CHAPTER 1

# LINEAR DIFFERENTIAL EQUATIONS OF HIGH ORDER

## 1 Basic Properties of Linear Differential Equations of High Order

### 1.1 Definition and Basic Existence Theorem

**Definition 1.1**

A linear ordinary differential equation of order $n$ is an equation that can be expressed in the form

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = b, \tag{1.1}$$

where $a_0, \ldots, a_n$ and $b$ are continuous functions on an interval $(a, b)$ and that $a_0(x) \neq 0$ for all $x \in (a, b)$. The right-hand member $b$ is called the non-homogeneous term. If $b$ is identically zero, the equation (1.1) is called homogeneous.

**Theorem 1.2**

Consider the linear ordinary differential equation (1.1), where $a_0, \ldots, a_n$ and $b$ continuous functions on an interval $(a, b)$ and that $a_0(x) \neq 0$ for any $x \in (a, b)$. For any $x_0 \in (a, b)$ and $c_0, \ldots, c_{n-1}$, $n$ arbitrary real constants, there exists a unique solution $y$ of the equation (1.1) such that $y(x_0) = c_0, \ldots, y^{(n-1)}(x_0) = c_{n-1}$. This solution is defined over the interval $(a, b)$.

**Remark 1.3**

If $y_1, \ldots, y_m$ are solutions of the homogeneous differential equation of (1.1), then for all $a_1, \ldots, a_m \in \mathbb{R}$, $a_1y_1 + \ldots + a_my_m$ is also solution of the homogeneous differential equation.

## 1.2 Fundamental Set of Solutions

**Definition 1.4**

Let $f_1, \ldots, f_m$, $m$ functions defined on the interval $(a, b)$. The functions are called linearly dependent on $(a, b)$ if there exist constants $a_1, \ldots, a_n$ not all zero, such that $a_1 f_1(x) + \ldots + a_m f_m(x) = 0$ for all $x \in (a, b)$. Otherwise, the functions are called linearly independent.

**Examples 1 :**

1. The functions $\sin x, \cos x$ are linearly independent on the interval $[0, \frac{\pi}{2}]$.
   If $a \sin x + b \cos x = 0$ for all $x \in [0, \frac{\pi}{2}]$, then for $x = 0$, $b = 0$ and for $x = \frac{\pi}{2}$, $a = 0$.
   We can also prove that $\sin x, \cos x$ are linearly independent on any non trivial interval.

2. The functions $e^x, \sin x, \cos(2x)$ are linearly independent on the interval $[0, \frac{\pi}{2}]$.
   If $ae^x + b \sin x + c \cos(2x) = 0$ for all $x \in [0, \frac{\pi}{2}]$, then for $x = 0$, $a + c = 0$ and for $x = \frac{\pi}{2}$, $ae^{\frac{\pi}{2}} + b = 0$. Also we can differentiate this function and we get: $ae^x + b \cos x - 2c \sin(2x) = 0$ for all $x \in [0, \frac{\pi}{2}]$. Also for $x = 0$ and $x = \frac{\pi}{2}$, we have, $a + b = 0$ and $a = 0$. Then $a = b = c = 0$.
   We can also prove that $e^x, \sin x, \cos x$ are linearly independent on any non trivial interval.

3. The functions $\sin x, \cos x, \sin(x + 1)$ are linearly dependent on any non trivial interval. Indeed, $\sin(x + 1) = \cos 1 \sin x + \sin 1 \cos x$.

**Theorem and Definition 1.5**

The $n^{th}$-order homogeneous linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0, \tag{1.2}$$

has $n$ solutions linearly independent. Further, if $f_1, \ldots, f_n$ are $n$ linearly independent solutions of (1.2), then every solution $f$ of (1.2) is a linear combination of $f_1, \ldots, f_n$:

$$f = c_1 f_1 + \ldots + c_n f_n, \tag{1.3}$$

for some $c_1, \ldots, c_n \in \mathbb{R}$.
The expression $f$ in (2.5) is called the general solution of the homogeneous equation (1.2) and $\{f_1, \ldots, f_n\}$ is called a fundamental set of solutions of this equation.

**Examples 2 :**

1. $\{\sin x, \cos x\}$ is a fundamental set of solutions of the homogeneous differential equation $y^{''} + y = 0$. Then the general solution of this equation is $y = a \sin x + b \cos x$, with $a, b \in \mathbb{R}$.

2. $\{e^x, xe^x\}$ is a fundamental set of solutions of the homogeneous differential equation $y^{''} - 2y' + y = 0$. Then the general solution of this equation is $y = (ax+b)e^x$, with $a, b \in \mathbb{R}$.

**Remark 1.6**

If $y_1, \ldots, y_m$ are linearly independent, we will construct a differential equation of order $m$ such that $\{y_1, \ldots, y_m\}$ is a fundamental set of solutions of this equation.

We give now a simple criterion for determining whether or not $n$ solutions of (1.2) are linearly independent.

**Definition 1.7**

Let $f_1, \ldots, f_n$, $n$ functions defined on an interval $(a, b)$ each of which has an $(n-1)$ derivative. The determinant

$$W = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix} \tag{1.4}$$

is called the Wronskian of the functions $f_1, \ldots, f_n$.

**Theorem 1.8**

Let $f_1, \ldots, f_n$ solutions of the $n^{th}$-order homogeneous linear differential equation (1.2). These functions are linearly independent on $(a, b)$ if and only if the Wronskian $W$ is not the zero function on the interval $(a, b)$.

We have further:

**Theorem 1.9**

The Wronskian of $n$ solutions $f_1, \ldots, f_n$ of (1.2) is either identically zero on $(a, b)$ or else is never zero on $(a, b)$.

**Proof** .

Using the fundamental properties of the determinant, we have

$$\begin{aligned} W'(x) &= \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n)} & f_2^{(n)} & \dots & f_n^{(n)} \end{vmatrix} \\ &= -\frac{a_1}{a_0} \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \\ &= -\frac{a_1}{a_0} W(x). \end{aligned}$$

Then

$$W(x) = W(x_0) e^{-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)} dt}.$$

□

**Examples 3 :**

1. Let $f(x) = \sin x$, $g(x) = \cos x$ on any non empty open interval $(a, b)$. The Wronskian of $f$ and $g$ is $W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$. Then $f, g$ are linearly independent on the interval $(a, b)$.

2. Let $f(x) = e^x$, $g(x) = \sin x$, $h(x) = \cos x$ on any non empty open interval $(a, b)$. The Wronskian of $f, g, h$ is $W = \begin{vmatrix} e^x & \sin x & \cos x \\ e^x & \cos x & -\sin x \\ e^x & -\sin x & -\cos x \end{vmatrix} = -2e^x$. Then $f, g, h$ are linearly independent on $(a, b)$.

3. Let $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = \sin(x+1)$ on any non empty open interval $(a, b)$. The Wronskian of $f, g, h$ is $W = \begin{vmatrix} \sin x & \cos x & \sin(x+1) \\ \cos x & -\sin x & \cos(x+1) \\ -\sin x & -\cos x & -\sin(x+1) \end{vmatrix} = 0$, since the first and the third row are proportional. Then $f, g, h$ are linearly dependent on $(a, b)$.

**Theorem 1.10**

Let $f_1, \dots, f_n$ be $n$ linearly independent functions $(a, b)$ of class $C^n$. There is a linear differential equation of order $n$ such that $\{f_1, \dots, f_n\}$ is a fundamental set of solutions of this equation.

**Proof** .

Consider the linear differential equation defined by:

$$\begin{vmatrix} y & f_1 & \dots & f_n \\ y' & f_1' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ y^{(n-1)} & f_1^{(n-1)} & \dots & f_n^{(n-1)} \\ y^{(n)} & f_1^{(n)} & \dots & f_n^{(n)} \end{vmatrix} = 0.$$

This equation is of order $n$ since $f_1, \dots, f_n$ are linearly independent.
By definition the set $\{f_1, \dots, f_n\}$ is a fundamental set of solutions of this equation.

□

**Examples 4 :**

1. Consider the functions $f = \sin x$ and $g = \cos x$ on $\mathbb{R}$. The functions $f, g$ are linearly independent. Consider the equation defined by

$$\begin{vmatrix} y & \sin x & \cos x \\ y' & \cos x & -\sin x \\ y'' & -\sin x & -\cos x \end{vmatrix} = y'' + y = 0.$$

   Then $\{f, g\}$ is a fundamental set of solutions of the equation $y'' + y = 0$.

2. Consider the functions $f = \sin x$, $g = \cos x$ and $h(x) = e^x$ on $\mathbb{R}$. The functions $f, g, h$ are linearly independent.

$$\begin{vmatrix} y & \sin x & \cos x & e^x \\ y' & \cos x & -\sin x & e^x \\ y'' & -\sin x & -\cos x & e^x \\ y^{(3)} & -\cos x & \sin x & e^x \end{vmatrix} = -2e^x(y^{(3)} - y'' + y' - y).$$

   Then $\{f, g, h\}$ is a fundamental set of solutions of the equation

$$y^{(3)} - y'' + y'|! - y = 0.$$

# 2 Non Homogeneous Equation

Any function $y_p$ that satisfies (1.1) is called a particular solution of the equation.
For example, $\sin x$ is a particular solution of the differential equation $xy'' + y' + xy = \cos x$.

**Remark 2.1**

If $y_1, \dots, y_m$ are solutions of the homogeneous equation (1.2) on an interval $I$ and $y_p$ is any particular solution of the non-homogeneous equation (1.1) on $I$, then the linear combination

$$c_1 y_1 + \dots + c_m y_m + y_p$$

is also a solution of the non-homogeneous equation (1.1).

**Theorem 2.2**

Let $y_p$ be any particular solution of the non-homogeneous differential equation (1.1) on an interval $I$, and let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of the associated homogeneous differential equation (1.2) on $I$. Then the general solution of the equation (1.1) on the interval $I$ is

$$y = c_1 y_1 + \ldots + c_n y_n + y_p$$

where the $c_1, \ldots, c_n \in \mathbb{R}$.

**Proof** .
Let $y$ be any solution of the differential equation (1.1), the function $y - y_p$ is a solution of the homogeneous equation (1.2). Then there is $c_1, \ldots, c_n \in \mathbb{R}$ such that $y = c_1 y_1 + \ldots + c_n y_n + y_p$. □

**Example 5 :**
Consider the differential equation $(\sin x - \cos x)y'' + 2y' \sin x + y(\cos x + \sin x) = 2$. The function $\cos x$ is a particular solution and $\{e^x, \sin x\}$ is a fundamental set of solutions of this equation. Then the general solution of this equation is $y = axe^x + b \sin x + \cos x$, $a, b \in \mathbb{R}$.

**Theorem 2.3**

Consider the differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b_1 + \ldots + b_m, \tag{2.5}$$

where $a_0, \ldots, a_n, b_1, \ldots, b_m$ continuous functions on an interval $(a, b)$.
If $y_k$ is a particular solution of the non-homogeneous differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b_k,$$

for all $1 \leq k \leq m$, then $y_1 + \ldots + y_m$ is a particular solution of the non-homogeneous differential equation (2.5).

**Example 6 :**
Consider the differential equation $y'' + 2y' + y = e^x + 2e^{-x} + \sin x$.
$\frac{1}{4}e^x$ is a particular solution of the differential equation $y'' + 2y' + y = e^x$.
$x^2 e^{-x}$ is a particular solution of the differential equation $y'' + 2y' + y = 2e^{-x}$.
$-\frac{1}{2}\cos x$ is a particular solution of the differential equation $y'' + 2y' + y = \sin x$.
Then $y_p = \frac{1}{4}e^x + x^2 e^{-x} - \frac{1}{2}\cos x$ is a particular solution of the differential equation $y'' + 2y' + y = e^x + 2e^{-x} + \sin x$.

# 3 Reduction of Order

Consider now the homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0, \tag{3.6}$$

where $a_1, \ldots, a_n$ are continuous functions on an interval $(a, b)$.

If $y_1$ is a known solution of the differential equation and $y_1(x) \neq 0$ for all $x \in (a, b)$. If $y_2 = uy_1$ is a solution, the function $u$ fulfills a linear differential equation of order $n$ in the form $y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' = 0$. This method is called reduction of order.

**Examples 7 :**

1. Consider the differential equation $y'' - 3y' + 2y = 0$.
   $y_1 = e^x$ is a solution of the differential equation. If $y_2 = uy_1$ is a solution, we must have $u'' - u' = 0$. Then $u = a + be^x$ and $y_2 = e^{2x}$ is a second solution of the equation $y'' - 3y' + 2y = 0$ and $y_1, y_2$ are linearly independent.

2. Consider the differential equation $(1 - x^2)y'' - xy' + y = 0$ on the interval $(1, +\infty)$.
   $y_1 = x$ is a solution. Consider a solution $y$ in the form $y = xu$, with $u$ not constant. The function $u$ fulfills the following differential equation:
   $x(x^2 - 1)u'' + (3x^2 - 2)u' = 0$. This yields that $u' = \dfrac{\lambda}{x^2\sqrt{x^2 - 1}}$ and $y_2 = \sqrt{x^2 - 1}$ is a solution.

3. Consider the differential equation $(x - 1)y'' - xy' + y = 0$.
   $y_1 = x$ is a solution of the differential equation. If $y_2 = xu$ is a solution, we must have $x(x - 1)u'' + (-x^2 + 2x - 2)u' = 0$. Then $u' = \left(\dfrac{e^x}{x}\right)'$ and $y_2 = e^x$ is a solution of the differential equation.

## 3.1 Linear Homogeneous Equations With Constant Coefficients

We consider now the differential equation (1.2) with $a_0, \ldots, a_n$ constants in $\mathbb{R}$. We seek for solutions in the form $y = e^{rx}$, where $r$ is constant.
$y$ is a solution of (1.2) if and only if $r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$. This equation is called the characteristic equation or the auxiliary equation of (1.2).

In what follows, we consider the linear homogeneous equations of order 2 and with constant coefficients.

$$y'' + ay' + b = 0$$

The characteristic equation is $r^2 + ar + b = 0$. We have three cases

1. If $\Delta = a^2 - 4b > 0$, the characteristic equation has two different solutions $r_1$ and $r_2$. $\{e^{r_1 x}, e^{r_1 x}\}$ is a fundamental set of solutions of the equation.

2. If $\Delta = 0$, the characteristic equation has one solution $r = -\frac{a}{2}$. $\{e^{rx}, xe^{rx}\}$ is a fundamental set of solutions of the equation.

3. If $\Delta < 0$, the characteristic equation has two different complex solutions $r_1$ and $r_2$. $r_1 = \alpha + \mathrm{i}\beta$, $r_2 = \alpha - \mathrm{i}\beta$. $\{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$ is a fundamental set of solutions of the equation.

**Examples 8 :**

1. $y" - 3y' + 2y = 0$, $\{e^x, e^{2x}\}$ is a fundamental set of solutions of the equation.
2. $y" + 4y' = 0$. $\{1, e^{-4x}\}$ is a fundamental set of solutions of the equation.
3. $y" + y' + y = 0$. $\{e^{\frac{-x}{2}}\cos\frac{\sqrt{3}}{2}x, e^{\frac{-x}{2}}\sin\frac{\sqrt{3}}{2}x\}$ is a fundamental set of solutions of the equation.
4. $y" + 2y' + y = 0$. $\{e^{-x}, xe^{-x}\}$ is a fundamental set of solutions of the equation.

## 3.2 Linear Non-Homogeneous Equations With Constant Coefficients

We consider the differential equation

$$y^{''} + ay' + by = f(x). \tag{3.7}$$

We give two methods to contruct a particular solution of the differential equation 2.

### 3.2.1 Change of Parameters Method

**Theorem 3.1**

1. Let $\{y_1, y_2\}$ be a fundamental set of solutions of the homogeneous equation. For any differentiable function $y$ on $I$, there exists a unique pair of differentiable functions $(U, V)$ on $I$ such that:

$$\begin{cases} y = Uy_1 + Vy_2 \\ y' = Uy_1' + Vy_2' \end{cases} \tag{3.8}$$

2. If $y$ is a solution of the differential equation (2), there exists a unique pair of differentiable functions $(U, V)$ on $I$ such that:

$$\begin{cases} U'y_1 + V'y_2 = 0 \\ U'y_1' + V'y_2' = f \end{cases}.$$

This method is called the change of parameters method.

**Proof** .

1. For any $x \in I$, the determinant of the linear system (3.8) is $W(x) \neq 0$, thus we have a unique solution

$$U(x) = \frac{\begin{vmatrix} y(x) & y_2(x) \\ y'(x) & y_2'(x) \end{vmatrix}}{W(x)}, \quad V(x) = \frac{\begin{vmatrix} y_1(x) & y(x) \\ y_1'(x) & y'(x) \end{vmatrix}}{W(x)}.$$

2. If $y$ is a is a solution of the differential equation . There exists a unique pair of differentiable functions $(U, V)$ on $I$ satisfying the system (3.8). If we differentiate the first equation of the system, we get:

$$U'y_1 + V'y_2 = 0. \tag{3.9}$$

$y$ is twice differentiable, then

$$y" = Uy_1" + Vy_2" + U'y_1' + V'y_2'. \tag{3.10}$$

Now $y$ is a solution of the differential equation (2) if and only if

$$\begin{cases} U'y_1 + V'y_2 = 0 \\ U'y_1' + V'y_2' = f \end{cases}.$$

This system is a Crammer system, so it has a unique solution $y = Uy_1 + Vy_2$. Thus the set of solutions of the differential equation (2) is the set $\{y = Uy_1 + Vy_2,\}$ where $U, V$ differentiable functions solutions of the following system:

$$\begin{cases} U'y_1 + V'y_2 = 0 \\ U'y_1' + V'y_2' = f \end{cases}.$$

□

**Examples 9 :**

1. Consider the differential equation $y'' + y = \dfrac{1}{3 + \cos(2x)}$.

The general solution of the homogenous differential equation is $y = a\cos x + b\sin x$. Using the change of parameters method, $y = U\cos x + V\sin x$, we find:

$$\begin{cases} U'\cos x + V'\sin x = 0 \\ -U'\sin x + V'\cos x = \frac{1}{3+\cos(2x)} \end{cases}.$$

Then

$$U = -\frac{1}{2}\tan^{-1}(\cos x) + a, \qquad V = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sin x}{\sqrt{2}}\right) + b.$$

2. Consider the differential equation $y''+4y'+5y=\cosh(2x)\cos x$. The characteristic equation is $r^2+4r+5=(r+2+\mathrm{i})(r+2-\mathrm{i})$. $\{e^{-2x}\cos(2x), e^{-2x}\sin(2x)\}$ is a fundamental set of solutions of the equation.
   Using the change of parameter method, the general solution of the equation takes the form: $y=Ue^{-2x}\cos(x)+Ve^{-2x}\sin(x)$, with $U'e^{-2x}\cos(x)+V'e^{-2x}\sin(x)=0$ and $U'e^{-2x}(-\sin(x)-2\cos(x))+V'e^{-2x}(\cos(x)-2\sin(x))=\cosh(2x)\cos(x)$.
   Then $U=-\dfrac{1}{8}\cos(2x)+\dfrac{1}{20}e^{4x}\sin(2x)-\dfrac{1}{40}e^{4x}\cos(2x)+a$ and $V=\frac{1}{40}e^{4x}\sin(2x)+\frac{1}{20}e^{4x}\cos(2x)+\frac{x}{4}+\frac{1}{8}\sin(2x)+\frac{1}{16}e^{4x}+b$.

# 4 The Cauchy-Euler equation

## 4.1 The Homogeneous Cauchy-Euler equation

A second order homogeneous linear ordinary differential equation of the form

$$ax^2y''+bxy'+cy=0, \tag{4.11}$$

with $a,b$ constants, is called an homogeneous **Euler's equations** or an homogeneous **Cauchy-Euler equations**. This equation can be reduced to linear homogeneous differential equation with constant coefficients. This conversion can be done in two ways.
The first way is to take the change of variables $x=e^t$. We denote $z(t)=y(e^t)=y(x)$. $z'=e^ty'(e^t)=xy'$ and $z''=z'+x^2y''(x)$. Then $ax^2y''+bxy'+cy=a(z''-z')+bz'+cz=az''+(b-a)z'+cz$. The Cauchy-Euler equation (4.11) becomes a linear differential equation

$$az''+(b-a)z'+cz=0.$$

The second way is to look for a solutions in the form $y=x^r$. Substituting into the differential equation gives the following: $x^r\left(ar(r-1)+br+c\right)=0$. For $x\neq 0$, we have $ar(r-1)+br+c=0$.

**Examples 10 :**

1. Consider the following Cauchy-Euler equation: $x^2y''+3xy'-3y=0$. The differential equation of $z$ is $z''+2z'-3z=0$. Then $z=ae^t+be^{-3t}$ and $y=ax+bx^{-3}$.

   If we use the second method, we get: $r^2+2r-3=0$. Then $r=1$ or $r=-3$ and $y=ax+bx^{-3}$.

2. Consider the following Cauchy-Euler equation: $x^2y''-3xy'+7y=0$. The differential equation of $z$ is $z''-4z'+7z=0$. Then $z=ae^{2t}\cos(\sqrt{3}t)+be^{2t}\sin(\sqrt{3}t)$ and $y=ax^2\cos(\sqrt{3}\ln x)+bx^2\sin(\sqrt{3}\ln x)$.

   If we use the second method, we get: $r^2-4r+7=0$. Then $r=2\pm\mathrm{i}\sqrt{3}$ and $y=ax^2\cos(\sqrt{3}\ln x)+bx^2\sin(\sqrt{3}\ln x)$.

## 4.2 The Non-Homogeneous Cauchy-Euler equation

The non-homogeneous Euler equation is written as

$$ax^2y'' + bxy' + cy = f. \tag{4.12}$$

To solve this equation, we look for a fundamental set of solutions of the homogeneous equation and use the change of parameter method.

**Example 11 :**
Consider the following Cauchy-Euler equation: $x^2y'' + 3xy' - 3y = e^x$.
$\{y_1 = x, y_2 = x^{-3}\}$ is a fundamental set of solutions. In use of the change of parameter method, $y = Ux + Vx^{-3}$, we find $U = \frac{1}{4}e^x$, $V' = -\frac{1}{4}x^4e^x$ and $V = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$. Then $y = ax + bx^{-3} + e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)x^{-3}$.

## 4.3 Action of Differential Operator on Elementary Functions

1. The exponential function:

$$De^{\lambda x} = \lambda e^{\lambda x}, \quad D^n e^{\lambda x} = \lambda^n e^{\lambda x}$$

In this case we say that the function $e^{\lambda x}$ is an eigenfunction of operator $D$ with eigenvalue $\lambda$.

2. The sine and cosine functions:

$$D^2 \sin(\lambda x) = -\lambda^2 \sin(\lambda x), \qquad D^2 \cos(\lambda x) = -\lambda^2 \cos(\lambda x).$$

We say that the functions $\sin(\lambda x)$ and $\cos(\lambda x)$ are eigenfunctions of the operator $D^2$ with eigenvalue $-\lambda^2$.

3. The power functions

$$D^n x^k = k(k-1)\cdots(k-n+1)x^{k-n},$$

where $k \in \mathbb{N}$. In particular $D^n x^k = 0$ for $k < n$.

## 4.4 Polynomial of the Differential Operator $D$

**Definition 4.1**

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n$ of real variable, where $a_0, a_1, \cdots, a_n$ are real constants. Then The operator

$$P(D) = a_n D^n + a_{n-1}D^{n-1} + a_1 D + \cdots + a_0$$

is called a polynomial of differential operator $D$ of degree $n$.

**Theorem 4.2**

If $P(D) = a_n D^n + a_{n-1}D^{n-1} + a_1 D + \cdots + a_0$, then

1. $P(D)e^{\lambda x} = P(\lambda)e^{\lambda x}$.
2. $P(D^2)\sin(\lambda x) = P(-\lambda^2)\sin(\lambda x)$ and
   $P(D^2)\cos(\lambda x) = P(-\lambda^2)\cos(\lambda x)$.

We prove this theorem by induction.

**Theorem 4.3**

If $P(D) = a_n D^n + a_{n-1}D^{n-1} + a_1 D + \cdots + a_0$ and $f$ $n$-times differentiable a function on $\mathbb{R}$, we have

1. $D^n\left(e^{\lambda x}f(x)\right)=e^{\lambda x}\left(D+\lambda\right)^n f(x)$.
2. $P(D)\left(e^{\lambda x}f(x)\right)=e^{\lambda x}P\left(D+\lambda\right)f(x)$.

**Proof** .
We prove the Theorem by induction.

For $n=1$: $D\left(e^{\lambda x}f(x)\right)=\lambda e^{\lambda x}f(x)+e^{\lambda x}f'(x)=e^{\lambda x}(Df(x)+\lambda)=e^{\lambda x}(D+\lambda)f(x)$.

Assume the result holds for $n$, then

$$\begin{aligned}D^{n+1}\left(e^{\lambda x}f(x)\right) &= D\left(D^n\left(e^{\lambda x}f(x)\right)\right)\\ &= D\left(e^{\lambda x}(D+\lambda)^n f(x)\right)\\ &= e^{\lambda x}(D+\lambda)((D+\lambda)^n f(x))\\ &= e^{\lambda x}(D+\lambda)^{n+1}f(x)\end{aligned}$$

The result of (2) follows from (1):

**Definition 4.4**

We interpret the previous Theorem as follows:

$$(D+\lambda)^n=e^{-\lambda x}D^n e^{\lambda x}.$$

the function $e^{\lambda x}$ and $e^{-\lambda x}$ are interpreted as operators.

**Definition 4.5**

Let $P(D)=a_nD^n+a_{n-1}D^{n-1}+\ldots+a_1D+a_0$ be a polynomial of differential operator $D$ of degree $n$.
The kernel of the polynomial $P(D)$ is the set of solutions of the linear $n$-th order ordinary homogeneous differential equation $P(D)y=0$.

**Theorem 4.6**

1. If $P(D)=(D-r_1)(D-r_2)\ldots(D-r_n)$ where $r_1,\ldots,r_n$ different, then the kernel of $P(D)$ is the vector space spanned by $\{e^{r_1x},\ldots,e^{r_nx}\}$.
2. If $P(D)=(D-r_1)^{n_1}(D-r_2)^{n_2}\cdots(D-r_k)^{n_k}=\prod_{j=1}^{k}(D-r_j)^{n_j}$, where $r_1,\ldots,r_k$ different and $\sum_{j=1}^{k}n_j=n$, then the kernel of $P(D)$ is

$$\oplus_{j=1}^{k}\text{Vect}\left(e^{r_jx},xe^{r_jx},\ldots,x^{n_j-1}e^{r_jx}\right),$$

where Vect $(e^{r_jx}, xe^{r_jx}, \dots, x^{n_j-1}e^{r_jx})$ is the vector space generated by the set of functions $(e^{r_jx}, xe^{r_jx}, \dots, x^{n_j-1}e^{r_jx})\}$.

3. If $P(D) = \left((D-\lambda_1)^2+\beta_1^2\right)\dots\left((D-\lambda_m)^2+\beta_m^2\right) = \prod_{j=1}^{m}\left((D-\lambda_j)^2+\beta_j^2\right)$, where $n = 2m$ and $\lambda_j \neq \lambda_k$ or $\beta_j \neq \beta_k$ for all $j \neq k$, $j,k = 1,\dots,m$, then the kernel of $P(D)$ is the vector space generated by the set of functions $\left(e^{\lambda_1 x}\sin(\beta_1 x), e^{\lambda_1 x}\cos(\beta_1 x), \dots, e^{\lambda_m x}\sin(\beta_m x), e^{\lambda_m x}\cos(\beta_m x)\right)$.

4. If $P(D) = \left((D-\lambda_1)^2+\beta_1^2\right)^{n_1}\dots\left((D-\lambda_m)^2+\beta_m^2\right)^{n_m} = \prod_{j=1}^{m}\left((D-\lambda_j)^2+\beta_j^2\right)^{n_j}$, where $n = \sum_{j=1}^{m} n_j$ and $\lambda_j \neq \lambda_k$ or $\beta_j \neq \beta_k$ for all $j \neq k$, $j,k = 1,\dots,m$, then the kernel of $P(D)$ is the vector space

$$\oplus_{j=1}^{m}\text{Vect}\left(e^{\lambda_j x}\sin(\beta_j x), e^{\lambda_j x}\cos(\beta_j x), \dots, x^{n_j-1}e^{\lambda_j x}\sin(\beta_j x), x^{n_j-1}e^{\lambda_j x}\cos(\beta_j x)\right).$$

# 5 Inverse of Differential Operator

**Definition 5.1**

Let $f$ be a continuous function on a finite interval $[a,b]$. According to the Fundamental Theorem of Calculus we have:

$$\frac{d}{dx}\int_a^x f(t)dt = D\int_a^x f(t)dt = f(x).$$

The inverse of differential operator $D^{-1}$ of the operator $D$ can be defined as follows:

$$D^{-1}f(x) = \int_a^x f(t)dt.$$

Further, we inductively define the higher order inverse differential operator of $D$ as follows:

$$D^{-n}f(x) = D^{-n+1}(D^{-1}f(x)).$$

**Examples 12 :**

1. $D^{-1}x^r = \dfrac{x^{r+1}}{r+1} + c,$

   $D^{-2}x^r = \dfrac{x^{r+2}}{(r+1)(r+2)} + a_1x + a_2,$

$D^{-n}x^r = \dfrac{x^{r+n}}{(r+1)\dots(r+n)} + P_{n-1}(x)$, where $P_{n-1}$ is a polynomial of degree less then $n-1$.

2. Let $\lambda \neq 0$,
$D^{-1}e^{\lambda x} = \frac{1}{\lambda}e^{\lambda x} + c$,
$D^{-n}e^{\lambda x} = \frac{1}{\lambda^n}e^{\lambda x} + P_{n-1}(x)$, where $P_{n-1}$ is a polynomial of degree less then $n-1$.

3. Let $\lambda \neq 0$,
$D^{-1}\cos(\lambda x) = \frac{1}{\lambda}\sin(\lambda x) + c = \frac{1}{\lambda}\cos(\lambda x - \frac{\pi}{2}) + c$,
$D^{-n}\cos(\lambda x) = \frac{1}{\lambda^n}\cos(\lambda x - \frac{n\pi}{2}) + P_{n-1}(x)$, where $P_{n-1}$ is a polynomial of degree less then $n-1$.

## 5.1 Differential Operator Method of Solving Non-homogeneous Linear Ordinary Differential Equation

The general form of non-homogeneous linear ordinary differential equation with constant coefficients takes the following form:

$$(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y = P_n(D)y = f.$$

Our purpose is to find a particular solution to the previous differential equation.

## 5.2 The case $f(x) = e^{\lambda x}$

**Theorem 5.2**

1. If $P(\lambda) \neq 0$, then
$$y(x) = \frac{1}{P(\lambda)}e^{\lambda x}$$
is a particular solution.

2. If $P(D) = (D-\lambda)^m Q(D)$, $1 \leq m \leq n$ and $Q(\lambda) \neq 0$, then
$$y = \frac{1}{Q(\lambda)}\left(\frac{1}{m!}x^m + P_{m-1}(x)\right)e^{\lambda x}$$
is a particular solution, where $P_{m-1}$ is a polynomial of degree less then $m-1$.

**Proof** .
We have $P(D)e^{\lambda x} = P(\lambda)e^{\lambda x}$.

1. If $P(\lambda) \neq 0$, then
$$\frac{1}{P(\lambda)}\left[P(D)\left(e^{\lambda x}\right)\right] = P(D)\left(\frac{e^{\lambda x}}{P(\lambda)}\right) = e^{\lambda x}.$$

Then $y = \frac{1}{P(\lambda)} e^{\lambda x}$ is a particular solution.

2. If $P(D) = (D-\lambda)^m Q(D)$, $1 \le m \le n$ and $Q(\lambda) \neq 0$, then the equation becomes $P(D)y = Q(D)(D-\lambda)^m y = e^{\lambda x}$. Then $y = \frac{1}{Q(\lambda)}(\frac{1}{m!}x^m + P_{m-1}(x))e^{\lambda x}$ is a particular solution, with $P_{m-1}$ a polynomial of degree less then $m-1$.

   Because

$$\begin{aligned}(D-\lambda)^m Q(D)\left[(\frac{1}{m!}x^m + P_{m-1})e^{\lambda x}\right] &= Q(\lambda)(D-\lambda)^m\left(e^{\lambda x}\frac{1}{m!}x^m\right) \\ &= Q(\lambda)e^{\lambda x}D^m\left(\frac{1}{m!}x^m\right) = Q(\lambda)e^{\lambda x}\end{aligned}$$

□

**Example 13 :**

1. Particular solution to the differential equation

$$y'' - 2y' + 6y = e^{3x}.$$

$$y_p = \frac{e^{3x}}{3^2 - 2\cdot 3 + 8} = \frac{1}{11}e^{3x}.$$

2. A particular solution to the differential equation

$$(D-1)^3(D+2)(D-2)y(x) = e^x.$$

   is

$$y_p = \frac{e^x}{(1+2)(1-2)}\left(\frac{1}{3!}x^3\right) = -\frac{x^3}{18}e^x.$$

# 6 Exercises

**Exercise 1 :**
Let $g\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ be a continuous function and consider the following differential equation

$$y'' + y = g \tag{6.13}$$

1. Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:
$$h(x) = \sin x \int_0^x g(t)\cos t dt - \cos x \int_0^x g(t)\sin t dt.$$
   (a) Prove that $h$ is a solution of the differential equation (6.13) .
   (b) Prove that $h(x) = \displaystyle\int_0^x g(t)\sin(x-t)dt.$
   (c) Prove that $h(x) + h(x+\pi) = \displaystyle\int_0^\pi g(x+t)\sin t dt$ and deduce that $h(x) + h(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.
2. (a) Prove that any solution $f$ of (6.13) on $\mathbb{R}$ fulfills $f(x) + f(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.
   (b) Deduce that if a function $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^2$ and $F''(x) + F(x) \geq 0$, $\forall x \in \mathbb{R}$, then $F(x) + F(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

**Exercise 2 :**
Solve the differential equation $y'' + y = \dfrac{1}{3+\cos(2x)}$.

**Exercise 3 :**
Consider the differential equation $y'' + \lambda y = 0$, with $\lambda \in \mathbb{R}$.

1. Give the general solutions for this differential equation.
2. Determine the values of $\lambda$ for which there exists a non zero solution $y$ of the differential equation $y'' + \lambda y = 0$ and fulfills $y(0) = y(1) = 0$.

**Exercise 4 :**

1. Determine the primitives of the functions $e^{\alpha x}\sin^2 x$ and $e^{\alpha x}\sin x\cos x$, with $\alpha \neq 0$.
2. Find two linearly independent solutions of the differential equation
$y'' - 2ky' + (k^2+1)y = 0$, where $k \in \mathbb{R}$.
3. Solve the differential equation $y'' - 2ky' + (k^2+1)y = e^x \sin x$.

**Exercise 5 :**
Solve the following differential equations:

1. $y'' - 5y' + 6y = 0$,
2. $4y'' + 4y' + y = 0$,
3. $y'' + y' + y = 0$,
4. $y'' + y' - 2y = 2x^2 - 3x + 1$,
5. $2y'' + 2y' + 3y = x^2 + 2x - 1$,
6. $y'' - 2y' + y = e^{-x}$,
7. $y'' - y' - 2y = x^2 e^{-3x}$,
8. $y'' - 2y' + 2y = e^x + x$,
9. $y'' + 4y = \sin(3x)$,
10. $y'' + 4y = \cos(2x) + \cos(4x)$,
11. $y'' + y = \dfrac{1}{1 + \sin^2 x}$,
12. $y'' + 4y' + 5y = \cosh(2x).\cos x$,
13. $y'' - 6y' + 9y = \sinh^3 x$.

**Exercise 6 :**
Consider the following differential equations: : $y'' - y = 1$ and $y'' + y = 1$.

1. Solve these differential equations.
2. Give the bounded solutions on $\mathbb{R}^+$ of these differential equations.
3. Give the even solutions of these differential equations.
4. Let $a \in \mathbb{R}^*$. Examine if there exist solutions of these differential equations which vanishes at 0 and at $a$.
   Discuss according to the values of $a$.

**Exercise 7 :**

1. Let $\alpha, \beta$ be different real numbers.
   Solve the following differential equation $y^{''} - (\alpha + \beta)y' + \alpha\beta y = 0$.
2. Determine the solutions of the differential equation $y^{''} + y = \cos x$.
3. Determine the solutions of the differential equation $y^{''} + y = \dfrac{1}{3 + \cos(2x)}$.

**Exercise 8 :**
We consider the following differential equations

$$y^{''} - y = 1 \tag{6.14}$$

and

$$y^{''} + y = 1 \tag{6.15}$$

1. Solve the differential equations (6.14) and (6.15).

   Give the solutions of (6.14) and (6.15) which have the same initial conditions $y(0) = \alpha, y'(0) = \beta$, $\alpha, \beta \in \mathbb{R}$.

2. Give if there exists
   (a) the bounded solutions on $\mathbb{R}^+$ for the differential equations (6.14) and (6.15),
   (b) the even solutions on $\mathbb{R}$ for the differential equations (6.14) and (6.15).

3. Let $a \in \mathbb{R}^*$. Say whether there exist solutions for the differential equations (6.14) and (6.15) vanishing at 0 and at $a$. Discuss according to the values of $a$.

4. (a) Let $\lambda \in \mathbb{R}$, $f$ and $g$ two differentiable functions on $\mathbb{R}^+$ such that $f'+\lambda f \leq g$. We set $h(x) = \displaystyle\int_0^x e^{\lambda t} g(t)dt$.
   Compute $h'$ and deduce that
   $$f(x) \leq e^{-\lambda x} f(0) + e^{-\lambda x} h(x), \quad \forall\, x \in \mathbb{R}^+.$$
   (b) Let $\varphi$ be function twice differentiable on $\mathbb{R}^+$ such that
   $$\forall x \in \mathbb{R}^+, \quad \varphi^{''}(x) - \varphi(x) \leq 1.$$
   Let $\psi$ be the solution of (6.14) such that $\psi(0) = \varphi(0)$, $\psi'(0) = \varphi'(0)$.
   Prove that $\forall\, x \in \mathbb{R}^+$, $\varphi(x) \leq \psi(x)$. (Hint: we can use the question a) with $f = \varphi' - \varphi$ and $\lambda = 1$).

5. Let $\varphi(x) = 1 - e^{-x}$.
   (a) Verify that $\varphi^{''} + \varphi \leq 1$.
   (b) Let $\psi$ be the solution of (6.15) such that $\psi(0) = \varphi(0) = 0$ and $\psi'(0) = \varphi'(0) = 1$.
   Is $\varphi(x) \leq \psi(x)$, $\forall\, x \in \mathbb{R}^+$?

**Exercise 9 :**
Consider the differential equation $y^{''} + \lambda y = 0$, with $\lambda \in \mathbb{R}$.

1. Give the real general solution of the equation according to the values of $\lambda$.
2. Determine the values of $\lambda$ for which there exists a non zero real solution of the equation such that $y(0) = y(1) = 0$.

**Exercise 10 :**

1. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Determine the anti-derivatives of $e^{\alpha x} \sin^2 x$ and of $e^{\alpha x} \sin x \cos x$.
2. Find two linearly independent solutions of the differential equation: $y'' - 2ky' + (k^2+1)y = 0$, with $k \in \mathbb{R}$.
3. Solve the differential equation $y'' - 2ky' + (k^2+1)y = e^x \sin x$.

**Exercise 11 :**
Solve the following differential equations:

1. $y'' - 5y' + 6y = 0$,
2. $4y'' + 4y' + y = 0$,
3. $y'' + y' + y = 0$.
4. $y'' + y' - 2y = 2x^2 - 3x + 1$,
5. $2y'' + 2y' + 3y = x^2 + 2x - 1$,
6. $y'' - 2y' + y = e^{-x}$,
7. $y'' - y' - 2y = x^2 e^{-3x}$,
8. $y'' - 2y' + 2y = e^x + x$,
9. $y'' + 4y = \sin(3x)$,
10. $y'' + 4y = \cos(2x) + \cos(4x)$,
11. $y'' + y = \dfrac{1}{1 + \sin^2 x}$,
12. $y'' + 4y' + 5y = \cosh(2x).\cos x$,
13. $y'' - 6y' + 9y = \sinh^3 x$.

## 6.1 Linear Differential Equation of Second Order

**Solution of Exercise 1:**

1. (a) $h'(x) = \cos x \int_0^x g(t)\cos t dt + \sin x \int_0^x g(t)\sin t dt$ and $h''(x) = -h(x) + g(x)\cos^2 x + g(x)\sin^2 x = g(x) - h(x)$. Then $h$ is a solution of the differential equation (6.13).

   (b) Since $\sin(x-t) = \sin x\cos t - \cos x\sin t$, then $h(x) = \int_0^x g(t)\sin(x-t)dt$.

   (c)
$$\begin{aligned} h(x) + h(x+\pi) &= \int_0^x g(t)\sin(x-t)dt - \int_0^{x+\pi} g(t)\sin(x-t)dt \\ &= -\int_x^{x+\pi} g(t)\sin(x-t)dt \\ &\overset{x-t=u}{=} \int_0^{\pi} g(x+u)\sin u du. \end{aligned}$$

   $h(x) + h(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$ because $g(x) \geq 0$, $\forall x \in \mathbb{R}$.

2. (a) Let $f$ be a solution of the differential equation (6.13) on $\mathbb{R}$. The function $k = f - h$ fulfills the differential equation $y'' + y = 0$. Then there is $a, b \in \mathbb{R}$ such that $f = h + a\cos x + b\sin x$. It results that $f(x) + f(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

   (b) The function $g = F'' + F$ is non negative and the function $F$ is a solution of the differential equation (6.13). Then $F(x) + F(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

**Solution of Exercise 2:**

The general solution of the homogenous differential equation is $y = a\cos x + b\sin x$. Using the classical method for solving this differential equation, $y = U\cos x + V\sin x$, we find: $U'\cos x + V'\sin x = 0$ and $-U'\sin x + V'\cos x = \dfrac{1}{3+\cos(2x)}$. Then

$$U = -\frac{1}{2}\tan^{-1}(\cos x) + a, \qquad V = \frac{1}{2\sqrt{2}}\tan^{-1}(\frac{\sin x}{\sqrt{2}}) + b.$$

**Solution of Exercise 3:**

1. The characteristic equation is $r^2 + \lambda = 0$.

   • If $\lambda = 0$, the general solutions of this equation is $y = ax + b$, with $a, b \in \mathbb{R}$.

   • If $\lambda > 0$, the general solutions of this equation is $y = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, with $a, b \in \mathbb{R}$.

   • If $\lambda < 0$, the general solutions of this equation is $y = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$, with $a, b \in \mathbb{R}$.

2. To have a solution $y$ of the differential equation $y'' + \lambda y = 0$ such that $y(0) = y(1) = 0$, we must have $\lambda > 0$ and $\lambda = k^2\pi^2$, with $k \in \mathbb{Z}$.

**Solution of Exercise 4:**

1. By integration by parts

$$\int e^{\alpha x} \sin^2 x dx = \frac{e^{\alpha x}}{2\alpha} - \frac{e^{\alpha x}}{2(\alpha^2+4)} (\alpha \cos(2x) + 2\sin(2x)) + c$$

and

$$\begin{aligned} \int e^{\alpha x} \sin x \cos x dx &= \frac{e^{\alpha x}}{2} \sin^2(x) + \frac{\alpha}{2} \int e^{\alpha x} \sin^2 x dx \\ &= \frac{e^{\alpha x}}{2} \sin^2(x) + \frac{\alpha e^{\alpha x}}{4} - \frac{\alpha e^{\alpha x}}{4(\alpha^2+4)} (\alpha \cos(2x) + 2\sin(2x)) + c \end{aligned}$$

2. The equation $r^2 - 2kr + (1+k^2)$ is the characteristic equation of the differential equation. Then $e^{(k+1)x}$ and $e^{(k-1)x}$ are solutions linearly independent of the differential equation.

3. Using the variation of the constant method, the general solution of the differential equation has the form $y = Ue^{(k+1)x} + Ve^{(k-1)x}$, with $U'e^{(k+1)x} + V'e^{(k-1)x} = 0$ and $(k+1)U'e^{(k+1)x} + (k-1)V'e^{(k-1)x} = e^x \sin x$. Then

$$U' = \frac{1}{2} e^{-kx} \sin x, \qquad U = \frac{1}{2} e^{-kx} \left( -\frac{1}{1+k^2} \cos x - \frac{k}{1+k^2} \sin x \right) + c_1,$$

$$V' = -\frac{1}{2} e^{(2-k)x} \sin x, \qquad V = \frac{1}{2} e^{(2-k)x} \left( \frac{1}{1+(k-2)^2} \cos x + \frac{2-k}{1+(k-2)^2} \sin x \right) + c_2.$$

**Solution of Exercise 5:**

1. The characteristic equation of the differential equation $y'' - 5y' + 6y = 0$ is $r^2 - 5r + 6 = 0 = (r-2)(r-3)$. The general solution of the differential equation is $y = ae^{3x} + be^{2x}$, $a, b \in \mathbb{R}$.

2. The characteristic equation of the differential equation $4y'' + 4y' + y = 0$ is $4r^2 + 4r + 1 = 0 = (2r+1)^2$. The general solution of the differential equation is $y = (ax+b)e^{-\frac{1}{2}x}$, $a, b \in \mathbb{R}$.

3. The characteristic equation of the differential equation $y'' + y' + y = 0$ is $r^2 + r + 1 = 0 = (r - e^{-\frac{1+i\sqrt{3}}{2}x})(r - e^{\frac{-1+i\sqrt{3}}{2}x})$. The general solution of the differential equation is $y = ae^{-\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2}x) + be^{-\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x)$, $a, b \in \mathbb{R}$.

4. The characteristic equation of the differential equation $y'' + y' - 2y = 2x^2 - 3x + 1$ is $r^2 + r - 2 = 0 = (r-1)(r+2)$. The general solution of the homogeneous differential equation is $y = ae^x + be^{-2x}$, $a, b \in \mathbb{R}$.
$-\dfrac{1}{2}x^2 - \dfrac{3}{2}x - \dfrac{3}{4}$ is a particular solution. Then $y = ae^x + be^{-2x} - \frac{1}{2}x^2 - \frac{3}{2}x - \frac{3}{4}$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

5. The characteristic equation of the differential equation $2y''+2y'+3y = x^2+2x-1$ is $2r^2 + 2r + 3 = 0 = (r - e^{-\frac{1+i\sqrt{5}}{2}x})(r - e^{\frac{-1+i\sqrt{5}}{2}x})$. The general solution of the homogeneous differential equation is $y = ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x) + be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x)$, $a, b \in \mathbb{R}$.
$\dfrac{1}{3}x^2+\dfrac{2}{9}x-\dfrac{25}{27}$ is a particular solution. Then $y = ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x)+be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x)+\frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{27}$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

6. The characteristic equation of the differential equation $y'' - 2y' + y = e^{-x}$ is $r^2 - 2r + 1 = 0 = (r - 1)^2$. The general solution of the homogeneous differential equation is $y = (ax + b)e^x$, $a, b \in \mathbb{R}$.
$\dfrac{1}{4}e^{-x}$ is a particular solution. Then $y = (ax+b)e^x + \frac{1}{4}e^{-x}$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

7. The characteristic equation of the differential equation $y'' - y' - 2y = x^2e^{-3x}$ is $r^2 - r - 2 = 0 = (r + 1)(r - 2)$. The general solution of the homogeneous differential equation is $y = ae^{-x} + be^{2x}$, $a, b \in \mathbb{R}$.
$\dfrac{1}{10}x^2+\dfrac{7}{50}x+\dfrac{39}{500}$ is a particular solution. Then $y = ae^{-x}+be^{2x}+\frac{1}{10}x^2+\frac{7}{50}x+\frac{39}{500}$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

8. The characteristic equation of the differential equation $y'' - 2y' + 2y = e^x + x$ is $r^2-2r+2 = 0 = (r-e^{(1+i)})(r-e^{(1-i)})$. The general solution of the homogeneous differential equation is $y = ae^x\cos(x) + be^x\sin(x)$, $a, b \in \mathbb{R}$.
$e^x + \dfrac{1}{2}(x + 1)$ is a particular solution. Then $y = ae^x\cos(x) + be^x\sin(x) + e^x + \frac{1}{2}(x + 1)$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

9. The characteristic equation of the differential equation $y'' + 4y = \sin(3x)$ is $r^2+4 = 0 = (r-2i)(r+2i)$. The general solution of the homogeneous differential equation is $y = a\cos(2x) + b\sin(2x)$, $a, b \in \mathbb{R}$.
$\sin(3x)$ is a particular solution. Then $y = a\cos(2x)+b\sin(2x)+\sin(3x)$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

10. The characteristic equation of the differential equation $y'' + 4y = \cos(2x) + \cos(4x)$ is $r^2+4 = 0 = (r-2i)(r+2i)$. The general solution of the homogeneous differential equation is $y = a\cos(2x) + b\sin(2x)$, $a, b \in \mathbb{R}$.
$\dfrac{x}{4}\sin(2x)-\dfrac{1}{12}\cos(4x)$ is a particular solution. Then $y = a\cos(2x)+b\sin(2x)+\frac{x}{4}\sin(2x)-\frac{1}{12}\cos(4x)$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

11. The characteristic equation of the differential equation $y'' + y = \dfrac{1}{1 + \sin^2 x}$ is $r^2 + 1 = (r + i)(r - i)$.The general solution of the homogeneous differential equation is $y = a\cos(x) + b\sin(x)$, $a, b \in \mathbb{R}$.
Using the change of parameter method, the general solution of the equation takes the form: $y = U\cos(x) + V\sin(x)$, with $U'\cos(x) + V'\sin(x) = 0$ and

$-U'\sin(x)+V'\cos(x) = \dfrac{1}{1+\sin^2 x}$. Then $U = \dfrac{1}{2\sqrt{2}}\ln\left(\dfrac{\sqrt{2}+\cos(x)}{\sqrt{2}-\cos(x)}\right)+a$ and $V = \tan^{-1}(\sin(x))+b$

12. The characteristic equation of the differential equation $y''+4y'+5y = \cosh(2x)\cos x$ is $r^2+4r+5 = (r+2+\mathrm{i})(r+2-\mathrm{i})$. The general solution of the homogeneous differential equation is $y = e^{-2x}(a\cos(2x)+b\sin(2x))$, $a, b \in \mathbb{R}$.
Using the change of parameter method, the general solution of the equation takes the form: $y = Ue^{-2x}\cos(x)+Ve^{-2x}\sin(x)$, with $U'e^{-2x}\cos(x)+V'e^{-2x}\sin(x) = 0$ and $U'e^{-2x}(-\sin(x)-2\cos(x))+V'e^{-2x}(\cos(x)-2\sin(x)) = \cosh(2x)\cos(x)$.
Then $U = -\dfrac{1}{8}\cos(2x)+\dfrac{1}{20}e^{4x}\sin(2x)-\dfrac{1}{40}e^{4x}\cos(2x)+a$ and $V = \frac{1}{40}e^{4x}\sin(2x)+\frac{1}{20}e^{4x}\cos(2x)+\frac{x}{4}+\frac{1}{8}\sin(2x)+\frac{1}{16}e^{4x}+b$.

13. The characteristic equation of the differential equation $y''-6y'+9y = \sinh^3 x$ is $r^2-6r+9 = (r-3)^3$. The general solution of the homogeneous differential equation is $y = (ax+b)e^{3x}$, $a, b \in \mathbb{R}$.
Using the change of parameter method, the general solution of the equation takes the form: $y = (U+xV)e^{3x}$, with $U'+xV' = 0$ and $3U'+(1+3x)V' = e^{-3x}\sinh^3 x$. Then $V = \dfrac{1}{8}\left(x+3e^{-x}-\dfrac{3}{4}e^{-4x}+\dfrac{1}{6}e^{-6x}\right)$ and
$U = \dfrac{1}{8}\left(-\dfrac{1}{2}x^2-3(1+x)e^{-x}+\dfrac{3}{16}(1+4x)e^{-4x}-\dfrac{1}{36}(1+6x)e^{-6x}\right)$.

**Solution of Exercise 6:**

1. The general solution of the differential equation : $y''-y = 1$ is $y = -1+ae^x+be^{-x}$, $a, b \in \mathbb{R}$.
The general solution of the differential equation : $y''+y = 1$ is $y = 1+a\cos x+b\sin x$, $a, b \in \mathbb{R}$.

2. The bounded solutions on $\mathbb{R}^+$ of the differential equation : $y''-y = 1$ are $y = -1+be^{-x}$, $b \in \mathbb{R}$.
All solutions of the differential equation : $y''+y = 1$ are bounded on $\mathbb{R}^+$.

3. The even solutions of the differential equation : $y''-y = 1$ are $y = -1+a\cosh(x)$, $a \in \mathbb{R}$.
The even solutions of the differential equation : $y''+y = 1$ are $y = 1+a\cos x$.

4. Let $y = -1+be^x+ce^{-x}$. $y(0) = 0 = y(a)$ yields that $b+c = 1$ and $be^a+ce^{-a}-1 = 0$. It results that $c = \dfrac{1-e^a}{2\sinh(a)}$.
Let $y = 1+b\cos x+c\sin x$. $y(0) = 0 = y(a)$ yields that $b = -1$ and $c\sin a = 1-\cos a$.
If $a \in 2\pi\mathbb{Z}$, $b = 1$ and $c \in \mathbb{R}$.
If $a \notin 2\pi\mathbb{Z}$, $b = 1$ and $c = \tan(\frac{a}{2})$.

**Solution of Exercise 7:**

1. The general solution of the differential equation $y'' - (\alpha+\beta)y' + \alpha\beta y = 0$ is $y = ae^{\alpha x} + be^{\beta x}$, with $a, b \in \mathbb{R}$.

2. The solutions of the differential equation $y'' + y = \cos x$ is $y = a\cos x + (b + \frac{x}{2})\sin x$, with $a, b \in \mathbb{R}$.

3. In use of the variation of parameter method, the general solution of the differential equation $y'' + y = \dfrac{1}{3+\cos(2x)}$ is

$$y = a\cos + b\sin x - \frac{1}{2}\tan^{-1}(\cos x)\cos x + \frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}+\sin x}{\sqrt{2}-\sin x}\right)\sin x, \qquad a, b \in \mathbb{R}.$$

**Solution of Exercise 8:**

1. The general solutions of (6.14) are $y = -1 + ae^{x} + be^{-x}$, with $a, b \in \mathbb{R}$.

   The general solutions of (6.15) are $y = 1 + a\cos x + b\sin x$, with $a, b \in \mathbb{R}$.

   If $y(0) = \alpha, y'(0) = \beta$, the solution of (6.14) is $y = -1 + \frac{\alpha+\beta+2}{2}e^{x} + \frac{\alpha-\beta}{2}e^{-x}$ and the solution of (6.15) is $y = 1 + (\alpha-1)\cos x + (\beta-1)\sin x$.

2. (a) The bounded solutions on $\mathbb{R}^{+}$ for the differential equation (6.14) are $y = -1 + be^{-x}$, with $b \in \mathbb{R}$ and the solutions of the differential equation (6.15) are all bounded.

   (b) The even solutions on $\mathbb{R}$ for the differential equation (6.14) are $y = -1 + a\cosh x$, with $a \in \mathbb{R}$ and the even solutions on $\mathbb{R}$ for the differential equation (6.15) are $y = 1 + a\cos x$, with $a \in \mathbb{R}$.

3. Let $y = -1 + ue^{x} + ve^{-x}$ be a solution of the differential equation (6.14). $y(0) = y(a) = 0 \iff u + v - 1 = 0$ and $-1 + ue^{a} + ve^{-a} = 0$, for $a \neq 0$. This system is Crammer and has a unique solution.

   Let $y = 1 + u\cos x + v\sin x$ be a solution of the differential equation (6.15). $y(0) = y(a) = 0 \iff u = 1$ and $v\sin a = 1 - \cos a$. This equation has solutions if and only if $a \neq (2n+1)\pi$, $n \in \mathbb{Z}$.

4. (a) $h'(x) = e^{\lambda x}g(x)$. Since $f' + \lambda f \leq g$, then $h'(x) \geq (f'(x) + \lambda f(x))e^{\lambda x} = (f(x)e^{\lambda x})'$. After integration, we get $f(x) \leq e^{-\lambda x}f(0) + e^{-\lambda x}h(x)$, $\forall\, x \in \mathbb{R}^{+}$.

   (b)

   $$\psi(x) = \frac{1}{2}(1 + \varphi(0) + \varphi'(0))e^{x} + \frac{1}{2}(1 + \varphi(0) - \varphi'(0))e^{-x} - 1.$$

   Let $f = \varphi' - \varphi$ and $\lambda = 1$, we have $f' + f = \varphi'' - \varphi \leq 1$, then using question a), we get

   $$\varphi'(x) - \varphi(x) \leq -e^{-x}(1 + \varphi(0) - \varphi'(0)) + 1.$$

   Also using the same question with $\lambda = -1$ and $g = -e^{-x}(1 + \varphi(0) - \varphi'(0)) + 1$, we find

   $$\varphi(x) \leq \frac{1}{2}(1 + \varphi(0) + \varphi'(0))e^{x} + \frac{1}{2}(1 + \varphi(0) - \varphi'(0))e^{-x} - 1 = \psi(x), \quad \forall\, x \in \mathbb{R}^{+}.$$

5. $\varphi'(x) = e^{-x}$, $\varphi''(x) = -e^{-x}$. Then $\varphi'' + \varphi = 1 - 2e^{-x} \leq 1$.
   $\psi(x) = -\cos x + \sin x + 1$.
   $\varphi(x) - \psi(x) = -e^{-x} + \cos x - \sin x$.
   $\varphi(2n\pi) - \psi(2n\pi) = 1 - e^{-2n\pi} \geq 0$ and $\varphi(\frac{\pi}{2} + 2n\pi) - \psi(\frac{\pi}{2} + 2n\pi) = -1 - e^{-2n\pi} \leq 0$.

**Solution of Exercise 9:**

1. If $\lambda > 0$, the real general solution of the equation is $y = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, with $a, b \in \mathbb{R}$.
   If $\lambda < 0$, the real general solution of the equation is $y = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$, with $a, b \in \mathbb{R}$.
   If $\lambda = 0$, the real general solution of the equation is $y = ax + b$, with $a, b \in \mathbb{R}$.
2. There exists a non zero real solution of the equation such that $y(0) = y(1) = 0$ on ly for $\lambda = (k\pi)^2$, with $k \in \mathbb{N}$.

**Solution of Exercise 10:**

1. $$\begin{aligned}\int e^{\alpha x}\sin^2 x dx &= \frac{1}{2}\int e^{\alpha x}(1 - \cos(2x))dx \\ &= \frac{1}{2\alpha}e^{\alpha x} + \frac{e^{\alpha x}}{\alpha^2 + 4}\left(\alpha\cos(2x) + 2\sin(2x)\right) + c,\end{aligned}$$
   with $c \in \mathbb{R}$.
   By integration by parts, $\displaystyle\int e^{\alpha x}\sin^2 x dx = \frac{e^{\alpha x}}{\alpha}\sin^2 x - \frac{2}{\alpha}\int e^{\alpha x}\sin x\cos x dx$.
   Then
   $$\int e^{\alpha x}\sin x\cos x dx = \frac{e^{\alpha x}}{2}\sin^2 x - \frac{1}{4}e^{\alpha x} - \frac{\alpha e^{\alpha x}}{2(\alpha^2 + 4)}\left(\alpha\cos(2x) + 2\sin(2x)\right) + c$$
2. $y_1 = e^{kx}\cos x$ and $y_2 = e^{kx}\sin x$ are linearly independent solutions of the differential equation: $y'' - 2ky' + (k^2 + 1)y = 0$.

   In use of the variation of constants method, the general solution of the equation is $y = Ue^{kx}\cos x + Ve^{kx}\sin x$, with
   $U'e^{kx}\cos x + V'e^{kx}\sin x = 0$ and $U'(e^{kx}(k\cos x - \sin x) + V'e^{kx}(k\sin x + \cos x) = e^x\sin x$. Then $U' = -e^{(1-k)x}\sin^2 x$ and $V' = e^{(1-k)x}\sin x\cos x$.
   If $k = 1$, $e^x\sin x$, $U = -\dfrac{x}{2} + \dfrac{1}{4}\sin(2x) + c_1$ and $V = -\dfrac{1}{4}\cos(2x) + c_2$.
   If $k \neq 1$, $U = -\dfrac{1}{2(1-k)}e^{(1-k)x} - \dfrac{e^{(1-k)x}}{(1-k)^2 + 4}\left((1-k)\cos(2x) + 2\sin(2x)\right) + c_1$
   and $V = \dfrac{e^{(1-k)x}}{2}\sin^2 x - \dfrac{1}{4}e^{(1-k)x} - \dfrac{(1-k)e^{(1-k)x}}{2((1-k)^2 + 4)}\left((1-k)\cos(2x) + 2\sin(2x)\right) + c_2$

**Solution of Exercise 11:**

1. The general solution of this homogeneous equation is $y = Ae^{3x} + Be^{2x}$, with $A, B \in \mathbb{R}$.

2. The general solution of this homogeneous equation is $y = (Ax + B)e^{-\frac{1}{2}x}$, with $A, B \in \mathbb{R}$.

3. The general solution of this homogeneous equation is
$y = Ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x) + Be^{-\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x)$, with $A, B \in \mathbb{R}$.

4. The general solution of the homogeneous equation is $y = Ae^{x} + Be^{-2x}$, with $A, B \in \mathbb{R}$.
The polynomial $P = -x^2 + \frac{1}{2}x - \frac{5}{4}$ is a particular solution of the equation $y'' + y' - 2y = 2x^2 - 3x + 1$.
Then the general solution of this equation is: $y = Ae^{x} + Be^{-2x} - x^2 + \dfrac{1}{2}x - \dfrac{5}{4}$, with $A, B \in \mathbb{R}$.

5. The general solution of the homogeneous equation is $y = Ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x) + Be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x)$, with $A, B \in \mathbb{R}$.
The polynomial $P = -\frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{9}$ is a particular solution of the equation $2y'' + 2y' + 3y = x^2 + 2x - 1$. Then the general solution of this equation is: $y = Ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x) + Be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x) - \dfrac{1}{3}x^2 + \dfrac{2}{9}x - \dfrac{25}{9}$, with $A, B \in \mathbb{R}$.

6. The general solution of the homogeneous equation is $y = (Ax + B)e^{x}$, with $A, B \in \mathbb{R}$.
$\dfrac{1}{4}e^{-x}$ is a particular solution of the equation $y'' - 2y' + y = e^{-x}$. Then the general solution of this equation is: $y = (Ax + B)e^{x} + \dfrac{1}{4}e^{-x}$, with $A, B \in \mathbb{R}$.

7. The general solution of the homogeneous equation is $y = Ae^{-x} + Be^{2x}$, with $A, B \in \mathbb{R}$.
$e^{-3x}(\dfrac{1}{10}x^2 + \dfrac{7}{50}x + \dfrac{39}{500})$ is a particular solution of the equation $y'' - y' - 2y = x^2e^{-3x}$. Then the general solution of this equation is: $y = Ae^{-x} + Be^{2x} + e^{-3x}(\dfrac{1}{10}x^2 + \dfrac{7}{50}x + \dfrac{39}{500})$, with $A, B \in \mathbb{R}$.

8. The general solution of the homogeneous equation is $y = Ae^{x}\cos x + Be^{x}\sin x$, with $A, B \in \mathbb{R}$.
$e^{x} + \dfrac{1}{2}x + \dfrac{1}{2}$ is a particular solution of the equation $y'' - 2y' + 2y = e^{x} + x$. Then the general solution of this equation is: $y = Ae^{x}\cos x + Be^{x}\sin x + e^{x} + \dfrac{1}{2}x + \dfrac{1}{2}$, with $A, B \in \mathbb{R}$.

9. The general solution of the homogeneous equation is $y = A\cos(2x) + B\sin(2x)$, with $A, B \in \mathbb{R}$.
$-\frac{1}{5}\sin(3x)$ is a particular solution of the equation $y'' + 4y = \sin(3x)$. Then the general solution of this equation is: $y = A\cos(2x) + B\sin(2x) - \frac{1}{5}\sin(3x)$, with $A, B \in \mathbb{R}$.

10. The general solution of the homogeneous equation is $y = A\cos(2x) + B\sin(2x)$, with $A, B \in \mathbb{R}$.
$-\frac{1}{12}\cos(4x) + \frac{1}{4}\sin(2x)$ is a particular solution of the equation $y'' + 4y = \cos(2x) + \cos(4x)$. Then the general solution of this equation is: $y = A\cos(2x) + B\sin(2x) - \frac{1}{12}\cos(4x) + \frac{1}{4}\sin(2x)$, with $A, B \in \mathbb{R}$.

11. The general solution of the homogeneous equation is $y = A\cos x + B\sin x$, with $A, B \in \mathbb{R}$.
Using the variation of constants method, the general solution of the equation is written as $y = U\cos x + V\sin x$, with

$$U' = \frac{-\sin x}{1+\sin^2 x}, \quad V' = \frac{\cos x}{1+\sin^2 x}.$$

Then $U = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}\right| + A$ and $V = \tan^{-1}(\sin x) + B$
Then the general solution of the equation is

$$y = A\cos x + B\sin x + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}\right|\cos x + (\sin x)\tan^{-1}(\sin x),$$

with $A, B \in \mathbb{R}$.

12. The general solution of the homogeneous equation is $y = Ae^{-2x}\cos x + Be^{-2x}\sin x$, with $A, B \in \mathbb{R}$.
Using the variation of constants method, the general solution of the equation is written as $y = Ue^{-2x}\cos x + Ve^{-2x}\sin x$, with

$$U' = -\frac{1}{2}e^{4x}\cosh(2x)\sin(2x), \quad V' = e^{4x}\cosh(2x)\cos^2 x = \frac{1}{4}(e^{6x}+e^{2x})(1+\cos(2x)).$$

Then

$$U = -\frac{1}{80}e^{6x}\left(-\cos(2x) + 3\sin(2x)\right) - \frac{1}{32}e^{2x}\left(-2\cos(2x) + \sin(2x)\right) + A$$

and

$$V = -\frac{1}{24}\left(e^{6x} + 3e^{2x}\right) - \frac{1}{80}e^{6x}\left(3\cos(2x) + \sin(2x)\right) - \frac{1}{16}e^{2x}\left(\cos(2x) + \sin(2x)\right) + B$$

13. The general solution of the homogeneous equation is $y = (Ax + B)e^{3x}$, with $A, B \in \mathbb{R}$.
The differential equation is $y'' - 6y' + 9y = \sinh^3 x = \frac{1}{8}\left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right)$.
$-\dfrac{3}{32}e^x - \dfrac{3}{128}e^{-x} - \dfrac{1}{288}e^{-3x} + \dfrac{1}{48}xe^{3x}$ is a particular solution of the equation.
Then the general solution of the equation is

$$y = (Ax + B)e^{3x} - \frac{3}{32}e^x - \frac{3}{128}e^{-x} - \frac{1}{288}e^{-3x} + \frac{1}{48}xe^{3x},$$

with $A, B \in \mathbb{R}$.