The Riemann Integral

Mongi BLEL

Department of Mathematics
King Saud University

2017-2018

Table of contents

Indefinite Integrals & Anti-derivative Function

Definition

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on an interval I. A function $F : I \longrightarrow \mathbb{R}$ is called an anti-derivative of f on I if F is differentiable on I and $F'(x) = f(x)$, $forall x \in I$.

There are many anti-derivatives of the function $f(x) = 2x$ on $\mathbb{R}$ such as

$$F_1(x) = x^2+1, \quad F_2(x) = x^2-7, \quad F_3(x) = x^2+\frac{11}{2}, \quad F_4(x) = x^2+c.$$

Proposition

Let F and G be two anti-derivatives of a function f on an interval I, then there is a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c, \quad \forall x \in I.$$

Definition

Let F be an anti-derivative of a function f on an interval I, we denote $\int f(x)dx$ any anti-derivative i.e.

$$\int f(x)dx = F(x) + c, \quad \forall x \in I \tag{1}$$

$\int f(x)dx$ is called the indefinite integral of f on I.
In the equation (1),

- *the constant c is called the constant of integration,*
- *x is called the variable of integration,*
- *$f(x)$ is called the integrand.*

The mapping $f \longmapsto \int f(x)dx$ is called as indefinite integral or evaluating the integral or integrating f.

Basic table of indefinite integrals

$f(x)$	$\int f(x)dx$
$x^r;\ r \neq -1, R \in \mathbb{Q}$	$\frac{x^{r+1}}{r+1} + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$
$\sec^2(x)$	$\tan(x) + c$
$\csc^2(x)$	$-\cot(x) + c$
$\sec(x)\tan(x)$	$\sec(x) + c$

Proposition

(Some important formulas)

- *If f is differentiable on an interval I, then*

$$\int \frac{d}{dx} f(x)dx = f(x) + c.$$

- *If f has an anti-derivative on an interval I, then*

$$\frac{d}{dx} \int f(x)dx = f(x).$$

$$\int \alpha f(x)dx = \alpha \int f(x)dx.$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx.$$

Substitution Method

Theorem

*(**Substitution**)*
If F is an anti-derivative of f, then $f(g(x))g'(x)$ has anti-derivative $F(g(x))$. Or,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This is obvious. It is called "substitution" since it can be obtained by substituting $u = g(x)$ and $du = g'(x)dx$ into
$\displaystyle\int f(u)du = F(u) + c.$

Remark

Substitution method is also called changing variable method.

Examples

1. $\displaystyle\int (x^2+1)^n 2x dx \overset{u=x^2+1}{=} \int u^n du = \frac{u^{n+1}}{n+1} = \frac{(x^2+1)^{n+1}}{n+1} + c.$
2. $\displaystyle\int \sin(2x+3) dx \overset{u=2x+3}{=} \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x+3) + c.$
3. $\displaystyle\int \frac{1}{\cos^2(\pi x)} dx \overset{u=\pi x}{=} \frac{1}{\pi}\int \frac{1}{\cos^2(u)} du = \frac{1}{\pi}\tan(\pi x) + c.$

Theorem

Let I be an interval, $r \in \mathbb{Q} \setminus \{-1\}$ and $f : I \to \mathbb{R}$ a differentiable function. Assume that $f^r(x)$ is defined for every $x \in I$. Then

$$\int f^r(x)f'(x)dx = \frac{f^{r+1}(x)}{r+1} + c.$$

Summation Notations

Definition

Given a set of numbers $\{a_1, a_2, ..., a_n\}$, *the symbol* $\sum_{k=1}^{n} a_k$ *represents their sum as follows*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

Theorem

For every $c \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} c = nc.$$

Theorem

Let $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$. For every $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$ we have

$$\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k.$$

Theorem

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{k=1}^{n} k^3 &= \left[\frac{n(n+1)}{2}\right]^2.
\end{aligned}$$

Example 1 :

$$\begin{aligned}
\sum_{k=1}^{n}(3k^2-2k+1) &= 3\sum_{k=1}^{n}k^2-2\sum_{k=1}^{n}k+\sum_{k=1}^{n}1,\\
&= 3.\frac{n(n+1)(2n+1)}{6}-2.\frac{n(n+1)}{2}+1.n,\\
&= \frac{n}{2}\Big[(n+1)(2n+1)-2(n+1)+2\Big],\\
&= \frac{n}{2}(2n^2+3n+1-2n-2+2),\\
&= \frac{n}{2}(2n^2+n+1).
\end{aligned}$$

Riemann Sums, Area and the Definite Integral

The approach of the integral of function by areas gives the geometrical sense of integration. The second approach consists in introducing a priori the anti-derivative of function. The idea of the first approach is to cut the interval $[a, b]$ by a subdivision in sub-intervals $[a_j, a_{j+1}]$, then to add the areas of rectangles based on the intervals $[a_j, a_{j+1}]$.

Definitions

1. *A partition P of the closed interval $[a, b]$ is a finite set of points $P = \{a_0, a_1, a_2, \dots, a_n\}$ such that $a = a_0 < a_1 < a_2 < \dots < a_{n-1} < a_n = b$. Each $[a_{j-1}, a_j]$ is called a subinterval of the partition and the number $h_j = a_j - a_{j-1}$ is called the amplitude of this interval.*
2. *The norm of a partition is defined to be the length of the longest subinterval $[a_j, a_{j+1}]$, that is, it is $||P|| = \max\{a_j - a_{j-1}, j = 1, \dots, n\}$.*
3. *A partition $P = \{a_0, a_1, a_2, \dots, a_n\}$ of the closed interval $[a, b]$ is a called uniform if $a_{k+1} - a_k = \dfrac{b-a}{n}$. Then in this case*

$b - a$

Definitions

1. *A mark on the partition* $P = \{a_0, a_1, a_2, \dots, a_n\}$ *is a set of points* $w = \{x_1, \dots, x_n\}$ *such that* $x_j \in [a_{j-1}, a_j]$.
2. *A pointed partition of an interval is a partition of an interval together with a finite sequence of numbers* $x_1, x_2, \dots, x_n$ *such that*

$$\forall j = 1, \dots, n, \qquad x_j \in [a_{j-1}, a_j].$$

This pointed partition will be denoted by

$$P = \{([a_{j-1}, a_j], x_j)\}_{1 \le j \le n}.$$

Definition

Let $P = \{([a_{j-1}, a_j], x_j)\}_{1\leq j\leq n}$ *be a pointed partition of the interval* $[a, b]$*. The Riemann sum of* f *with respect to the pointed partition* P *is the number*

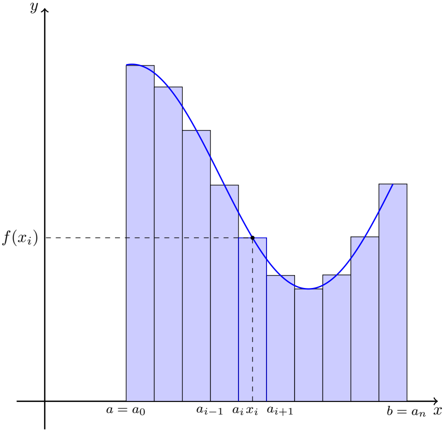
$$R(f, P) = \sum_{j=1}^{n} f(x_j)(a_j - a_{j-1}) = \sum_{j=1}^{n} f(x_j)\Delta_j \qquad (2)$$

Each term in the sum is the product of the value of the function at a given point and the length of an interval. Consequently, each term represents the area of a rectangle with height $f(x_j)$ *and length* $a_j - a_{j-1}$*. The Riemann sum is the signed area under all the rectangles.*

The Riemann sum $s_D(f)$ is the algebraic area of the union of the rectangles of width Δ_j and height $f(x_j)$. This is an algebraic area since $f(x_j)\Delta_j$ is counted positively if $f(x_j) > 0$ and negatively if $f(x_j) < 0$.
Intuitively the algebraic area A under the graph of f is the limit of $s_D(f)$ when the Δ_j tend to 0. One possible choice is a uniform partition of the interval $[a, b]$

$$a_j = a + j\frac{b-a}{n}, \quad 0 \leq j \leq n \quad \text{where} \quad \Delta_j = h = \frac{b-a}{n},$$

which could be combined with any choice of x_j.


y
f(x_i)
a = a_0
a_{i-1}
a_i x_i
a_{i+1}
b = a_n
x

Definition

For any bounded function f defined on the bounded and closed interval $[a, b]$, the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x_k, \ (\|P\| \to 0)$$

whenever the limit exists. (The limit is over all pointed partitions $P = \{([x_{j-1}, x_j], w_j)\}_{1\leq j\leq n}$). When the limit exists, we say that f is Riemann integrable (or integrable) on $[a, b]$.

Example 2 :
Let $f\colon [0,1] \longrightarrow \mathbb{R}$ defined by $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$ and $f(x) = 0$ if $x \notin \mathbb{Q} \cap [0,1]$. If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ a partition of the closed interval $[0,1]$, we take the marks $t = \{t_0, t_1, t_2, \ldots, t_n\}$, $t' = \{t'_0, t'_1, t'_2, \ldots, t'_n\}$ such that $t_k \in [x_k, x_{k+1}] \cap \mathbb{Q}$ and $t'_k \in [x_k, x_{k+1}] \cap (\mathbb{R} \setminus \mathbb{Q})$, for all $k = 0, \ldots, n-1$. Then $R(f, P, t) = 1$ and $R(f, P, t') = 0$. Then f is not Riemann integrable.

Theorem

Any continuous function $f : [a, b] \longrightarrow \mathbb{R}$ *is Riemann integrable and*

$$\int_a^b f(x)dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k).$$

$x_k \in [a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}]$

Definition

A function $f : [a, b] \longrightarrow \mathbb{R}$ *is called piecewise continuous on a closed interval* $[a, b]$ *if there exists a partition* $P = \{x_0, x_1, x_2, \ldots, x_n\}$ *of the closed interval* $[a, b]$ *such that* f *is continuous on every interval* $]x_k, x_{k+1}[$, $\lim_{x\to x_k^+} f(x)$ *and* $\lim_{x\to x_{k+1}^-} f(x)$ *exist in* $\mathbb{R}$, *for all* $k = 0, \ldots, n-1$.

Theorem

Any piecewise continuous function $f : [a, b] \longrightarrow \mathbb{R}$ *is Riemann integrable and*

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k).$$

$x_k \in [a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}]$

Fundamental Properties.

1. Linearity. If $f, g\colon [a, b] \to \mathbb{R}$ are two functions and α,β two reals numbers, then

$$R(\alpha f + \beta g, P) = \alpha R(f, P) + \beta R(g, P).$$

2. Monotony. If $f, g\colon [a, b] \to \mathbb{R}$ are two functions, then

$$f \leq g \quad \Rightarrow \quad R(f, P) \leq R(g, P).$$

In particular, if $f \geq 0$, then $R(f, P) \geq 0$.

3. Chasles's Formula. If $a < c < b$ are three reals numbers and f be a function defined on $[a, b]$. If P_1 is a pointed partition of $[a, c]$ and P_2 be a pointed partition of $[b, c]$, then $P_1 \cup P_2$ is a pointed partition of $[a, b]$ and

$$R(f, P_1 \cup P_2) = R(f, P_1) + R(f, P_2).$$

Theorem

If $f : [a, b] \to \mathbb{R}$ *is integrable and*

$$f(x) \geq 0, \quad \forall x \in [a, b],$$

then the area A *of the region under the graph of* f *from* a *to* b *is*

$$A = \int_a^b f(x)dx.$$

Theorem

We have the following properties of the the definite integrals:
(P_1) *If α is a real number, then*

$$\int_a^b \alpha dx = \alpha(b - a).$$

(P_2) *If α is a real number and $f : [a, b] \to \mathbb{R}$ is an integrable function, then αf is integrable on $[a, b]$ and*

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$

Theorem

(P_3)*If f and g are two integrable functions on $[a, b]$, then $f + g$ is integrable on $[a, b]$ and*

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(P_4)*If f and g are two integrable functions on $[a, b]$, then $f - g$ is integrable on $[a, b]$ and*

$$\int_a^b f(x) - g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

Theorem

(P_5) *If $a < c < b$ and if f is an integrable function on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$, moreover*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(P_6)*If f is integrable on a closed interval I and if a, b and c three numbers in I, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem

(P_7) *If f is integrable on $[a, b]$ and*

$$\forall x \in [a, b], \quad f(x) \geq 0.$$

Then

$$\int_a^b f(x)dx \geq 0.$$

(P_8) *If f and g are integrable on $[a, b]$ and*

$$\forall x \in [a, b], \quad f(x) \geq g(x).$$

Then

$$\int_a^b f(x)dx \geq \int_a^b f(x)dx.$$

Example 3 :
Verify the following inequality

$$\int_1^4 (2x+2)dx \leq \int_1^4 (3x+1)dx.$$

Solution. Here $[a, b] = [1, 4]$ and

$$f(x) = 3x+1, \quad g(x) = 2x+2.$$

We have

$$f(x) - g(x) = (3x+1) - (2x+2) = x - 1 \geq 0, \quad \forall x \in [1, 4],$$

then

$$f(x) \geq g(x), \quad \forall x \in [1, 4].$$

Using the property **(P_8)**, we obtain

$$\int_1^4 g(x)dx \leq \int_1^4 f(x)dx$$

or

$$\int_1^4 (2x+2)dx \leq \int_1^4 (3x+1)dx.$$

Example 4 :
Compute the following integral

$$\int_{-3}^{3} |x^2 - x - 2| dx.$$

Solution. $x^2 - x - 2 = (x-2)(x+1)$, then

$$\begin{aligned}\int_{-3}^{3} |x^2 - x - 2| dx &= \int_{-3}^{-1} x^2 - x - 2 dx + \int_{-1}^{2} 2 - x^2 + x dx \\ &\quad + \int_{2}^{3} x^2 - x - 2 dx \\ &= 9.\end{aligned}$$

Theorem

(Mean Value Theorem for the definite integrals)
If f is continuous on $[a, b]$, then there is a number $c \in [a, b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c).$$

Proof

Let $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$.

Since $m \leq f \leq M$, $m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$. By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c).$$ □

Definition

Let f be a continuous on $[a, b]$. Then the average value f_{av} of f is given by

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 5 :

Let $f(x) = 3x + 7$ on the interval $[0, 1]$. We know that $\int_0^1 (3x+7)dx = \frac{17}{2}$. Then the point c where f assumed its average value verify $3c + 7 = 172$, then $c = \frac{1}{2}$.

The Fundamental Theorem of Calculus

Theorem

(The Fundamental Theorem of Calculus) part I

If f is a continuous function on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on $[a, b]$ and its derivative is $F'(x) = f(x)$.

Proof

Let $x \in [a, b]$ and $h \neq 0$ such that $x + h \in [a, b]$. Then it results fron the The Mean Value Theorem for Definite Integrals that there exists $c \in [x, x + h]$ or $c \in [x + h, x]$ such that

$f(c) = \dfrac{F(x+h) - F(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)dt$. As f is continuous $\lim_{h\to 0} f(c) = f(x) = F'(x)$. □

Theorem

(The Fundamental Theorem of Calculus) part II

If f is a continuous function on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proof

Let $G(x) = \int_a^x f(t)dt$. We know that $G'(x) = f(x)$, then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + C$ for some constant C for $a \leq x \leq b$. Since $G(a) = 0$, then $C = F(a)$, and $G(x) = F(x) - F(a)$, for all $x \in [a, b]$. $\square$

Theorem

Let f be continuous on the closed interval $[a, b]$. Let $c \in [a, b]$ and

$$G(x) = \int_c^x f(t)dt; \quad x \in [a, b].$$

Then

$$G'(x) = f(x); \quad \forall x \in [a, b].$$

Proof

$G(x) = \int_a^x f(t)dt - \int_a^c f(t)dt$, then $G'(x) = f(x)$. □

Theorem

Let f be continuous on an interval I. If v and u be two differentiable functions on an interval J such that $v(J) \subset I$ and $u(J) \subset I$, then the function

$$x \longmapsto \int_{u(x)}^{v(x)} f(t)dt$$

is differentiable on the interval J. Moreover

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

Proof

Let $F(x) = \int_a^x f(t)dt$, where $a \in I$.
$\int_{u(x)}^{v(x)} f(t)dt = F(u(x)) - F(v(x))$. Since $F'(x) = f(x)$, the Chain Rule Formula yields

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

Definition

1. *A function* $f : [-a, a] \longrightarrow \mathbb{R}$ *is odd if* $f(-x) = -f(x)$ *for all* $x \in [-a, a]$.
2. *A function* $f : (-a, a) \longrightarrow \mathbb{R}$ *is even if* $f(-x) = f(x)$ *for all* $x \in [-a, a]$.
3. *A function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is* $T-$*periodic if* $f(x + T) = f(x)$ *for all* $x \in \mathbb{R}$

Theorem

1. *If f is an odd function on* $[-a, a]$, *then*

$$\int_{-a}^{a} f(x)\, dx = 0.$$

2. *If f is an even function on* $[-a, a]$, *then*

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

3. *If f is T−periodic, then* $\int_{a}^{a+T} f(x)\, dx = \int_{0}^{T} f(x)\, dx$, *for all* $a \in \mathbb{R}$.

Example 1 :

- $\displaystyle\int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = 2\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}.$
- $\displaystyle\int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^1 = 0.$
- $\displaystyle\int_{5-\pi}^{5+\pi} \sin(x)\,dx = \int_{-\pi}^{\pi} \sin(x)\,dx = 0.$

Numerical Integration

Very often definite integrations cannot be done in closed form. When this happens we need some simple and useful techniques for approximating definite integrals. In this section we discuss three such simple and useful methods.

The Trapezoidal Rule

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a non negative continuous function. In this method, to approximate the area under the graph of f, we join the point $(x_j, f(x_j))$ with the point $(x_{j+1}, f(x_{j+1}))$ for each sub-interval $[x_j, x_{j+1}]$, by a straight line and find the area under this line. Which means that we replace f on $[x_j, x_{j+1}]$ by the polynomial P of degree 1 such that $P(x_j) = f(x_j)$ and $P(x_{j+1}) = f(x_{j+1})$. We say that P interpolates f at the points x_j and x_{j+1}. Then

$$P(x) = f(x_j)\frac{x_{j+1} - x}{x_{j+1} - x_j} + f(x_{j+1})\frac{x - x_j}{x_{j+1} - x_j}.$$

The area under the graph of P on the interval $[x_j, x_{j+1}]$ is the area of a trapezoid with value equal to

$$\frac{1}{2}(x_{j+1} - x_j)(f(x_{j+1}) + f(x_{j+1})).$$

The area under the graph of f is approximated by

$$\sum_{j=1}^{n} \frac{1}{2}(x_{j+1} - x_j)(f(x_{j+1}) + f(x_j)).$$

In the case where $x_{j+1} - x_j = \dfrac{b-a}{n}$, this area is approximated by

$$\frac{b-a}{2n} \sum_{j=1}^{n} (f(x_{j+1}) + f(x_{j+1})) = \frac{b-a}{2n} \left(f(a) + 2 \sum_{j=1}^{n-1} f(x_j) \quad + f(b) \right).$$

$$\int_a^b f(x)dx \approx \frac{b-a}{2n} \left(f(a) + 2 \sum_{j=1}^{n-1} f(x_j) \quad + f(b) \right). \qquad (3)$$

This formula (3) is called the **Trapezoidal Rule**.
This formula is exact for polynomials of degree at most 1.

If the function f is C^2 on the interval $[a, b]$. The reminder for this method is approximated as follows

$$|R_n| \leq \frac{(b-a)^3 M_2}{12n^2}, \qquad M_2 = \sup_{x \in [a,b]} |f^{(2)}(x)|.$$

The Simpson Method

In this method, we replace f on $[x_j, x_{j+1}]$ by the polynomial P of degree 2 which interpolates f at the points x_j, x_{j+1} and the middle point $m_j = \frac{x_j+x_{j+1}}{2}$.

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_j(x)dx = \frac{x_{j+1}-x_j}{6}(f(x_j)+f(x_{j+1})+4f(m_j)).$$

$$\begin{aligned} P_j(x) &= f(x_j)\frac{(x_{j+1}-x)(x-m_j)}{(x_{j+1}-x_j)(x_j-m_j)} + f(m_j)\frac{(x_{j+1}-x)(x-x_j)}{(x_{j+1}-m_j)(m_j-x_j)} \\ &+ f(x_{j+1})\frac{(x-x_j)(x-m_j)}{(x_{j+1}-x_j)(x_{j+1}-m_j)}. \end{aligned}$$

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_2(x)dx = \frac{x_{j+1}-x_j}{6}(f(x_j)+f(x_{j+1})+4f(m_j)).$$

If the partition is uniform, $h = x_{j+1} - x_j = \dfrac{b-a}{n}$, then

$$\begin{aligned} S_n(f) &= \sum_{j=0}^{n-1} \frac{b-x}{6n}\left(f(x_j) + f(x_{j+1}) + 4f(m_j)\right) \\ &= \frac{h}{6}\Big(f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j) + 4\sum_{j=0}^{n-1} f(m_j)\Big). \end{aligned}$$

$$\int_a^b f(x)dx \approx \frac{(b-a)}{6n}\Big(f(a)+f(b)+2\sum_{j=1}^{n-1} f(x_j)+4\sum_{j=0}^{n-1} f(m_j)\Big). \quad (4)$$

In other words if we take $n = 2m$, the Simpson formula is given by the following:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{3n}\Big(f(a)+4\sum_{j=0}^{m-1} f(x_{2j+1})+2\sum_{j=1}^{m-1} f(x_{2j})+f(b)\Big). \tag{5}$$

$x_j = a + j\frac{b-a}{2m}$.
This formula (5) is called **The Simpson Formula** and it is exact for polynomials of degree at most 3.

If the function f is C^4 on the interval $[a, b]$, The reminder for this method is approximated as follows

$$|R_n| \leq \frac{(b-a)^5 M_4}{2880 n^4}, \qquad M_4 = \sup_{x \in [a,b]} |f^{(4)}(x)|.$$