The Transcendental Functions

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Table of contents

The Natural Logarithm Function

• For $\alpha \in \mathbb{Q}$, the function $x \longmapsto x^\alpha$ is continuous on $(0, +\infty)$, then it is Riemann integrable on any interval $[a, b] \subset (0, +\infty)$.

• For $\alpha \in \mathbb{Q}$, $\alpha \neq -1$, $\int x^\alpha dx = \dfrac{x^{\alpha+1}}{\alpha+1} + c$.

Definition

For $x > 0$, we define the function

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$

$\ln(x)$: *called* **the Natural Logarithmic Function of x.**

Remark

For $x > 0$, we define the function

$$\ln(x) = \int_1^x \frac{1}{t}dt = \begin{cases} \text{the area of the region between the graph} \\ \text{of the function } f(t) = \frac{1}{t} \\ \text{the } x-\text{axis and the lines} \\ t = 1 \text{ and } t = x, \text{ if } x > 1 \\ \\ -\Big(\text{the area of the region between} \\ \text{the graph of the function} \\ f(t) = \frac{1}{t} \text{ the } x-\text{axis and the} \\ \text{lines } \; t = x \text{ and } t = 1, \text{ if } 0 < x < 1\Big). \end{cases}$$

Theorem

For all x, y *in* $]0, +\infty[$, *we have*

1. $\ln xy = \ln x + \ln y$.
2. $\ln \frac{1}{x} = -\ln x$.
3. $\ln x^n = n \ln x$, *for all* $n \in \mathbb{N}$.
4. $\ln x^r = r \ln x$, *for all* $r \in \mathbb{Q}$.

Theorem

1. $\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1$
2. $\lim_{x\to +\infty} \ln x = +\infty.$
3. $\lim_{x\to 0^+} \ln x = -\infty$
4. $\lim_{x\to +\infty} \frac{\ln x}{x} = 0.$
5. $\lim_{x\to +\infty} \frac{\ln x}{x^s} = 0;\ \forall\ s \in \mathbb{Q}_+^*.$

Corollary

The Logarithmic function is bijective from $]0,+\infty[$ *onto* $\mathbb{R}$. *There exists a unique real number denoted e such that* $\ln(e)=1$, $(2<e<3)$, *e is called the basis of the natural Logarithmic function.*

Theorem

- $\ln(x)>0,\ \forall x>1$ *and* $\ln(x)<0,\ \forall 0<x<1.$
- $\dfrac{d}{dx}(\ln(x))=\dfrac{1}{x}>0,\ \ \forall x>0,$ *i.e. the function* $x\longmapsto\ln(x)$ *is increasing on* $(0,\infty)$.
- $\dfrac{d^2}{dx^2}(\ln(x))=-\dfrac{1}{x^2}>0;\ \forall x>0.$ *i.e. the function* $x\longmapsto\ln(x)$ *is concave on* $(0,\infty)$.
- $\lim\limits_{x\to 0^+}\ln(x)=-\infty$ *and* $\lim\limits_{x\to\infty}\ln(x)=\infty.$

The Logarithmic Differentiation

In some cases, to compute the derivative of a function f we can find the derivative of the function $\ln|f|$.

Theorem

(The Logarithmic Differentiation)
Let $u\colon I \longrightarrow \mathbb{R}\setminus\{0\}$ *be a differentiable function, then*

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}.$$

In particular if $u(x) > 0$ *for every* $x \in I$*,*

$$\frac{d}{dx}(\ln(u(x))) = \frac{u'(x)}{u(x)}.$$

($\frac{u'}{u}$ *is called the Logarithmic differentiation of* u*).*

Examples

1. If $f(x) = \ln(x^2 + 2x + 4)$, $f'(x) = \frac{2x+2}{x^2+2x+4}$.
2. If $f(x) = \ln(|2 - 3x|^5)$,
$$f'(x) = \frac{5 \times (-3) \times (2 - 3x)^4}{(2 - 3x)^5} = -\frac{15}{2 - 3x}.$$
3. If $f(x) = \ln(|(2 - 3x)^5|) = 5 \ln |2 - 3x|$, then
$$f'(x) = 5.\frac{-3}{2 - 3x} = -\frac{15}{2 - 3x}.$$

4. If $f(x) = \ln(|(2-3x)^5|) = 5\ln|2-3x|$, then $f'(x) = 5.\dfrac{-3}{2-3x} = -\dfrac{15}{2-3x}$.
5. Let $f(x) = \ln(\sqrt{\dfrac{4+x^2}{4-x^2}})$.

$$\begin{aligned} f(x) &= \ln(\sqrt{\frac{4+x^2}{4-x^2}}) = \ln(\frac{(4+x^2)^{\frac{1}{2}}}{(4-x^2)^{\frac{1}{2}}}) \\ &= \ln((4+x^2)^{\frac{1}{2}}) - \ln((4-x^2)^{\frac{1}{2}}) \\ &= \frac{1}{2}\ln(4+x^2) - \frac{1}{2}\ln(4-x^2). \end{aligned}$$

Then

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\frac{2x}{4+x^2} - \frac{1}{2}\frac{(-2x)}{4-x^2} = \frac{x}{4+x^2} + \frac{x}{4-x^2} = \frac{8x}{(4+x^2)(4-x^2)}.$$

6. We can use implicit differentiation to find y'

$$y^2 + \ln(\frac{x}{y}) - 4x = -3.$$

$$y^2 + \ln|x| - \ln|y| - 4x = -3.$$

Differentiating with respect to x,

$$2yy' + \frac{1}{x} - \frac{y'}{y} - 4 = 0,$$

Then

$$2yy' - \frac{y'}{y} = 4 - \frac{1}{x} \Rightarrow (2y - \frac{1}{y})y' = 4 - \frac{1}{x}$$

and

$$y' = \frac{4xy - y}{2y^2x - x}.$$

7. If $f(x) = \sqrt{(3x^2+2)\sqrt{6x-7}}$.

$$\begin{aligned}\ln(f(x)) &= \ln(\sqrt{(3x^2+2)\sqrt{6x-7}})\\ &= \frac{1}{2}\ln((3x^2+2)\sqrt{6x-7})\\ &= \frac{1}{2}\ln(3x^2+2) + \frac{1}{2}\ln(\sqrt{6x-7})\\ &= \frac{1}{2}\ln(3x^2+2) + \frac{1}{4}\ln(6x-7).\end{aligned}$$

Now differentiating with respect to x

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\frac{6x}{3x^2+2} + \frac{1}{4}\frac{6}{6x-7} = \frac{3x}{3x^2+2} + \frac{1}{2}\frac{3}{6x-7}.$$

Then

$$f'(x) = \left(\frac{3x}{3x^2+2} + \frac{1}{2}\frac{3}{6x-7}\right)\sqrt{(3x^2+2)\sqrt{6x-7}}.$$

8. If $f(x) = (x+1)^2(x+2)^3(x-5)^7$.

$$\begin{aligned} \ln|f(x)| &= \ln\left|(x+1)^2(x+2)^3(x-5)^7\right| \\ &= \ln\left|(x+1)^2\right| + \ln\left|(x+2)^3\right| + \ln\left|(x-5)^7\right| \\ &= 2\ln|x+1| + 3\ln|x+2| + 7\ln|x-5|. \end{aligned}$$

Now differentiating with respect to x

$$\frac{f'(x)}{f(x)} = 2\frac{1}{x+1} + 3\frac{1}{x+2} + 7\frac{1}{x-5}.$$

So

$$f'(x) = \Big(\frac{2}{x+1} + \frac{3}{x+2} + \frac{7}{x-5}\Big)(x+1)^2(x+2)^3(x-5)^7$$

The Exponential Function

Definition

The Logarithmic function is increasing from $]0, +\infty[$ *onto* $\mathbb{R}$ *and bijective. It has an inverse function denoted* $\exp(x)$, $\ln^{-1}(x)$ *or* e^x *and called the* **Natural Exponential Function**.

Theorem

1. *The exponential function* $e\colon \mathbb{R} \longrightarrow]0, +\infty[$ *is increasing.*
2. $\frac{d}{dx} e^x = e^x$,
3. $e^{x+y} = e^x e^y$.
4. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
5. $\displaystyle\lim_{x\to -\infty} e^x = 0, \qquad \lim_{x\to +\infty} e^x = +\infty.$
6. $\lim_{x\to +\infty} \frac{e^x}{x} = +\infty.$
7. $\displaystyle\lim_{x\to -\infty} xe^x = 0.$

Corollary

If $u\colon I \longrightarrow \mathbb{R}$ *is a differentiable function*

$$\frac{d}{dx}(e^{u(x)}) = u'(x)e^{u(x)}.$$

Example 1 :
If $f(x) = e^{1-x^2}$, $f'(x) = -2xe^{1-x^2}$.

Example 2 :
If $f(x) = e^{x\ln(x)}$, $f'(x) = (\ln(x) + x.\frac{1}{x})e^{x\ln(x)} = (\ln(x) + 1)e^{x\ln(x)}$.

Example 3 :
If $xe^y + 2x - \ln(y+1) = 3$, in use of the implicit differentiation, differentiating with respect to x, we have

$$\begin{aligned}
xe^y + 2x - \ln(y+1) = 3 \quad &\Rightarrow \quad e^y + xy'e^y + 2 - \frac{y'}{y+1} = 0 \\
&\Rightarrow \quad xy'e^y - \frac{y'}{y+1} = -(2+e^y) \\
&\Rightarrow \quad (xe^y - \frac{1}{y+1})y' = -(2+e^y) \\
&\Rightarrow \quad y' = -\frac{2+e^y}{xe^y - \frac{1}{y+1}}.
\end{aligned}$$

Exercise

Find the equation of the tangent line to the graph of the function $f(x) = x - e^{-x}$ *that is parallel to the line* (D) *of equation* $6x - 2y = 7$.

Recall that : if

$(D_1): \; y = ax + c$

$(D_2): \; y = a'x + c'$

then

$(D_1) \parallel (D_2) \iff a = a'$

$(D_1) \perp (D_2) \iff a.a' = -1.$

The required tangent line equation is

$$y - y_1 = m(x - x_1), \quad \text{with } m = f'(x_1). \tag{1}$$

The equation of D is $y = 3x - \frac{7}{2}$, then (1) is parallel to D if and only if $m = 3$. Now, it suffices to find x such that $f'(x) = 3$.

$$f'(x) = 1+e^{-x} = 3 \Rightarrow e^{-x} = 2 \Rightarrow -x = \ln(2) \Rightarrow x = -\ln(2) = \ln(\frac{1}{2}).$$

Then $x_1 = -\ln(2)$ and $f(x_1) = -\ln(2) - e^{\ln(2)} = 2 - \ln(2)$.
Therefore

$$y - 2 + \ln(2) = 3(x + \ln(2)). \tag{2}$$

Integration using "ln" and "exp" functions

Theorem

By using the last properties of the functions ln *and* exp, *we have*

$$\int \frac{1}{x} dx = \ln|x| + c,$$

$$\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c,$$

$$\int e^x dx = e^x + c,$$

$$\int u'(x) e^{u(x)} dx = e^{u(x)} + c.$$

Examples 1 :

1. $\displaystyle\int \frac{dx}{2x+7} = \frac{1}{2}\int \frac{2}{2x+7}dx = \frac{1}{2}\ln|2x+7| + c.$
2. $\displaystyle\int x^2 e^{3x^3} dx = \frac{1}{9}\int (9x^2)e^{3x^3} dx = \frac{1}{9}e^{3x^3} + c.$
3. $\displaystyle\int \frac{x-2}{x^2-4x+9}dx = \frac{1}{2}\int \frac{2(x-2)}{x^2-4x+9}dx =$ $\displaystyle\frac{1}{2}\ln|x^2-4x+9| + c.$

4. $I = \int \frac{(2+\ln(x))^{10}}{x} dx = \int (2+\ln(x))^{10} \frac{1}{x} dx$. If $u = 2 + \ln(x)$,

$$I = \int \frac{(2+\ln(x))^{10}}{x} dx = \int u^{10} du = \frac{(2+\ln(x))^{11}}{11} + c.$$

5. $\int \frac{dx}{x(\ln(x))^2} = \int \frac{(\ln(x))^{-2}}{x} dx = \int (\ln(x))'(\ln(x))^{-2} dx = \frac{-1}{\ln(x)} + c.$

6. $\displaystyle\int \frac{e^x}{(e^x+1)^2}dx = \int (e^x+1)'(e^x+1)^{-2}dx = -(e^x+1)^{-1}+c.$
7. $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \int \frac{(e^x+e^{-x})'}{e^x+e^{-x}}dx = \ln(e^x+e^{-x})+c.$
8. $\displaystyle\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\ln|\cos(x)|+c =$ $\ln|\sec(x)|+c.$

9. $\displaystyle\int \cot(x)dx = \int \frac{\cos(x)}{\sin(x)}dx = \ln|\sin(x)| + c.$

10. To compute the integral, $\displaystyle\int \frac{\tan(e^{-3x})}{e^{3x}}dx$ we set $u = e^{-3x}$, then $du = -3e^{-3x}dx$ and

$$\begin{aligned}\int \frac{\tan(e^{-3x})}{e^{3x}}dx &= \int \frac{\tan(e^{-3x})}{e^{3x}}dx\\ &= -\frac{1}{3}\int \tan(u)du = -\frac{1}{3}\ln|\sec(u)| + c\\ &= -\frac{1}{3}\ln|\sec(e^{-3x})| + c.\end{aligned}$$

Theorem

$$\begin{aligned}
\int \sec(x)dx &= \ln|\sec(x)+\tan(x)|+c \\
\int \csc(x)dx &= \ln|\csc(x)-\cot(x)|+c \\
&= -\ln|\csc(x)+\cot(x)|+c'.
\end{aligned}$$

Proof:

$(\sec)'(x) = \sec(x)\tan(x)$ and $(\tan)'(x) = \sec^2(x)$, then $(\sec + \tan)'(x) = \sec(x)(\sec(x) + \tan(x))$ and

$$\frac{(\sec + \tan)'(x)}{(\sec(x) + \tan(x))} = \sec(x).$$

We deduce that

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + c.$$

$(\csc)'(x) = -\csc(x)\cot(x)$ and $(\cot)'(x) = -\csc^2(x)$, then
$(\csc - \cot)'(x) = \csc(x)(\csc(x) - \cot(x))$,
$(\csc + \cot)'(x) = -\csc(x)(\csc(x) + \cot(x))$ and

$$\frac{(\csc - \cot)'(x)}{(\csc(x) - \cot(x))} = \csc(x),$$

$$\frac{(\csc + \cot)'(x)}{(\csc(x) + \cot(x))} = -\csc(x).$$

We deduce that

$$\int \csc(x)dx = \ln|\csc(x) - \cot(x)| + c = -\ln|\csc(x) + \cot(x)| + c'$$

The General Exponential Function

Definition

For $a > 0$, the function $f(x) = e^{x\ln(a)}$ defined on $\mathbb{R}$ is called the exponential function with base a and denoted by a^x.

Theorem

Let $a > 0$ and $b > 0$. If x and y are two real numbers, we have the following properties

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $(\dfrac{a}{b})^x = \dfrac{a^x}{b^x}$.

Theorem

(Derivative of the General Exponential Function)

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(a^{u(x)}) = a^{u(x)} \ln(a) u'(x).$$

Example 1 :

$$\frac{d}{dx}(5^x) = 5^x \ln(5)$$

$$\frac{d}{dx}(6^{\sqrt{x}}) = 6^{\sqrt{x}} \ln(6) \frac{1}{2\sqrt{x}}.$$

Theorem

(Properties of the General Exponential Functions)

• *If* $a > 1$, $(a^x)' = a^x \ln(a) > 0$ *for all* $x \in \mathbb{R}$. *Hence* a^x *is an increasing function on* $\mathbb{R}$.

• *If* $0 < a < 1$, $(a^x)' = a^x \ln(a) < 0$ *for all* $x \in \mathbb{R}$. *Hence* a^x *is a decreasing function on* $\mathbb{R}$.

Theorem

If $a > 0$ *and* $a \neq 1$, *then*

$$\int a^x dx = \frac{a^x}{\ln(a)} + c.$$

Examples 2 :

1. $$\int 3^x dx = \frac{3^x}{\ln(3)} + c.$$
2. $$\int_{-1}^{0} 3^x dx = \Big[\frac{3^x}{\ln(3)}\Big]_{-1}^{0} = \frac{1-\frac{1}{3}}{\ln(3)} = \frac{2}{3\ln(3)}.$$
3. $$\int \frac{(2^x+1)^2}{2^x} dx = \int \frac{(2^x)^2 + 2.2^x + 1}{2^x} dx =$$
$$\int 2^x + 2 + (\frac{1}{2})^x dx = \frac{2^x}{\ln 2} + 2x + \frac{(\frac{1}{2})^x}{\ln(\frac{1}{2})} + c.$$
4. $$\int \frac{5^{\tan(x)}}{\cos^2(x)} dx = \int 5^{\tan(x)} \sec^2(x) dx = \int 5^{\tan(x)} \tan'(x) dx =$$
$$\frac{5^{\tan(x)}}{\ln(5)} + c.$$

Theorem

$$\lim_{h \to 0} (1+h)^{\frac{1}{h}} = \lim_{x \to \infty} (1 + \frac{1}{x})^x = e.$$

The General Logarithm Function

Definition

(The General Logarithm Function)
If $a \in (0, \infty)$ *and* $a \neq 1$, *the function* $f : \mathbb{R} \longrightarrow (0, \infty)$ *defined by* $f(x) = a^x$ *is bijective. Its inverse function* f^{-1} *is denoted by* $\log_a$ *and called the logarithm function with base* a. *For* $y \in (0, \infty)$ *and* $x \in \mathbb{R}$,

$$x = \log_a(y) \iff y = a^x.$$

Example 2 :

$$\begin{array}{ccccccc} 9 & = & 3^2 & \iff & 2 & = & \log_3(9) \\ 16 & = & 4^2 & \iff & 2 & = & \log_4(16) \\ 64 & = & 4^3 & \iff & 3 & = & \log_4(64). \end{array}$$

Theorem

For $a \in (0, \infty) \setminus \{1\}$,

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)},$$

$$\log_a(x) = \frac{\ln(x)}{\ln(a)},\ \forall x > 0,$$

$$\log_e(x) = \ln(x).$$

Notation. For $a = 10$ the function $\log_{10}$ is denoted by Log.

Examples

1. If $f(x) = \mathrm{Log}|\ln(x)|$,
   $$f'(x) = \frac{\frac{1}{x}}{\ln(10)\ln(x)} = \frac{1}{x\ln(10)\ln(x)}.$$
2. If $f(x) = \ln|\mathrm{Log}(x)|$,
   $$f'(x) = \frac{\frac{1}{x}}{\ln(x)} = \frac{1}{x\ln(x)}.$$
3. If $f(x) = x^{4+x^2}$, $\ln(f(x)) = (4+x^2)\ln(x)$. Then
   $$\frac{f'(x)}{f(x)} = 2x\ln(x) + (4+x^2)\frac{1}{x}.$$
   Then
   $$f'(x) = \Big(2x\ln(x) + (4+x^2)\frac{1}{x}\Big)x^{4+x^2}.$$

Example 3 :
If

$$f_1(x) = x^4, \quad f_2(x) = \pi^x, \quad f_3(x) = x^\pi, \quad f_4(x) = \pi^\pi, \quad f_5(x) = x^x.$$

$$f_1'(x) = 4x^3, \quad f_2'(x) = \pi^x \ln(\pi), \quad f_3'(x) = \pi x^{\pi-1}, \quad f_4'(x) = 0.$$

$$\ln(f_5(x)) = x\ln(x) \Longrightarrow \frac{f_5'(x)}{f_5(x)} = \ln(x) + 1 \Rightarrow f_5'(x) = (\ln(x)+1)x^x.$$

We get two very interesting relations, namely

$$x = \log_{10}(10^x) \qquad \text{and} \quad N = 10^{\log_{10} N}.$$

If $b > 0$ and $b \neq 1$, we get $x = \log_b(b^x)$ and $y = b^{(\log_b y)}$. The Logarithmic function with base b, $b > 0$, $b \neq 1$, satisfies the following important properties

1. $\log_b(b) = 1$, $\log_b(1) = 0$, and $\log_b(b^x) = x$ for all $x \in \mathbb{R}$.
2. $\log_b(xy) = \log_b x + \log_b y$, $x > 0$, $y > 0$.
3. $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, $x > 0$, $y > 0$.
4. $\log_b(x^y) = y \log_b x$, $x > 0$, $x \neq 1$, for all $y \in \mathbb{R}$.
5. $b^x b^y = b^{x+y}$
6. $\dfrac{b^x}{b^y} = b^{x-y}$
7. $(b^x)^y = b^{xy}$.

Inverse Trigonometric Functions

The sine Function

The function $f: I = [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow J = [-1, 1]$ defined by $f(x) = \sin x$, is continuous and bijective. Its inverse function is denoted by $f^{-1}(x) = \sin^{-1} x$.
$f^{-1}: [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a continuous function.
For $x \in]-1, 1[$,

$$(f^{-1})'(x) = (\sin^{-1})'(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}.$$

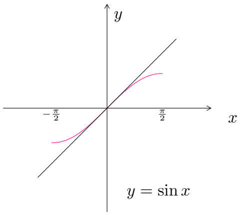
then $\sin(\sin^{-1} x) = x$; $\forall x \in [-1, 1]$, $\sin^{-1}(\sin x) = x$, $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark

$\sin^{-1}(\sin x) = x$ *only for* $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Graph

1



Exercise

Find $\cos(\sin^{-1} x)$ *for* $x \in [-1, 1]$. $\tan(\sin^{-1} x)$ *for* $x \in]-1, 1[$.
$\cot(\sin^{-1} x)$ *for* $x \in [-1, 1], x \neq 0$.

The Cosine Function

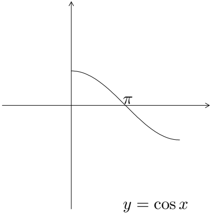
The function $f\colon [0,\pi] \longrightarrow [-1,1]$ defined by $f(x) = \cos x$ is continuous and bijective. f is decreasing. Its inverse function f^{-1} is denoted by $\cos^{-1}\colon [-1,1] \longrightarrow [0,\pi]$.

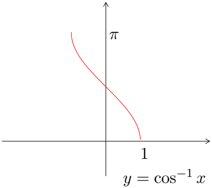
Remarks

1. $\cos(\cos^{-1} x) = x$, *if* $x \in [-1,1]$. $\cos^{-1}(\cos x) = x$, *if* $x \in [0,\pi]$. $\sin(\cos^{-1} x) = \sqrt{1-x^2}$, *if* $x \in [-1,1]$.
2. *For* $x \in]-1,1[$,
$$(f^{-1})'(x) = (\cos^{-1})'(x) = \frac{-1}{\sin(\cos^{-1}(x))} = \frac{-1}{\sqrt{1-x^2}}.$$

Graph

1



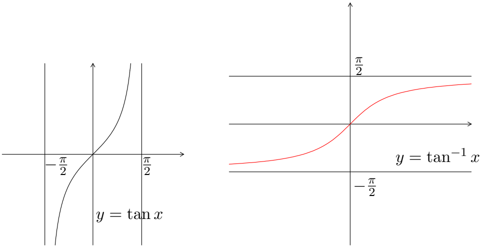


The Tangent Function The function $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ defined by $f(x) = \tan x$ is increasing and continuous. Its inverse function f^{-1} is denoted by $\tan^{-1}$.
$y = \tan^{-1} x \iff x = \tan y$, $\forall x \in \mathbb{R}$ and $\forall y \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then $\tan(\tan^{-1} x) = x$ if $x \in \mathbb{R}$. $\tan^{-1}(\tan x) = x$; $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

$$(f^{-1})'(x) = (\tan^{-1})'(x) = \frac{1}{1 + \tan^2(\tan^{-1}(x))} = \frac{1}{1 + x^2}.$$

Graph

1



The cotangent function

In the same way we define the function $\cot^{-1}\colon \mathbb{R} \longrightarrow]0, \pi[$, as the inverse function of $\cot\colon]0, \pi[\longrightarrow \mathbb{R}$.

$$(\cot^{-1})'(x) = \frac{-1}{1+\cot^2(\cot^{-1}(x))} = \frac{-1}{1+x^2}.$$

The Secant Function

The function $f\colon [0,\frac{\pi}{2}[\cup]\frac{\pi}{2},\pi]$ defined by $f(x)=\dfrac{1}{\cos x}=\sec x$ is increasing and C^∞. Its inverse function is denoted by $f^{-1}(x)=\sec^{-1}x$ for $x\in]-\infty,-1]\cup[1,+\infty[$.
$\sec'(x)=\sec(x)\tan(x)$ and

$$(\sec^{-1})'(x)=\frac{1}{|x|\sqrt{x^2-1}}$$

for $x\in]-\infty,-1]\cup[1,+\infty[$.

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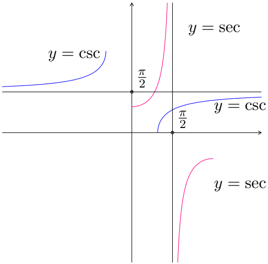

The Cosecant Function The function $f\colon [-\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}]$ defined by $f(x) = \dfrac{1}{\sin x} = \csc x$ is decreasing and C^∞. Its inverse function is denoted by $f^{-1}(x) = \csc^{-1} x$ for $x \in]-\infty, -1] \cup [1, +\infty[$.
$\csc'(x) = -\csc(x)\cot(x)$ and $(\csc^{-1})'(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$, for $x \in]-\infty, -1] \cup [1, +\infty[$.

Exercise

Prove that
a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
b) $\tan^{-1} x + \tan^{-1} \dfrac{1}{x} = \dfrac{\pi}{2}$ *if* $x > 0$ *and* $\tan^{-1} x + \tan^{-1} \dfrac{1}{x} = -\dfrac{\pi}{2}$ *if* $x < 0$.

Basic Trigonometric Formulas

• $\sin^{-1}(x)$ is the angle in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x.
For $x \in [-1, 1]$ and $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$;

$$y = \sin^{-1}(x) \Longleftrightarrow x = \sin(y).$$

• $\cos^{-1}(x)$ is the angle in the interval $[0, \pi]$ whose cosine is x.
For $x \in [-1, 1]$ and $y \in [0, \pi]$;

$$y = \cos^{-1}(x) \Longleftrightarrow x = \cos(y).$$

• $\tan^{-1}(x)$ is the angle in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ whose tan is x.
For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $y \in \mathbb{R}$;

$$y = \tan^{-1}(x) \Longleftrightarrow x = \tan(y).$$

Theorem

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}; \quad \frac{d}{dx}(\sin^{-1}(u(x))) = \frac{u'(x)}{\sqrt{1-u(x)^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}; \quad \frac{d}{dx}(\cos^{-1}(u(x))) = -\frac{u'(x)}{\sqrt{1-u(x)^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}; \quad \frac{d}{dx}(\tan^{-1}(u(x))) = \frac{u'(x)}{1+u(x)^2}$$

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}; \quad \frac{d}{dx}(\cot^{-1}(u(x))) = -\frac{u'(x)}{1+u(x)^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}; \quad \frac{d}{dx}(\sec^{-1}(u(x))) = \frac{u'(x)}{|u(x)|\sqrt{u(x)^2-1}}.$$

Corollary

For $a > 0$, we have

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \sin^{-1}\left(\frac{x}{a}\right) + c \\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \\
\int \frac{dx}{x\sqrt{x^2 - a^2}} &= \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c, \text{for } x > 0
\end{aligned}$$

Remark

The three given substitutions are very useful in calculus. In general, we use the following substitutions for the given radicals

1. $\sqrt{a^2 - x^2}$, $x = a\sin\theta$,
2. $\sqrt{x^2 - a^2}$, $x = a\sec\theta$,
3. $\sqrt{a^2 + x^2}$, $x = a\tan\theta$.

Examples 3 :

Evaluate the following integrals:

1. $I_1 = \displaystyle\int \frac{e^x}{1+e^{2x}}dx,$
2. $I_2 = \displaystyle\int_0^1 \frac{e^x}{1+e^{2x}}dx$
3. $I_3 = \displaystyle\int \frac{x}{\sqrt{1-x^4}}dx$
4. $I_4 = \displaystyle\int_0^{2^{-\frac{1}{4}}} \frac{x}{\sqrt{1-x^4}}dx =$

5. $I_5 = \displaystyle\int \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}dx$
6. $I_6 = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}dx$
7. $I_7 = \displaystyle\int \frac{1}{x\sqrt{x^6-1}}dx$
8. $I_8 = \displaystyle\int \frac{e^x}{\sqrt{e^{2x}-1}}dx.$

The Hyperbolic and Inverse Hyperbolic functions

- Hyperbolic sine function: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$; $x \in \mathbb{R}$.
- Hyperbolic cosine function: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$; $x \in \mathbb{R}$.
- Hyperbolic tangent function:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}};\ x \in \mathbb{R}.$$

- Hyperbolic cotangent function:

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}};\ x \in \mathbb{R} \setminus \{0\}.$$

- Hyperbolic secant function:

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}};\ x \in \mathbb{R}.$$

- Hyperbolic cosecant function:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}};\ x \in \mathbb{R} \setminus \{0\}.$$

Theorem

1. $\cosh^2(x) - \sinh^2(x) = 1, \ \forall x \in \mathbb{R}$,
2. $1 - \tanh^2(x) = \mathrm{sech}^2(x), \ \forall x \in \mathbb{R}$,
3. $\coth^2(x) - 1 = \mathrm{csch}^2(x), \ \forall x \in \mathbb{R} \setminus \{0\}$,
4. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$,
5. $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

Theorem

(Derivative of Hyperbolic Functions)

$$\begin{aligned}
\frac{d}{dx}(\sinh(x)) &= \cosh(x) \\
\frac{d}{dx}(\cosh(x)) &= \sinh(x) \\
\frac{d}{dx}(\tanh(x)) &= \operatorname{sech}^2(x) \\
\frac{d}{dx}(\coth(x)) &= -\operatorname{csch}^2(x) \\
\frac{d}{dx}(\operatorname{sech}(x)) &= -\operatorname{sech}(x)\tanh(x) \\
\frac{d}{dx}(\operatorname{csch}(x)) &= -\operatorname{csch}(x)\coth(x).
\end{aligned}$$

Theorem

(Integration of Hyperbolic Functions)

$$\begin{aligned}
\int \sinh(x)dx &= \cosh(x) + c \\
\int \cosh(x)dx &= \sinh(x) + c \\
\int \operatorname{sech}^2(x)dx &= \tanh(x) + c \\
\int \operatorname{csch}^2(x)dx &= -\coth(x) + c \\
\int \operatorname{sech}(x)\tanh(x)dx &= -\operatorname{sech}(x) + c \\
\int \operatorname{csch}(x)\coth(x)dx &= -\operatorname{csch}(x) + c.
\end{aligned}$$

Examples

1. If $f(x) = \sinh(x^2+1)$, $f'(x) = 2x\cosh(x^2+1)$.
2. $$\int \frac{\sinh(\sqrt{x})}{\sqrt{x}}dx \overset{u=\sqrt{x}}{=} 2\int \sinh(u)du = 2\cosh(\sqrt{x}) + c.$$
3. $$\int \cosh(x)\mathrm{csch}^2(x)dx = -\cosh(x) + c.$$

Example

Find the points on the graph of $f(x) = \sinh(x)$ at which the tangent line has slope 2.
The slope of the tangent line is

$$m = \frac{dy}{dx} = f'(x) = \cosh(x).$$

Then $m = 2 \iff \cosh(x) = 2 \iff e^{2x} - 4e^x + 1 = 0$.
Put $X = e^x$, then X satisfies $X^2 - 4X + 1 = 0$.
There is two solutions of this equation: $X_1 = 2 + \sqrt{3}$ and $X_2 = 2 - \sqrt{3}$.
As $X = e^x$, then $x_1 = \ln(2 + \sqrt{3})$ and $x_2 = \ln(2 - \sqrt{3})$.

Example

Compute $f(x_1) = \sinh(\ln(2+\sqrt{3}))$ and $f(x_2) = \sinh(\ln(2-\sqrt{3}))$:

$$\begin{aligned} f(x_1) &= \frac{e^{\ln(2+\sqrt{3})} - e^{-\ln(2+\sqrt{3})}}{2} \\ &= \frac{e^{\ln(2+\sqrt{3})} - e^{\ln(\frac{1}{2+\sqrt{3}})}}{2} \\ &= \frac{2+\sqrt{3} - \frac{1}{2+\sqrt{3}}}{2} = \sqrt{3} \end{aligned}$$

$$\begin{aligned} f(x_2) &= \frac{e^{\ln(2-\sqrt{3})} - e^{-\ln(2-\sqrt{3})}}{2} \\ &= \frac{e^{\ln(2-\sqrt{3})} - e^{\ln(\frac{1}{2-\sqrt{3}})}}{2} \\ &= \frac{2-\sqrt{3} - \frac{1}{2-\sqrt{3}}}{2} = -\sqrt{3}. \end{aligned}$$

The Inverse Hyperbolic Functions

- **The inverse sine hyperbolic function**

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \sinh(x)$ is bijective and increasing since $f'(x) = \cosh x > 0$. The function f is odd.
$\lim_{x\to+\infty} \sinh x = +\infty$ and $\lim_{x\to+\infty} \frac{\sinh x}{x} = +\infty$.
Its inverse function is denoted by $\sinh^{-1}$. If $x, y \in \mathbb{R}$;

$$y = \sinh^{-1}(x) \iff x = \sinh(y).$$

$$(\sinh^{-1})'(x) = \frac{1}{\sqrt{1+x^2}}, \qquad \forall x \in \mathbb{R}.$$

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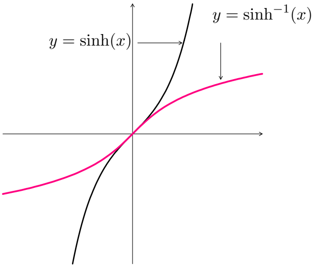
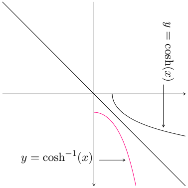


• The inverse cosine hyperbolic function

The function $f\colon [0,+\infty[\longrightarrow [1,+\infty[$ defined by $f(x)=\cosh(x)$ is bijective and increasing since $f'(x)=\sinh x>0$.
$f(x)=\cosh x,\ f'(x)=\sinh x.$ The function f is even.
$\lim_{x\to+\infty}\cosh x=+\infty$ and $\lim_{x\to+\infty}\frac{\cosh x}{x}=+\infty.$
Its inverse function is denoted by $\cosh^{-1}$. If $x\in[1,+\infty[$ and $y\in[0,+\infty[$,

$$y=\cosh^{-1}(x)\iff x=\cosh(y).$$

$$(\cosh^{-1})'(x)=\frac{1}{\sqrt{x^2-1}},\qquad \forall x\in]1,+\infty[.$$



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• **The inverse tangent hyperbolic function**

The function $f : \mathbb{R} \longrightarrow]-1, 1[$ defined by $f(x) = \tanh(x)$ is bijective and increasing since $f'(x) = 1 - \tanh^2 x > 0$. The function f is odd.

$\lim_{x\to+\infty} \tanh x = 1$.

Its inverse function is denoted by $\tanh^{-1}$. If $x \in]-1, 1[$ and $y \in \mathbb{R}$,

$$y = \tanh^{-1}(x) \iff x = \tanh(y).$$

$$(\tanh^{-1})'(x) = \frac{1}{1 - x^2}, \qquad \forall x \in]-1, 1[.$$

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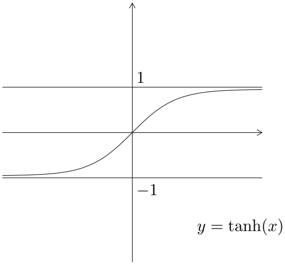

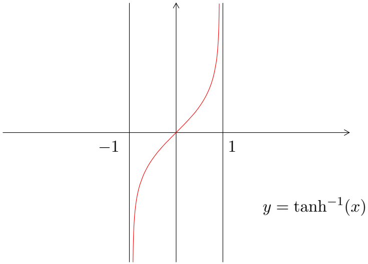

- **The inverse secant hyperbolic function**

The function $f : [0, +\infty[\longrightarrow]0, 1]$ defined by $f(x) = \mathrm{sech}(x) = \dfrac{2}{e^x + e^{-x}}$ is bijective and decreasing since $f'(x) = -\mathrm{sech}(x)\tanh(x) < 0$.
Its inverse function is denoted by sech^{-1}. If $x \in]0, 1]$ and $y \in [0, +\infty[$,

$$y = \mathrm{sech}^{-1}(x) \iff x = \mathrm{sech}(y).$$

$$\mathrm{sech}^{-1}(x) = \frac{1 + \sqrt{1 - x^2}}{x} \qquad \forall x \in]0, 1[.$$

$$(\mathrm{sech}^{-1})'(x) = \frac{-1}{x\sqrt{1 - x^2}}, \qquad \forall x \in]0, 1[.$$

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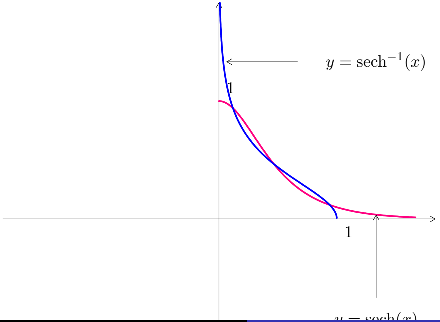

- **The inverse cosecant hyperbolic function**

The function $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ defined by $f(x) = \mathrm{csch}(x) = \dfrac{2}{e^x - e^{-x}}$ is bijective and decreasing since $f'(x) = -\mathrm{csch}(x)\coth(x) < 0$.
Its inverse function is denoted by csch^{-1}. If $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$,

$$y = \mathrm{csch}^{-1}(x) \iff x = \mathrm{csch}(y).$$

$$\mathrm{csch}^{-1}(x) = \frac{1 + \sqrt{1 + x^2}}{x} \qquad \forall x \in]0, +\infty[.$$

$$(\mathrm{csch}^{-1})'(x) = \frac{-1}{x\sqrt{1 + x^2}}, \qquad \forall x \in]0, +\infty[.$$

Theorem

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \ \forall x \in \mathbb{R}.$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1}), \ \forall x \in [1, \infty[.$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \ \forall x \in]-1, 1[.$$

$$\operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \ \forall x \in]0, 1].$$

$$\operatorname{csch}^{-1}(x) = \frac{1+\sqrt{1+x^2}}{x} \qquad \forall x \in]0, +\infty[.$$

Theorem

(Derivative of Inverse Hyperbolic Functions)

$$\begin{aligned}
\frac{d}{dx}\sinh^{-1}(x) &= \frac{1}{\sqrt{x^2+1}}, \ \forall x \in \mathbb{R}\\
\frac{d}{dx}\cosh^{-1}(x) &= \frac{1}{\sqrt{x^2-1}}, \ \forall x \in (1,\infty)\\
\frac{d}{dx}\tanh^{-1}(x) &= \frac{1}{1-x^2}, \ \forall x \in]-1,1[\\
\frac{d}{dx}\mathrm{sech}^{-1}(x) &= -\frac{1}{x\sqrt{1-x^2}}, \ \forall x \in]0,1[.\\
\frac{d}{dx}\mathrm{csch}^{-1}(x) &= \frac{-1}{x\sqrt{1+x^2}}, \qquad \forall x \in]0,+\infty[.
\end{aligned}$$

Corollary

(Some important formulas)
If $a > 0$,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+a^2}} &= \sinh^{-1}\left(\frac{x}{a}\right) + c, \ \forall x \in \mathbb{R} \\
\int \frac{dx}{\sqrt{x^2-a^2}} &= \cosh^{-1}\left(\frac{x}{a}\right) + c, \ \forall x \in (a, \infty) \\
\int \frac{1}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c, \ \forall x \in]-a, a[\\
\int \frac{1}{x\sqrt{a^2-x^2}} &= -\frac{1}{a}\mathrm{sech}^{-1}\left(\frac{|x|}{a}\right) + c, \ \forall x \in]-a, 0[\cup]0, a[. \\
\int \frac{1}{x\sqrt{a^2+x^2}} &= -\frac{1}{a}\mathrm{csch}^{-1}\left(\frac{|x|}{a}\right) + c, \ \forall x \in \mathbb{R}\setminus\{0\}.
\end{aligned}$$

Examples

1. If $f(x) = \cosh^{-1}(\sqrt{x})$,
$$f'(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{x-1}}.$$
2. If $f(x) = \tanh^{-1}(\sin(3x))$,
$$f'(x) = \frac{3\cos(3x)}{\cos^2(3x)} = \frac{3}{\cos(3x)}.$$
3. If $f(x) = \ln(\cosh^{-1}(4x))$,
$$f'(x) = \frac{\frac{4}{\sqrt{(4x)^2-1}}}{\cosh^{-1}(4x)} = \frac{4}{\sqrt{(4x)^2-1}\cosh^{-1}(4x)}.$$

4.

$$\begin{aligned}\int \frac{dx}{\sqrt{81+16x^2}} &\overset{u=4x}{=} \frac{1}{4}\int \frac{du}{\sqrt{9^2+u^2}}\\ &= \frac{1}{4}\sinh^{-1}(\frac{u}{9})+c\\ &= \frac{1}{4}\sinh^{-1}(\frac{4x}{9})+c.\end{aligned}$$

5.

$$\begin{aligned}\int \frac{dx}{\sqrt{5-e^{2x}}} &\overset{u=e^x}{=} \int \frac{du}{u\sqrt{\sqrt{5}^2-u^2}}\\ &= -\frac{1}{\sqrt{5}}\mathrm{sech}(\frac{|u|}{\sqrt{5}})+c\\ &= -\frac{1}{\sqrt{5}}\mathrm{sech}(\frac{e^x}{\sqrt{5}})+c.\end{aligned}$$

We resume the following

1. $\displaystyle\int x dx = \frac{x^2}{2} + c,$
2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1,$
3. $\displaystyle\int \frac{1}{x} dx = \ln|x| + c,$
4. $\displaystyle\int \sin(x) dx = -\cos(x) + c,$
5. $\displaystyle\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c,\ a \neq 0,$
6. $\displaystyle\int \cos(x) dx = \sin(x) + c,$
7. $\displaystyle\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c,\ a \neq 0,$

8. $$\int \tan(x)dx = \ln|\sec(x)| + c,$$
9. $$\int \tan(ax)dx = \frac{1}{a}\ln|\sec(ax)| + c,\ a \neq 0,$$
10. $$\int \cot(x)dx = \ln|\sin(x)| + c,$$
11. $$\int \cot(ax)dx = \frac{1}{a}\ln|\sin(ax)| + c,\ a \neq 0,$$
12. $$\int e^{x}dx = e^{x} + c,$$
13. $$\int e^{-x}dx = -e^{-x} + c,$$
14. $$\int e^{ax}dx = \frac{1}{a}e^{ax} + c,\ a \neq 0,$$

15. $$\int \sinh(x)dx = \cosh(x) + c,$$
16. $$\int \cosh(x)dx = \sinh(x) + c,$$
17. $$\int \tanh(x)dx = \ln\cosh(x) + c,$$
18. $$\int \coth(x)dx = \ln|\sinh(x)| + c,$$
19. $$\int \sinh(ax) = \frac{1}{a}\cosh(ax) + c,\ a \neq 0,$$
20. $$\int \cosh(ax)dx = \frac{1}{a}\sinh(ax) + c,\ a \neq 0,$$
21. $$\int \tanh(ax)dx = \frac{1}{a}\ln\cosh(ax) + c,\ a \neq 0$$

22. $$\int \coth(ax)dx = \frac{1}{a}\ln|\sinh(ax)| + c,\ a \neq 0,$$
23. $$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + c,$$
24. $$\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + c,$$
25. $$\int \sec(ax)dx = \frac{1}{a}\ln|\sec(ax) + \tan(ax)| + c,\ a \neq 0,$$
26. $$\int \sec^2(x)dx = \tan(x) + c,$$
27. $$\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + c,\ a \neq 0,$$
28. $$\int \csc^2(x)dx = -\cot(x) + c,$$

29. $$\int \csc^2(ax)dx = -\frac{1}{a}\cot(ax) + c,\ a \neq 0,$$
30. $$\int \tan^2(x)dx = \tan(x) - x + c,$$
31. $$\int \cot^2(x)dx = -\cot(x) - x + c,$$
32. $$\int \sec(x)\tan(x)dx = \sec(x) + c,$$
33. $$\int \csc(x)\cot(x)dx = -\csc(x) + c$$
34. $$\int \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\cos(x)) + c = \frac{1}{2}(x - \frac{\sin(2x)}{2}) + c,$$
35. $$\int \csc(ax)dx = -\frac{1}{a}\ln|\csc(ax) + \cot(ax)| + c,\ a \neq 0,$$
36. $$\int \cos^3(x)dx = \frac{1}{2}(x + \sin(x)\cos(x)) + c = \frac{1}{2}(x + \frac{\sin(2x)}{2}) + c,$$

Indeterminate Forms and L'Hôpital Rule

The indeterminate forms arise from the fact that $(\bar{\mathbb{R}}, +, .)$, is not a field, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The only operations that are wrong are $0.\infty$ and $+\infty + (-\infty)$. These operations are obtained for example within the real sequences or the limits of functions. For example if a sequence $(u_n)_n$ converges to 0 and the sequences $(v_n)_n$ tends to ∞, we can not decide about the limit of the sequence $(u_n.v_n)_n$.

The only indeterminate forms are $0.\infty$ and $+\infty + (-\infty)$. The other indeterminate forms a different form of writing theses forms. For examples we have

$$\frac{0}{0} = 0.\infty, \quad \frac{\infty}{\infty} = 0.\infty, \quad 1^{\infty} = e^{\infty \ln(1)} = e^{0.\infty}, \quad 0^0 = e^{0 \ln(0)} = e^{0.\infty}.$$

Example

$$\lim_{x\to 2}\frac{(x-2)^2}{(x-2)} = \lim_{x\to 2}(x-2) = 0,$$

$$\lim_{x\to 2}\frac{3(x-2)}{(x-2)} = \lim_{x\to 2} 3 = 3,$$

$$\lim_{x\to 2}\frac{(x-2)^2}{(x-2)^4} = \lim_{x\to 2}\frac{1}{(x-2)^2} = +\infty.$$

In each of above cases the functions are undefined at $x = 2$. And both numerator and denominator in each example approach to 0 as $x \to 0$.

Examples

1. $$\lim_{x\to 0} \frac{\ln(\cos(x))}{\sin(x)} = \frac{1}{2}\lim_{x\to 0} \frac{\ln(1-\sin^2(x))}{\sin(x)} = 0,$$
2. $$\lim_{x\to+\infty} e^{3x}\ln(1+\frac{1}{x}) = \lim_{x\to+\infty} \frac{e^{3x}}{x}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = +\infty,$$
3. $$\lim_{x\to+\infty}(1+x)^2 - \sqrt{x^4+x+2} = \lim_{x\to+\infty} \frac{4x^3+6x^2+3x-1}{(1+x)^2+\sqrt{x^4+x+2}} = \lim_{x\to+\infty} \frac{4x+6+\frac{3}{x}-\frac{1}{x^2}}{(1+\frac{1}{x})^2+\sqrt{1+\frac{1}{x^3}+\frac{2}{x^4}}} = +\infty,$$
4. $$\lim_{x\to+\infty}(1+\frac{1}{x})^x = \lim_{x\to+\infty} e^{\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}} = e.$$

Theorem

L'Hôpital's Rule

Let f, g *be two differentiable functions on* $]a, b[\setminus\{c\}$. *Assume that* $g'(x) \neq 0$ *for all* $x \in]a, b[\setminus\{c\}$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$.

If $\lim_{x\to c} \dfrac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{-\infty, +\infty\}$, *then* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \ell$.

Remarks

1. Theorem 4.1 is valid for one-sided limits as well as the twosided limit. This theorem is also true if $c = +\infty$ or $c = -\infty$.
2. Theorem 4.1 is valid for the case when $\lim\limits_{x\to c} f(x) = \infty$ or $-\infty$ and $\lim\limits_{x\to c} g(x) = \infty$ or $-\infty$.

Examples

Find the following limits if it exist

1. $\lim_{x\to 0} \dfrac{2\sinh x - \sinh 2x}{2x(\cos x - 1)}$.
2. $\lim_{x\to 0} \dfrac{\sin 3x}{3x^3} - \dfrac{\sin x}{x^3}$.
3. $\lim_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$,
4. $\lim_{x\to 0} \dfrac{\tan 2x}{\tan 3x}$,
5. $\lim_{x\to 0} \dfrac{\sin x}{x}$,
6. $\lim_{x\to 0} \dfrac{x}{\sin x}$,
7. $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$,

8. $\lim_{x\to 0} x \ln x.$
9. $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25},$
10. $\lim_{x\to 1} \frac{x^3-3x+2}{x^2-2x+1},$
11. $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin(x)}{\cos(x)},$
12. $\lim_{x\to \infty} \frac{x}{\ln(x)},$
13. $\lim_{x\to \infty} (1+\frac{1}{x})^{5x},$
14. $\lim_{x\to \infty} x^x,$
15. $\lim_{x\to 1^+} \left(\frac{3}{\ln(x)} - \frac{2}{x-1}\right).$

Solutions

1.
$$\frac{2\sinh x - \sinh 2x}{2x(\cos x - 1)} = \frac{2\sinh x - \sinh 2x}{x^3}\frac{x^2}{2(\cos x - 1)}.$$

From L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0}\frac{2\sinh x - \sinh 2x}{x^3} &= \lim_{x\to 0}\frac{2\cosh x - 2\cosh 2x}{3x^2}\\ &= \lim_{x\to 0}\frac{2\sinh x - 4\sinh 2x}{6x}\\ &= \lim_{x\to 0}\frac{2\cosh x - 8\cosh 2x}{6} = -1,\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{2(\cos x-1)} &= \lim_{x\to 0}\frac{2x}{-2\sin x}\\ &= \lim_{x\to 0}\frac{2}{-2\cos x}=-1,\end{aligned}$$

then $\lim_{x\to 0}\frac{2\sinh x-\sinh 2x}{2x(\cos x-1)}=1.$

2.
$$\frac{\sin 3x}{3x^3} - \frac{\sin x}{x^3} = \frac{\sin 3x - 3\sin x}{3x^3}.$$

From L'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 3x - 3\sin x}{3x^3} &= \lim_{x\to 0} \frac{3\cos 3x - 3\cos x}{9x^2} \\
&= \lim_{x\to 0} \frac{-9\sin 3x + 3\sin x}{18x} \\
&= \lim_{x\to 0} \frac{-27\cos 3x + 3\cos x}{18} \\
&= -\frac{4}{3}.
\end{aligned}$$

3. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$,
4. $\lim_{x\to 0} \frac{\tan 2x}{\tan 3x} = \lim_{x\to 0} \frac{2\sec^2 2x}{3\sec^2 3x} = \frac{2}{3}$,
   item $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x = 1$,
5. $\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$,
6. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$,
7. $\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} -x = 0$.

8. $$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25}=\lim_{x\to 5}\frac{1}{4x\sqrt{x-1}}=\frac{1}{40},$$
9. $$\lim_{x\to 1}\frac{x^3-3x+2}{x^2-2x+1}=\lim_{x\to 1}\frac{3x^2-3}{2x-2}=\lim_{x\to 1}\frac{3(x+1)}{2}=3,$$
10. $$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin(x)}{\cos(x)}=\lim_{x\to\frac{\pi}{2}}\frac{\cos(x)}{\sin(x)}=0,$$
11. $$\lim_{x\to\infty}\frac{x}{\ln(x)}=\lim_{x\to\infty}x=+\infty,$$
12. $$\lim_{x\to\infty}(1+\frac{1}{x})^{5x}=\lim_{x\to\infty}e^{5\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}=e^5,$$
13. $$\lim_{x\to\infty}x^x=\lim_{x\to\infty}e^{x\ln(x)}=+\infty,$$
14. $$\lim_{x\to 1^+}\left(\frac{3}{\ln(x)}-\frac{2}{x-1}\right)=\lim_{x\to 1^+}\frac{3(x-1)-2\ln x}{(x-1)\ln(x)}=$$
$$\lim_{x\to 1^+}\frac{3-\frac{2}{x}}{\ln(x)+\frac{x-1}{x}}=\infty.$$

Exercises

Exercise 1 Compute y' in each of the following

1. $y = 2\sinh(3x) + 4\cosh(2x)$,
2. $y = 4\tanh(5x) - 6\coth(3x)$,
3. $y = x\mathrm{sech}(2x) + x^2\mathrm{csch}(5x)$,
4. $y = 3\sinh^2(4x+1)$,
5. $y = 4\mathrm{csch}^2(2x-1)$,
6. $y = \sinh(2x)\mathrm{csch}(3x)$,
7. $y = (x^2+1)^{\sin(2x)}$,

8. $y = x^2 e^{-x^3}$,
9. $y = 2^{x^2}$,
10. $y = \ln(x^2 + 1)$,
11. $y = \log_2(\sec x + \tan x)$,
12. $y = 10^{(x^3+1)}$,
13. $y = x \ln x - x$,

14. $y = \ln(x + \sqrt{x^2 - 4})$,
15. $y = \ln(x + \sqrt{4 + x^2})$,
16. $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$,
17. $y = \sinh^{-1}(3x)$,
18. $y = \cosh^{-1}(3x)$.

Solution

1. $y' = 6\cosh(3x) + 8\sinh(2x)$,
2. $y' = 20(1 - \tanh^2(5x)) - 18(1 - \coth^2(3x))$,
3. $y' = \mathrm{sech}(2x) - 2x\mathrm{sech}(2x)\tanh(2x) + 2x\mathrm{csch}(5x) - 5x^2\mathrm{csch}(5x)\coth(5x)$,
4. $y' = 24\sinh(4x+1)\cosh(4x+1) = 12\sinh 2(4x+1)$,
5. $y' = -16\mathrm{csch}(2x-1)\mathrm{coth}(2x-1)$,
6. $y' = 2\cosh(2x)\mathrm{csch}(3x) - 3\sinh(2x)\mathrm{csch}(3x)\mathrm{coth}(3x)$,
7. $y' = (x^2+1)^{\sin(2x)}\left(\dfrac{2x\sin(2x)}{1+x^2} + 2\cos(2x)\ln(1+x^2)\right)$,

8. $y' = 2xe^{-x^3} - 3x^4e^{-x^3}$,
9. $y' = 2^{x^2+1}x\ln 2$,
10. $y' = \dfrac{2x}{x^2+1}$,
11. $y' = \dfrac{\sec x \tan x + \sec^2 x}{(\sec x + \tan x)\ln 2}$
12. $y' = 3x^2 10^{(x^3+1)} \ln 10$,
13. $y' = \ln x$,

14. $y' = \dfrac{1}{\sqrt{x^2 - 4}}$,
15. $y' = \dfrac{1}{\sqrt{4 + x^2}}$,
16. $y' = \dfrac{1}{1 - x^2}$,
17. $y' = \dfrac{3}{1 - 9x^2}$,
18. $y' = \dfrac{3}{9x^2 - 1}$.

Exercise 2

Find y' in each of the following, using logarithmic differentiation.

1. $y = \dfrac{(x^2+1)^3(x^2+4)^{10}}{(x^2+2)^5(x^2+3)^4}$,
2. $y = (x^2+4)^{(x^3+1)}$,
3. $y = (\sin x + 3)^{(4\cos x + 7)}$,
4. $y = (3\sinh x + \cos x + 5)^{(x^3+1)}$,
5. $y = (e^{x^2}+1)^{(2x+1)}$,
6. $y = x^2(x^2+1)^{(x^3+1)}$,
7. $y = \dfrac{(x+1)^3(2x-3)^{\frac{3}{4}}}{(1+7x)^{\frac{1}{3}}(2x+3)^{\frac{3}{2}}}$.

Solution

1. $\dfrac{y'}{y} = \dfrac{6x}{x^2+1} + \dfrac{20}{x^2+4} - \dfrac{10}{x^2+2} - \dfrac{8}{x^2+3}$,
2. $\dfrac{y'}{y} = 3x^2 \ln(x^2+4) + \dfrac{2x(x^3+1)}{x^2+4}$,
3. $\dfrac{y'}{y} = -4\sin x \ln(\sin x + 3) + \dfrac{\cos x(4\cos x + 7)}{\sin x + 3}$,
4. $\dfrac{y'}{y} = 3x^2 \ln(3\sinh x + \cos x + 5) + \dfrac{(x^3+1)(3\cosh x - \sin x)}{3\sinh x + \cos x + 5}$,
5. $\dfrac{y'}{y} = 2\ln(e^{x^2}+1) + \dfrac{2x(2x+1)e^{x^2}}{e^{x^2}+1}$,
6. $\dfrac{y'}{y} = \dfrac{2}{x} + 3x^2\ln(x^2+1) + \dfrac{2x(x^3+1)}{x^2+1}$,
7. $\dfrac{y'}{y} = \dfrac{3}{x+1} + \dfrac{3}{2(2x-3)} - \dfrac{7}{3(1+7x)} - \dfrac{3}{2x+3}$.

Exercise 3

Solve the following equations for x

1. $\log_3(x^4) + \log_3(x^3) - 2\log_3(x^{\frac{1}{2}}) = 5.$
2. $\dfrac{e^x}{1+e^x} = \dfrac{1}{3}.$

Solution

1. $$\frac{\ln(x^4) + \ln(x^3) - 2\ln(x^{\frac{1}{2}})}{\ln 3} = 5 \iff \ln x^6 = 5\ln 3 \iff x = 3^{\frac{5}{6}}.$$
2. $\dfrac{e^x}{1+e^x} = \dfrac{1}{3} \iff e^x = \dfrac{1}{2} \iff x = -\ln 2.$

Exercise 4

Compute $f'(x)$ for the following functions f.

1. $f(x) = \int_1^x \sinh^3(t)dt$,
2. $f(x) = \int_x^{x^2} \cosh^5(t)dt$,
3. $f(x) = \int_{\sinh x}^{\cosh x} (1+t^2)^{\frac{3}{2}} dt$,
4. $f(x) = \int_{\tanh x}^{\mathrm{sech} x} (1+t^3)^{\frac{1}{2}} dt$,
5. $f(x) = \int_{\ln x}^{(\ln x)^2} (4+t^2)^{\frac{5}{2}} dt$,
6. $f(x) = \int_{(e^x)^2}^{e^{x^2}} (1+4t^2)^{\pi} dt$,

7. $f(x) = \displaystyle\int_{e^{\sin x}}^{e^{\cos x}} \frac{1}{(1+t^2)^{\frac{3}{2}}}\,dt,$
8. $f(x) = \displaystyle\int_{2^x}^{3^x} \frac{1}{(4+t^2)^{\frac{5}{2}}}\,dt,$
9. $f(x) = \displaystyle\int_{4^{2x}}^{5^{3x}} (1+2t^2)^{\frac{3}{2}}\,dt,$
10. $f(x) = \displaystyle\int_{\log_2 x}^{\log_3 x} \frac{1}{(1+5t^3)^{\frac{1}{2}}}\,dt,$
11. $f(x) = \displaystyle\int_{\sinh^{-1} x}^{\cosh^{-1} x} \frac{1}{(1+t^2)^{\frac{3}{2}}}\,dt,$
12. $f(x) = \displaystyle\int_{4x^3}^{2x^2} e^{t^2}\,dt,$
13. $f(x) = \displaystyle\int_{4\sin x}^{5\cos x} e^{-t^2}\,dt,$

$\int^{\cosh(x^2)}$

Solution

1. $f'(x) = \sinh^3(x)$,
2. $f'(x) = \; 2x\cosh^5(x^2) - \cosh^5(x)$,
3. $f'(x) = \sinh x(1+\cosh^2 x)^{\frac{3}{2}} - \cosh x(1+\sinh^2 x)^{\frac{3}{2}} = \sinh x(1+\cosh^2 x)^{\frac{3}{2}} - \cosh^4 x$,
4. $f'(x) = -\mathrm{sech} x \tanh x(1+\mathrm{sech}^3 x)^{\frac{1}{2}} - (1-\tanh^2)(1+\tanh^3 x)^{\frac{1}{2}}$.
5. $f'(x) = 2\frac{\ln x}{x}(4+\ln^4 x)^{\frac{5}{2}} - \frac{1}{x}(4+\ln^2 x)^{\frac{5}{2}}$.
6. $f'(x) = 2xe^{x^2}(1+4e^{2x^2})^{\pi} - 2(e^x)^2(1+4(e^x)^4)^{\pi}$,

7. $f'(x) = -\dfrac{\sin x e^{\cos x}}{(1+e^{2\cos x})^{\frac{3}{2}}} - \dfrac{\cos x e^{\sin x}}{(1+e^{2\sin x})^{\frac{3}{2}}}$,
8. $f'(x) = \dfrac{3^x \ln 3}{(4+3^{2x})^{\frac{5}{2}}} - \dfrac{2^x \ln 2}{(4+2^{2x})^{\frac{5}{2}}}$,
9. $f'(x) = 5^{3x}.3.\ln 5(1+2.5^{6x})^{\frac{3}{2}} - 4^{2x}.4.\ln 2(1+2.4^{4x})^{\frac{3}{2}}$,
10. $f'(x) = \dfrac{1}{x\ln 3(1+5\log_3^3 x)^{\frac{1}{2}}} - \dfrac{1}{x\ln 2(1+5\log_2^3 x)^{\frac{1}{2}}}$,
11. $f'(x) =$
$$\frac{1}{\sqrt{x^2-1}(1+(\cosh^{-1})^2(x))^{\frac{3}{2}}} - \frac{1}{\sqrt{x^2+1}(1+(\sinh^{-1})^2(x))^{\frac{3}{2}}},$$
12. $f'(x) = 4xe^{4x^4} - 12x^2 e^{16x^6}$,,
13. $f'(x) = -5\sin x e^{-25\cos^2 x} - 4\cos x e^{-16\sin^2 x}$,
14. $f'(x) = 2x\sinh(x^2)e^{-\cosh^3(x^2)} - 2x\cosh(x^2)e^{-\sinh^3(x^2)}$.

Exercise 5

Compute $\frac{dy}{dx}$ for each of the following

1. $y = \ln(x^2 + 1)$,
2. $y = \ln\left(\frac{1-x}{1+x}\right)$, $-1 < x < 1$,
3. $y = \log_2(x)$,
4. $y = \log_5(x^3 + 1)$,
5. $y = \log_{10}(3x + 1)$.
6. $y = \log_{10}(x^2 + 4)$,
7. $y = 2e^{-x}$,

8. $y = e^{x^2}$,
9. $y = \dfrac{1}{2}(e^{x^2} - e^{-x^2})$,
10. $y = \dfrac{1}{2}(e^{x^2} + e^{-x^2})$,
11. $y = \dfrac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$,
12. $y = \dfrac{2}{e^{x^2} + e^{-x^2}}$,
13. $y = \dfrac{2}{e^{x^3} - e^{-x^3}}$,
14. $y = \dfrac{2}{e^{x^4 + e^{-x^4}}}$,

15. $y = \sin^{-1}(\frac{x}{2})$,
16. $y = \cos^{-1}(\frac{x}{3})$, 1
17. $y = \tan^{-1}(\frac{x}{5})$,
18. $y = \cot^{-1}(\frac{x}{7})$,
19. $y = \sec^{-1}(\frac{x}{2})$,
20. $y = \csc^{-1}(\frac{x}{3})$,
21. $y = 3\sinh(2x) + 4\cosh(3x)$,

22. $y = e^{x}(3\sin(2x) + 4\cos(2x))$,
23. $y = e^{-x}(4\sin(3x) - 3\cos(3x))$,
24. $y = 4\sinh(2x) + 3\cosh(2x)$,
25. $y = 3\tanh(2x) - 7\coth(2x)$,
26. $y = 3\text{sech}(5x) + 4\text{csch}(3x)$,
27. $y = 10^{x^2}$,
28. $y = 2^{(x^3+1)}$,
29. $y = 5^{(x^4+x^2)}$.

Solution

1. $y' = \dfrac{2x}{x^2+1}$,
2. $y' = -\dfrac{1}{1-x} - \dfrac{1}{1+x} = -\dfrac{2}{1-x^2}$,
3. $y' = \dfrac{1}{x\ln 2}$,
4. $y' = \dfrac{3x^2}{(x^3+1)\ln 5}$,
5. $y' = \dfrac{3}{(3x+1)\ln(10)}$.
6. $y' = \dfrac{2x}{(x^2+4)\ln(10)}$,
7. $y' = -2e^{-x}$,

8. $y' = 2xe^{x^2}$,
9. $y' = x(e^{x^2} + e^{-x^2})$,
10. $y' = x(e^{x^2} - e^{-x^2})$,
11. $y' = \dfrac{e^{x^2} + e^{-x^2}}{e^{x^2} - e^{-x^2}}$,
12. $y' = -2x\mathrm{sech}(\mathrm{x}^2)\tanh(\mathrm{x}^2)$,
13. $y' = -3x^2\mathrm{csch}(x^3)\coth(x^3)$,
14. $y' = -4x^3\mathrm{sech}(\mathrm{x}^4)\tanh(\mathrm{x}^4)$,

15. $y' = \dfrac{1}{\sqrt{4-x^2}},$
16. $y' = -\dfrac{1}{\sqrt{9-x^2}},$ 1
17. $y' = \dfrac{5}{25+x^2},$
18. $y' = -\dfrac{7}{49+x^2},$
19. $y' = \dfrac{2}{|x|\sqrt{x^2-4}},$
20. $y' = -\dfrac{3}{|x|\sqrt{x^2-9}},$
21. $y' = 6\cosh(2x) + 12\sinh(3x),$

22. $y' = e^x(-5\sin(2x) + 10\cos(2x))$,
23. $y' = 5e^{-x}(\sin(3x) + 3\cos(3x))$,
24. $y' = 8\cosh(2x) + 6\sinh(2x)$,
25. $y' = 6\text{sech}^2(2\text{x}) + 14\text{csch}^2(2\text{x})$,
26. $y' = -15\text{sech}(5x)\tanh(5x) - 12\text{csch}(3x)\text{coth}(3x)$,
27. $y' = 2x10^{x^2}\ln(10)$,
28. $y' = 3x^2 2^{(x^3+1)}\ln 2$,
29. $y' = (4x^3 + 2x)5^{(x^4+x^2)}\ln 5$.

Exercise 6

Compute $\dfrac{dy}{dx}$ in the following equations

1. $x^3 + y^3 = 4xy$,
2. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$,
3. $y^2 + 3xy + 2x^2 = 1$.

Solution

1. $3x^2 + 3y^2y' = 4y + 4xy' \Rightarrow y'(3y^2 - 4x) = 4y - 3x^2,$
2. $y' = -x^{-\frac{1}{3}}y^{\frac{1}{3}},$
3. $y'(3x + 2y) = -(3y + 4x).$

Exercise 7

Find the equation of the tangent line to the graph of the following functions at the given points.

1. $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ at $(2, \dfrac{2\sqrt{5}}{3})$,
2. $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$ at $(\dfrac{3}{2}\sqrt{5}, 1)$.

Solution

1. $\frac{2x}{9}+\frac{yy'}{2}=0$, then $y'(2)=-\frac{4}{3\sqrt{5}}$ and the equation of the tangent line at $(2, \frac{2\sqrt{5}}{3})$ is
$y=-\frac{4}{3\sqrt{5}}(x-2)+\frac{2\sqrt{5}}{3}=-\frac{4}{3\sqrt{5}}x+\frac{6}{\sqrt{5}}$.
2. $y'=\frac{2\sqrt{5}}{3}$ and the equation of the tangent line at $(\frac{3}{2}\sqrt{5},1)$ is
$y=\frac{2\sqrt{5}}{3}x-4$.

Exercise 9

Evaluate the following integrals.

1. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$,
2. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$,
3. $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$,
4. $\displaystyle\int \frac{e^{\tan^{-1} x}}{1+x^2}dx$,
5. $\displaystyle\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}dx$,
6. $\displaystyle\int e^{\sin(2x)}\cos(2x)dx$,

7. $\displaystyle\int x^2 e^{x^3}\, dx,$
8. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\, dx,$
9. $\displaystyle\int e^x \cos(1+2e^x) dx,$
10. $\displaystyle\int e^{3x} \sec^2(2+e^{3x}) dx,$
11. $\displaystyle\int 10^{\cos x} \sin x dx,$
12. $\displaystyle\int \frac{4^{\sec^{-1} x}}{x\sqrt{x^2-1}} dx,$
13. $\displaystyle\int x 10^{x^2+3} dx.$

Solution

1. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}(\frac{x}{2}) + c,$
2. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}} = \sinh^{-1}(\frac{x}{2}) + c,$
3. $\displaystyle\int \frac{dx}{\sqrt{x^2-4}} = \cosh^{-1}(\frac{x}{2}) + c,$
4. $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}dx \overset{t=\tan^{-1}x}{=} \int e^t dt = e^{\tan^{-1}x} + c,$
5. $\displaystyle\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx \overset{t=\sin^{-1}x}{=} \int e^t dt = e^{\sin^{-1}x} + c,$
6. $\displaystyle\int e^{\sin(2x)}\cos(2x)dx \overset{t=\sin(2x)}{=} \frac{1}{2}\int e^t dt = \frac{1}{2}e^{\sin(2x)} + c,$

7. $$\int x^2 e^{x^3} dx \overset{t=x^3}{=} \frac{1}{3}\int e^t dt = \frac{1}{3}e^{x^3} + c,$$
8. $$\int \frac{e^{2x}}{1+e^{2x}} dx \overset{t=1+e^{2x}}{=} \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln(1+e^{2x}) + c,$$
9. $$\int e^x \cos(1+2e^x) dx \overset{t=1+2e^x}{=} \frac{1}{2}\int \cos(t) dt = \frac{1}{2}\sin(1+2e^x)+c,$$
10. $$\int e^{3x}\sec^2(2+e^{3x}) dx \overset{t=2+e^{3x}}{=} \frac{1}{3}\int \sec^2(t) dt = \frac{1}{3}\tan(2+e^{3x}) + c,$$
11. $$\int 10^{\cos x}\sin x dx \overset{t=\cos x}{=} -\int 10^t dt = -\frac{10^{\cos x}}{\ln 10} + c,$$
12. $$\int \frac{4^{\sec^{-1}x}}{x\sqrt{x^2-1}} dx \overset{t=\sec^{-1}x}{=} \int 4^t dt = \frac{4^{\sec^{-1}}\ln 4x}{+}c,$$
13. $$\int x10^{x^2+3} dx \overset{t=x^2+3}{=} \frac{1}{2}\int 10^t dt = \frac{10^{x^2+3}}{2\ln(10)} + c.$$

Exercise 10

Evaluate each of the following:

1. $3\sin^{-1}(\frac{1}{2}) + 2\cos^{-1}(\frac{\sqrt{3}}{2})$,
2. $4\tan^{-1}(\frac{1}{\sqrt{3}}) + 5\cot^{-1}(\frac{1}{\sqrt{3}})$,
3. $2\sec^{-1}(-2) + 3\cos^{-1}(-\frac{\sqrt{3}}{2})$,
4. $\cos(2\cos^{-1}(x))$,
5. $\sin(2\cos^{-1}(x))$.

Solution

1. $3\sin^{-1}(\frac{1}{2}) + 2\cos^{-1}(\frac{\sqrt{3}}{2}) = 3\frac{\pi}{6} + 2\frac{\pi}{6} = 5\frac{\pi}{6}$,
2. $4\tan^{-1}(\frac{1}{\sqrt{3}}) + 5\cot^{-1}(\frac{1}{\sqrt{3}}) = 4\frac{\pi}{6} + 5\frac{\pi}{3} = 7\frac{\pi}{3}$,
3. $2\sec^{-1}(-2) + 3\cos^{-1}(-\frac{\sqrt{3}}{2}) = 2\frac{2\pi}{3} + 3\frac{5\pi}{6} = 3\pi$,
4. $\cos(2\cos^{-1}(x)) = 2\cos^2(\cos^{-1}(x)) - 1 = 2x^2 - 1$,
5. $\sin(2\cos^{-1}(x)) = 2x\sin(\cos^{-1}(x)) = 2x\sqrt{1-x^2}$.

Exercise 11

Simplify each of the following expressions by eliminating the radical by using an appropriate trigonometric substitution.

1. $\dfrac{x}{\sqrt{9-x^2}}$,
2. $\dfrac{3+x}{\sqrt{16+x^2}}$,
3. $\dfrac{x-2}{x\sqrt{x^2-25}}$,
4. $\dfrac{1+x}{\sqrt{x^2+2x+2}}$,
5. $\dfrac{2-2x}{\sqrt{x^2-2x-3}}$.

Solution

1. $\dfrac{x}{\sqrt{9-x^2}} \overset{x=3\sin\theta}{=} \dfrac{\sin\theta}{|\cos\theta|}$,
2. $\dfrac{3+x}{\sqrt{16+x^2}} \overset{x=4\tan\theta}{=} \dfrac{3+4\tan\theta}{4|\sec\theta|}$,
3. $\dfrac{x-2}{x\sqrt{x^2-25}} \overset{x=5\sec\theta}{=} \dfrac{5\sec\theta-2}{5|\tan\theta|}$,
4. $\dfrac{1+x}{\sqrt{x^2+2x+2}} = \dfrac{1+x}{\sqrt{(x+1)^2+1}} \overset{x=\tan\theta-1}{=} \dfrac{\tan\theta}{|\sec\theta|}$,
5. $\dfrac{2-2x}{\sqrt{x^2-2x-3}} = \dfrac{2-2x}{\sqrt{(x-1)^2-4}} \overset{x=2\sec\theta+1}{=} \dfrac{1-2\sec\theta}{|\tan\theta|}$.

Exercise 12

Find the exact value of y in each of the following

1. $y = \cos^{-1}(-\frac{1}{2})$,
2. $y = \sin^{-1} \frac{\sqrt{3}}{2}$,
3. $y = \tan^{-1}(-\sqrt{3})$,
4. $y = \cot^{-1}(-\frac{\sqrt{3}}{3})$,
5. $y = \sec^{-1}(-\sqrt{2})$,
6. $y = \csc^{-1}(-\sqrt{2})$
7. $y = \sec^{-1}(-\frac{2}{\sqrt{3}})$,

8. $y = \csc^{-1}(-\frac{2}{\sqrt{3}})$,
9. $y = \sec^{-1}(-2)$,
10. $y = \csc^{-1}(-2)$,
11. $y = \tan^{-1}(\frac{-1}{\sqrt{3}}$,
12. $y = \cot^{-1}(-\sqrt{3})$.

Solution

1. $y = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$,
2. $y = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$,
3. $y = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$,
4. $y = \cot^{-1}(-\frac{\sqrt{3}}{3}) = \frac{2\pi}{3}$,
5. $y = \sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$,
6. $y = \csc^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$
7. $y = \sec^{-1}(-\frac{2}{\sqrt{3}}) = \frac{5\pi}{6}$,

8. $y = \csc^{-1}(-\frac{2}{\sqrt{3}}) = -\frac{\pi}{3}$,
9. $y = \sec^{-1}(-2) = \frac{2\pi}{3}$,
10. $y = \csc^{-1}(-2) = -\frac{\pi}{6}$,
11. $y = \tan^{-1}(\frac{-1}{\sqrt{3}} = \frac{5\pi}{6}$,
12. $y = \cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$.

Exercise 13

Prove the following identities

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \forall x \in [-1, 1].$
2. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall x \in \mathbb{R}$
3. $\tan^{-1} x + \tan^{-1}(\frac{1}{x}) = \frac{\pi}{2}, \quad x > 0.$
4. $\tan^{-1} x + \tan^{-1}(\frac{1}{x}) = -\frac{\pi}{2}, \quad x < 0.$
5. $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}, \forall x \in \mathbb{R} \setminus [-1, 1]$

Solution

1. $\sin^{-1} 0 + \cos^{-1} 0 = \frac{\pi}{2}$ and the derivative of the function $\sin^{-1} x + \cos^{-1} x$ is 0 for all $x \in]-1, 1[$. Then $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \forall x \in [-1, 1]$.
2. $\tan^{-1} 0 + \cot^{-1} 0 = \frac{\pi}{2}$ and the derivative of the function $\tan^{-1} x + \cot^{-1} x$ is 0 for all $x \in \mathbb{R}$. Then $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall x \in \mathbb{R}$.
3. $\tan^{-1} 1 + \tan^{-1}(1) = \frac{\pi}{2}$ and the derivative of the function $\tan^{-1} x + \tan^{-1}(\frac{1}{x})$ is 0. Then $\tan^{-1} x + \tan^{-1}(\frac{1}{x}) = \frac{\pi}{2}, \quad x > 0$.

4. $\tan^{-1}(-1) + \tan^{-1}(-1) = -\frac{\pi}{2}$ and the derivative of the function $\tan^{-1} x + \tan^{-1}(\frac{1}{x})$ is 0. Then $\tan^{-1} x + \tan^{-1}(\frac{1}{x}) = -\frac{\pi}{2}, \quad x < 0.$
5. $\sec^{-1}(\frac{1}{\sqrt{2}}) + \csc^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$, $\sec^{-1}(-\frac{1}{\sqrt{2}}) + \csc^{-1}(-\frac{1}{\sqrt{2}}) = -\frac{\pi}{4} + \pi - \frac{\pi}{4} = \frac{\pi}{2}$ and the derivative of the function $\sec^{-1} x + \csc^{-1} x$ is 0. Then $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$.

Exercise 14

The Chebyshev polynomials are defined by

$$T_n(x) = \cos(n\cos^{-1}x), \quad x \in [-1, 1], n = 0, 1, 2, \dots$$

1. Prove the recurrence relation for the Chebyshev polynomials

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1.$$

2. Show that $T_0(x) = 1$, $T_1(x) = x$ and generate $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ using the recurrence relation.

3. Show that for all integers m and n,

$$T_n(x)T_m(x) = \frac{1}{2}\left(T_{m+n}(x) + T_{|m-n|}(x)\right).$$

4. Determine the zeros of $T_n(x)$ and determine where $T_n(x)$ has its absolute maximum or minimum values for all $n \in \mathbb{N}$.

Solution

$$T_n(x) = \cos(n\cos^{-1} x), \quad x \in [-1, 1], n = 0, 1, 2, \dots$$

1. $T_{n+1}(x) + T_{n-1}(x) = \cos((n+1)\cos^{-1} x) + \cos((n+1)\cos^{-1} x) = 2\cos(n\cos^{-1} x)\cos(\cos^{-1} x) = 2xT_n(x)$. Then

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1.$$

2. $T_0(x) = \cos 0 = 1$, $T_1(x) = \cos(\cos^{-1} x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$ and $T_5(x) = 16x^5 - 20x^3 + 5x$.

3. $T_{m+n}(x) + T_{|m-n|}(x) = \cos((m+n)\cos^{-1} x) + \cos((m-n)\cos^{-1} x) = 2\cos(n\cos^{-1} x)\cos(m\cos^{-1} x) = 2T_n(x)T_m(x)$. Then

$$T_n(x)T_m(x) = \frac{1}{2}\left(T_{m+n}(x) + T_{|m-n|}(x)\right).$$

4. $\cos(n\cos^{-1} x) = 0 \iff n\cos^{-1} x = \frac{\pi}{2} + k\pi$, for $k \in \mathbb{Z}$. Then the zeros of T_n are $x_k = \cos(\frac{(2k+1)\pi}{2n})$, for $k = 0, \dots (n-1)$.
$\cos(n\cos^{-1} x) = 1 \iff n\cos^{-1} x = 2k\pi$, for $k \in \mathbb{Z}$. Then the maximum of T_n is reached at the points $y_k = \cos(\frac{2k\pi}{n})$, for $k =\in \mathbb{Z}$.
$\cos(n\cos^{-1} x) = -1 \iff n\cos^{-1} x = (2k+1)\pi$, for $k \in \mathbb{Z}$. Then the minimum of T_n is reached at the points $z_k = \cos(\frac{(2k+1)\pi}{n})$, for $k =\in \mathbb{Z}$.