sqrt{\frac{\widehat{p}\widehat{q}}{n}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Example

In a random sample of $n = 500$ families owning television sets in the city of Hamilton, Canada, it is found that $x = 340$ subscribe to HBO. Find a 95% confidence interval for the actual proportion of families with television sets in this city that subscribe to HBO.

Solution

The point estimate of p is $\widehat{p} = 340/500 = 0.68$. Using Table A.3, we find that $z_{0.025} = 1.96$. Therefore, the 95% confidence interval for p is

$$0.68 - 1.96\sqrt{\frac{(0.68)(0.32)}{500}} < p < 0.68 + 1.96\sqrt{\frac{(0.68)(0.32)}{500}}$$

which simplifies to $0.6391 < p < 0.7209$.

Theorem

If $\widehat{p}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\sqrt{\frac{\widehat{p}\widehat{q}}{n}}$.

Choice of Sample Size

Theorem

If $\widehat{p}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error will be less than a specified amount e when the sample size is approximately

$$n = \frac{z_{\alpha/2}^2 \widehat{p}\widehat{q}}{e^2}$$

Example

In a random sample of 500 families owning television sets in the city of Hamilton, Canada, it is found that $x = 340$ subscribe to HBO. How large a sample is required if we want to be 95% confident that our estimate of p is within 0.02 of the true value?

Solution

Let us treat the 500 families as a preliminary sample, providing an estimate $\widehat{p} = 0.68$. Then,

$$n = \frac{(1.96)^2(0.68)(0.32)}{0.02^2} = 2089.8 \approx 2090$$

Occasionally, it will be impractical to obtain an estimate of p to be used for determining the sample size for a specified degree of confidence. If this happens, we use the following theorem.

Theorem

If $\widehat{p}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error will not exceed than a specified amount e when the sample size is approximately

$$n = \frac{z^2_{\alpha/2}}{4e^2}$$

Example

In a random sample of 500 families owning television sets in the city of Hamilton, Canada, it is found that $X = 340$ subscribe to HBO. How large a sample is required if we want to be at least 95% confident that our estimate of p is within 0.02 of the true value?

Solution

Let assume that no preliminary sample has been taken to provide an estimate of p. Consequently, we can be at least 95% confident that our sample proportion will not differ from the true proportion by more than 0.02 if we choose a sample of size

$$n = \frac{(1.96)^2}{4(0.02)^2} = 2401$$

Comparing the results of Examples 28 and 29, we see that information concerning p, provided by a preliminary sample or from experience, enables us to choose a smaller sample while maintaining our required degree of accuracy.

Plan

Two Samples: Estimating the Difference between Two Proportions

Theorem (Large-Sample Confidence Interval for $p_1 - p_2$)

If $\widehat{p}_1$ and $\widehat{p}_2$ are the proportions of successes in random samples of large sizes n_1 and n_2, respectively, $\widehat{q}_1 = 1 - \widehat{p}_1$, and $\widehat{q}_2 = 1 - \widehat{p}_2$, an approximate $100(1-\alpha)\%$ confidence interval for the difference of two binomial parameters, $p_1 - p_2$, is given by

$$(\widehat{p}_1 - \widehat{p}_2) - z_{\frac{\alpha}{2}} \sqrt{\frac{\widehat{p}_1\widehat{q}_1}{n_1} + \frac{\widehat{p}_2\widehat{q}_2}{n_2}} < p_1 - p_2 < (\widehat{p}_1 - \widehat{p}_2) + z_{\frac{\alpha}{2}} \sqrt{\frac{\widehat{p}_1\widehat{q}_1}{n_1} + \frac{\widehat{p}_2\widehat{q}_2}{n_2}}$$

Example

A certain change in a process for manufacturing component parts is being considered. Samples are taken under both the existing and the new process so as to determine if the new process results in an improvement. If 75 of 1500 items from the existing process are found to be defective and 80 of 2000 items from the new process are found to be defective, find a 90% confidence interval for the true difference in the proportion of defective between the existing and the new process.

Solution

Let p_1 and p_2 be the true proportions of defective for the existing and new processes, respectively. Hence, $\widehat{p}_1 = 75/1500 = 0.05$ and $\widehat{p}_2 = 80/2000 = 0.04$, and the point estimate of $p_1 - p_2$ is

$$\widehat{p}_1 - \widehat{p}_2 = 0.05 - 0.04 = 0.01$$

Using Table A.3, we find $z_{0.05} = 1.645$. We find the 90% confidence interval to be

$$0.01 \pm 1.645\sqrt{\frac{(0.05)(0.95)}{1500} + \frac{(0.04)(0.96)}{2000}}$$

which simplifies to:

$$-0.0017 < p_1 - p_2 < 0.0217.$$

Plan

Single Sample: Estimating the Variance

If a sample of size n is drawn from a normal population with variance σ^2 and the sample variance S^2 is computed, we obtain a value of the statistic S^2. This computed sample variance is used as a point estimate of σ^2. Hence, the statistic S^2 is called an estimator of σ^2. An interval estimate of σ^2 can be established by using the statistic

$$X = \frac{(n-1)S^2}{\sigma^2}$$

the statistic X has a chi-squared distribution with $n-1$ degrees of freedom when samples are chosen from a normal population.

Theorem (Confidence Interval for σ^2)

If S^2 is the variance of a random sample of size n from a normal population, a $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are χ^2-values with $\nu = n-1$ degrees of freedom, leaving areas of $\alpha/2$ and $1-\alpha/2$, respectively, to the right.

Remark: An approximate $100(1-\alpha)\%$ confidence interval for σ is obtained by taking the square root of each endpoint of the interval for σ^2.

Example

The following are the weights, in decagrams, of 10 packages of grass seed distributed by a certain company: $46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, 46.0$. Find a 95% confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal population.

Solution

First we find $S^2 = 0.286$. To obtain a 95% confidence interval, we choose $\alpha = 0.05$. Then, using Table A.5 with $\nu = 9$ degrees of freedom, we find $\chi^2_{.025} = 19.023$ and $\chi^2_{.975} = 2.700$. Therefore, the 95% confidence interval for σ^2 is

$$\frac{(9)(0.286)}{19.023} < \sigma^2 < \frac{(9)(0.286)}{2.700}$$

or simply

$$0.135 < \sigma^2 < 0.953.$$

Plan

Two Samples: Estimating the Ratio of Two Variances

A point estimate of the ratio of two population variances σ_1^2/σ_2^2 is given by the ratios S_1^2/S_2^2 of the sample variances. Hence, the statistic S_1^2/S_2^2 is called an estimator of σ_1^2/σ_2^2 . If σ_1^2 and σ_2^2 are the variances of normal populations, we can establish an interval estimate of σ_1^2/σ_2^2 by using the statistic

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

Where the random variable F has an F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom.

Theorem (Confidence Interval for $\frac{\sigma_1^2}{\sigma_2^2}$)

If s_1^2 and s_2^2 are the variances of independent samples of sizes n_1 and n_2, respectively,from normal populations, then a $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is

$$\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(\nu_1,\nu_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}f_{\alpha/2}(\nu_2,\nu_1)$$

where $f_{\alpha/2}(\nu_1,\nu_2)$ is an f-value with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom, leaving an area of $\frac{\alpha}{2}$ to the right, and $f_{\alpha/2}(\nu_2,\nu_1)$ is a similar f-value with $\nu_2 = n_2 - 1$ and $\nu_1 = n_1 - 1$ degrees of freedom.

Remark:An approximate $100(1-\alpha)\%$ confidence interval for σ_1/σ_2 is obtained by taking the square root of each endpoint of the interval for $\frac{\sigma_1^2}{\sigma_2^2}$.

Example

A study was conducted to estimate the difference in the amounts of the chemical orthophosphorus measured at two different stations. Fifteen samples were collected from station 1, and 12 samples were obtained from station 2. The 15 samples from station 1 had an average orthophosphorus content of 3.84 milligrams per liter and a standard deviation of 3.07 milligrams per liter, while the 12 samples from station 2 had an average content of 1.49 milligrams per liter and a standard deviation of 0.80 milligram per liter. Determine a 98% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ and for $\frac{\sigma_1}{\sigma_2}$, where σ_1^2 and σ_2^2 are the variances of the populations of orthophosphorus contents at station 1 and station 2, respectively.

Solution

We have $n_1 = 15$, $n_2 = 12$, $S_1 = 3.07$, and $S_2 = 0.80$. For a 98% confidence interval, $\alpha = 0.02$. Interpolating in Table A.6, we find $f_{0.01}(14, 11) \approx 4.33$ and $f_{0.01}(11, 14) \approx 3.87$. Therefore, the 98% confidence interval for σ_1^2/σ_2^2 is

$$\left(\frac{3.07^2}{0.80^2}\right)\left(\frac{1}{4.33}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \left(\frac{3.07^2}{0.80^2}\right)(3.87),$$

which simplifies to $3.4 < \frac{\sigma_1^2}{\sigma_2^2} < 56.991$. Taking square roots of the confidence limits, we find that a 98% confidence interval for σ_1/σ_2 is

$$1.844 < \frac{\sigma_1}{\sigma_2} < 7.549.$$

Since this interval does not allow for the possibility of σ_1/σ_2 being equal to 1, we were correct in assuming that $\sigma_1 \neq \sigma_2$ (and $\sigma_1^2 \neq \sigma_2^2$).