1 Indefinite Integrals & Anti-derivative Function

Exercise 1 :
Evaluate the following indefinite integrals:

1. $\int x^\alpha dx, \quad \alpha \in \mathbb{Q} \setminus \{-1\}$,
2. $\int \sec^2 x dx$,
3. $\int \csc^2 x dx$,
4. $\int \tan^2 x dx$,
5. $\int \cot^2 x dx$,
6. $\int \sec x \tan x dx$,
7. $\int \csc x \cot x dx$,
8. $\int (x - \frac{1}{x^{\frac{2}{3}}} + \frac{1}{x^2}) dx$,
9. $\int x + 2 + \frac{4}{(x+1)^2} dx$,
10. $\int (\frac{1}{\sec x} - \frac{1}{\csc x}) dx$.

2 Change of variables, Substitution Method

Exercise 2 :
Evaluate the following integrals

1. $\int x\sqrt{x+1} dx$,
2. $\int \frac{x}{\sqrt{3-4x^2}} dx$,
3. $\int \frac{1}{\sqrt{x}\cos^2(\sqrt{x})} dx$,
4. $\int \frac{x^2+3x+6}{\sqrt{x+1}} dx$,
5. $\int (x^2+1)^n 2x dx$,
6. $\int \sin(2x+3) dx$,
7. $\int \frac{1}{\cos^2(\pi x)} dx$,
8. $\int \frac{1}{\sqrt{x}\cos^2(\sqrt{x})} dx$,
9. $\int x e^{-x^2} dx$,
10. $\int \frac{\sin(\ln x)}{x} dx$.

3 Riemann Sums, Area and the Definite Integral

We recall that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$

Exercise 3 :

1. Find the value of n such that $\sum_{k=1}^{n}(2k^2 - k + 1) = 147.$

2. Find the value of α such that $\sum_{k=1}^{6}\left(k^2 + 3k + 2\alpha\right) = 130,$

Exercise 4 :
Express the sum $\sum_{k=1}^{n} k(k+1)$ in terms of n.

Exercise 5 :
Find the following limits.

1. $\lim_{n \to +\infty} \frac{1}{n^2} \sum_{k=1}^{n}(3k - 2)$

2. $\lim_{n \to +\infty} \sum_{k=1}^{n}(2\frac{k}{n^2} - \frac{3}{n})$

3. $\lim_{n \to +\infty} \frac{1}{n^3} \sum_{k=1}^{n}(3k^2 - 2k + 1)$

Exercise 6 :
Find the Riemann sum R_P for the function f defined by $f(x) = 3x - 2$ on the interval $[-2, 2]$ with respect to the partition $P = \{-2, 0, 1, 1.5, 2\}$ by choosing on each sub-interval of the partition

1. The left-hand end point $w_k = x_{k-1}$

2. The right-hand end point $w_k = x_k$

3. The mid-point $w_k = \frac{x_{k-1}+x_k}{2}$

Exercise 7 :
Use the Riemann sum to find the following integrals:

1. $\int_0^1 (3x + 7)dx,$

2. $\int_1^4 (x^2 + x + 2)dx,$

3. $\int_0^2 (6x^3 + 1)dx,$

Exercise 8 :
Evaluate the following limits:

$$\lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}\frac{1}{n+k},\quad \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{2n+k},\quad \lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2},$$

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=1}^{n}\sin(\frac{kx}{n}),\ x\in\mathbb{R},\ \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{k}{n^2+k^2},\quad \lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2},$$

$$\lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}},\quad \lim_{n\longrightarrow+\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2\sin(\frac{k\pi}{n}),$$

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n}\cos\left(\frac{k\pi}{n}\right),\quad \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2},\quad \lim_{n\to+\infty}\sum_{k=1}^{2^n}\frac{k^3}{2^{4n}}.$$

4 The Fundamental Theorem of Calculus

Exercise 9 :
Find the following limits.

1. $\displaystyle\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\sec^2\left(\frac{k}{n}\right).$

2. $\displaystyle\lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n}(k-1)(k+2).$

Exercise 10 :
Evaluate the following integrals:

1. $\displaystyle\int_{-\pi/2}^{\pi}f(t)dt$ where $f(t)=\begin{cases}\cos(t) & \text{for} -\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\\ \sin(t) & \text{for}\frac{\pi}{2}\leq t\leq\pi.\end{cases}$,

2. Evaluate the integral $\displaystyle\int_0^2|x-1|\,dx.$

Exercise 11 :
Let I and J the integrals defined by:
$I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx$ and $J=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx.$

1. Prove that $I=J$, (we can use the substitution $t=\frac{\pi}{2}-x$).

2. Give the value of $I+J$.

3. Deduce the values of I and J.

Exercise 12 :
Give the value of the following integrals with the indicated change of variable:

1. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}dx, \quad (t = \cos x),$

2. $\displaystyle\int_1^2 \frac{dx}{x(x^3+1)}, \quad (t = x^3+1),$

Exercise 13 :
If $F(x) = x\displaystyle\int_{\sqrt{\pi}}^{x} \cos(t^2)dt$, find $F'(\sqrt{\pi})$.

Exercise 14 :
Compute $f'(x)$ for each f.

1) $f(x) = \displaystyle\int_1^x \sin^3(t)dt,$ 2) $f(x) = \displaystyle\int_x^{x^2} \cos^5(t)dt,$

3) $f(x) = \displaystyle\int_{\sin x}^{\cos x} (1-t^2)^{\frac{3}{2}}dt,$ 4) $f(x) = \displaystyle\int_{\tan x}^{\sec x} (1+t^3)^{\frac{1}{2}}dt,$

5) $f(x) = \displaystyle\int_{\frac{1}{x}}^{2} (4+t^2)^{\frac{5}{2}}dt,$ 6) $f(x) = \displaystyle\int_{e^x}^{e^{2x}} \sqrt{1+4t^2}dt,$

Exercise 15 :
Let f be a continuous function on $[a,b]$, $b \neq a$ and if $\displaystyle\int_a^b f(x)dx = 0$. Prove that there is $c \in [a,b]$ such that $f(c) = 0$.

Exercise 16 :
Find the number c that satisfies the conclusion of the Mean Value Theorem for the following functions

1. $f(x) = 3x + 7$ on the interval $[0,1]$.
2. $f(x) = x^2 + x + 2$ on the interval $[1,4]$.
3. $f(x) = 6x^3 + 1$ on the interval $[0,2]$.
4. $f(x) = ax + b, \quad a \neq 0$, on $[\alpha, \beta]$.

5 Numerical Integration

Exercise 17 :

1. Approximate the integral $\displaystyle\int_0^{\pi} \sqrt{1+\sin x}dx$ using Trapezoidal rule with $n = 4$ and the regular partition.
   Give an approximation of the error.

2. Approximate the integral $\int_0^2 \frac{x}{\sqrt{x+1}}dx$ using Simpson's rule for $n = 4$ and $n = 8$.
   Give an approximate of the reminder in each case.

Exercise 18 :
Let $f(x) = 2x - 1$ and $g(x) = x^2 + 3x - 1$ defined on the interval $[1, 3]$. Use the trapezoidal Method for $n = 5$ to give an approximation of the integrals $\int_1^3 f(x)dx$ and $\int_1^3 g(x)dx$.

Exercise 19 :
Let $g(x) = x^2 + 3x - 1$ and $h(x) = x^3$ defined on the interval $[1, 3]$. Use the Simpson Method for $n = 8$ to give an approximation of the integrals $\int_1^3 (x^2 + 3x - 1)dx$ and $\int_1^3 x^3dx$.