CHAPTER 1

LAPLACE TRANSFORMATIONS

1 Laplace Transform

Definition 1.1

1. Let f be a function defined on the interval $[a, b]$. The function f is said to be piecewise continuous if there is a finite numbers $a_1 = a < \ldots < a_n = b$ such that the function f is continuous on the intervals (a_j, a_{j+1}), for all $j = 1, \ldots n-1$ and $\lim\limits_{x \to a^+} = f(a^+)$, $\lim\limits_{x \to b^-} = f(b^-)$, $\lim\limits_{x \to a_j^-} = f(a_j^-)$ and $\lim\limits_{x \to a_j^+} = f(a_j^+)$ exist and finite for all $j = 2, \ldots n-1$.
2. Let f be a function defined on the interval $[0, +\infty)$. The function f is said to be piecewise continuous if f is piecewise continuous on any interval $[a, b] \subset [0, +\infty)$.

Definition 1.2

Let f be a piecewise continuous function on the interval $[0, +\infty)$. The Laplace transform $F = \mathcal{L}(f)$ is the function defined by

$$F(s) = \int_0^\infty e^{-sx} f(x) dx = \lim_{N \to +\infty} \int_0^N e^{-sx} f(x) dx.$$

Definition 1.3

A function f is said to be of exponential order if there exist constants $c, M > 0$, and $T \geq 0$ such that $|f(x)| \leq M e^{cx}$ for all $x \geq T$.

Theorem 1.4 *(Sufficient Conditions for Existence of $\mathcal{L}(f(t))$) If f is piecewise continuous on $[0, +\infty)$ and of exponential order c, then for all $s > c$, $\mathcal{L}(f)(s)$ is well* *defined.*

Theorem 1.5 *If f is piecewise continuous on $[0,+\infty)$ and of exponential order and $F(s)=\mathcal{L}(f(x))$, then $\lim_{s\to+\infty} F(s)=0$.*

1.1 Basic Properties of Laplace transform

Theorem 1.6: Basic Properties of Laplace transform

1. Linearity $\mathcal{L}\left(af(x)+bf(x)\right)=a\mathcal{L}\left(f(x)\right)+b\mathcal{L}\left(g(x)\right)$.
2. $\mathcal{L}\left(e^{ax}(f(x)\right)(s)=\mathcal{L}\left(f(x)\right)(s-a)$,
3. $\mathcal{L}\left(f(bx)\right)(s)=\frac{1}{b}\mathcal{L}\left(f(x)\right)(\frac{s}{b})$.

Theorem 1.7: Transforms of Some Basic Functions

1. $\mathcal{L}(1)=\frac{1}{s}$.
2. $\mathcal{L}(x^n)=\frac{n!}{s^{n+1}}$, $n\in\mathbb{N}$.
3. $\mathcal{L}(e^{ax})=\frac{1}{s-a}$, $n\in\mathbb{N}$.
4. $\mathcal{L}\left(e^{\mathrm{i}x}\right)=\frac{1}{s-\mathrm{i}}$,
5. $\mathcal{L}(\sin(ax))=\frac{a}{s^2+a^2}$.
6. $\mathcal{L}(\cos(ax))=\frac{s}{s^2+a^2}$.
7. $\mathcal{L}(\sinh(ax))=\frac{a}{s^2-a^2}$.
8. $\mathcal{L}(\cosh(ax))=\frac{s}{s^2-a^2}$.

Corollary 1.8

1. $\mathcal{L}\left(e^{ax}\sin(bx)\right)=\frac{b}{(s-a)^2+b^2}$,
2. $\mathcal{L}\left(e^{ax}\cos(x)\right)=\frac{s-a}{(s-a)^2+b^2}$.

Theorem 1.9 $\mathcal{L}\left(x^n f(x)\right)(s)=(-1)^n\frac{d^n}{ds^n}\mathcal{L}\left(f(x)\right)(s)$.

For $n = 1$ this formula follows taking the derivative inside the integral. The general case follows by induction.

Examples 1 :

1. If $f = 1$
$$\mathcal{L}\left(x^n\right)(s) = -\frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}}.$$
2. If $f(x) = \sin(x)$ and $f(x) = \cos(x)$ we get
$$\mathcal{L}\left(x\sin(bx)\right)(s) = -\frac{d}{ds}\frac{b}{s^2+b^2} = \frac{2bs}{(s^2+b^2)^2},$$
$$\mathcal{L}\left(x\cos(bx)\right)(s) = -\frac{d}{ds}\frac{s}{s^2+b^2} = \frac{s^2-b^2}{(s^2+b^2)^2} = \frac{1}{s^2+b^2} - \frac{2b^2}{(s^2+b^2)^2}.$$

Theorem 1.10 *1.* $\mathcal{L}\left(f'(x)\right)(s) = s\mathcal{L}\left(f(x)\right)(s) - f(0)$.

2. $\mathcal{L}\left(f^{(n)}\right)(s) = s^n\mathcal{L}\left(f\right)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$.

Proof .

1. This follows by integration by parts:
$$\begin{aligned}\mathcal{L}\left(f'(x)\right)(s) &= \int_0^\infty e^{-sx}f'(x)dx \\ &= e^{-sx}f(x)\Big]_0^\infty + s\int_0^\infty e^{-sx}f(x)dx \\ &= s\int_0^\infty e^{-sx}f(x)dx - f(0).\end{aligned}$$
2. We prove the formula by induction.

These formulas will useful to find the Laplace transform for the functions that are annihilated by a constant coefficient differential operator.

Example 2 :
If $f(x) = \sin x$, $f'' + f = 0$. Then $\mathcal{L}\left(f''(x) - f(x)\right)(s) = s^2\mathcal{L}\left(f(x)\right)(s) - 1 - \mathcal{L}\left(f(x)\right)(s)$. We get
$$\mathcal{L}\left(\sin x\right)(s) = \frac{1}{1+s^2}.$$

2 Inverse Laplace Transform

Theorem 2.1 *If* $\mathcal{L}\left(f(x)\right) = \mathcal{L}\left(g(x)\right)$ *then* $f = g$.

Definition 2.2

If F is the Laplace of a piecewise continuous function f, then f is called the **inverse Laplace transform** of F and denoted by

$$F = \mathcal{L}^{-1}(f).$$

The inverse Laplace transform is also linear. We have for example

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{1}{2}x\sin(x), \quad \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x).$$

Theorem 2.3 *Some Inverse Transforms*

1. $\mathcal{L}^{-1}(\frac{1}{s}) = 1.$
2. $\mathcal{L}^{-1}(\frac{n!}{s^{n+1}}) = t^n$, $n \in \mathbb{N}$.
3. $\mathcal{L}^{-1}(\frac{1}{s-a}) = e^{at}$, $n \in \mathbb{N}$.
4. $\mathcal{L}^{-1}(\frac{k}{s^2+k^2}) = \sin(kt).$
5. $\mathcal{L}^{-1}(\frac{s}{s^2+k^2}) = \cos(kt).$
6. $\mathcal{L}^{-1}(\frac{k}{s^2-k^2}) = .$
7. $\mathcal{L}^{-1}(\frac{s}{s^2-k^2}) = \cosh(kt).$

Example 3 :
$\mathcal{L}^{-1}(\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}).$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}.$$

$$\mathcal{L}^{-1}(\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}) = -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

Theorem 2.4 *laplace-50 If $f \in \mathcal{C}^{n-1}$ on $[0,+\infty)$ and $f^{(k)}$ are of exponential order and if $f^{(n)}$ is piecewise continuous on $[0,+\infty)$, then*

$$\mathcal{L}(f^{(n)}(x)) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}(f(x))$.

Remark 2.5:

e use Theorem (**??**) to solve linear ordinary differential equations.
If
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = g(t)$$
$y(0) = y_0, y^{'}(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$, where a_j, y_j are constants, for $0 \le j \le n-1$. By the linearity property, the Laplace transform of this linear combination is a linear combination of Laplace transforms:

$$a_n \mathcal{L}(y^{(n)}) + a_{n-1} \mathcal{L}(y^{(n-1)}) + \ldots + a_0 \mathcal{L}(y) = \mathcal{L}(g(t))$$

Example 4 :
Consider the differential equation

$$y' + 3y = 13\sin(2t), \quad y(0) = 6.$$

We take the transform of each member of the differential equation: $\mathcal{L}(y') + 3\mathcal{L}(y) = 13\mathcal{L}(\sin(2t))$. Then $sF(s) - 6 + 3F(s) = 6 + \frac{26}{s^2+4}$. Then $F(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$ and $y = 6\mathcal{L}^{-1}(\frac{1}{s+3}) - 2\mathcal{L}^{-1}(\frac{s}{s^2+4}) + 6\mathcal{L}^{-1}(\frac{1}{s^2+4}) = 8e^{-3t} - 2\cos(2t) + 3\sin(2t)$.

Example 5 :
Consider the differential equation

$$y^{''} - 3y' + 2y = e^{-4t}, \quad y(0) = 1, y'(0) = 5.$$

We take the transform of each member of the differential equation: $\mathcal{L}(y^{''}) - 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(e^{-4t})$.

Then $F(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4}$ and $y = -\frac{16}{5}e^t + \frac{25}{5}e^{2t} + \frac{1}{30}e^{-4t}$.

Example 6 :
Consider the initial value problem

$$y'' + y' + y = \sin(x), \quad y(0) = 1, \; y'(0) = -1.$$

Let $Y(s) = \mathcal{L}\left(y(x)\right)$, we have

$$\mathcal{L}\left(y'\right) = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\left(y''\right) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Taking Laplace transforms of the differential equation, we get

$$(s^2 + s + 1)Y(s) - s = \frac{1}{s^2+1}.$$

Then
$$Y(s) = \frac{s}{s^2+s+1} + \frac{1}{(s^2+s+1)(s^2+1)}.$$

Finding the inverse Laplace transform.

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s^2+s+1)(s^2+1)}\right). \end{aligned}$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

we have

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right) = e^{-\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x) - \frac{1}{\sqrt{3}}e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{s+1}{s^2+s+1} - \frac{s}{s^2+1}.$$

Then

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+s+1)(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+s+1}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right).$$

Since

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+s+1}\right) = \frac{2}{\sqrt{3}}e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x), \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(x)$$

we obtain

$$y(x) = 2e^{-\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x) - \cos(x).$$

Example 7 :

Consider the following system

$$\begin{cases} \dfrac{dx}{dt} &= -x+2y, \\ \dfrac{dy}{dt} &= 2x+y \end{cases}$$

with the initial conditions $x(0) = 1$, $y(0) = -1$.

Taking Laplace transform, the system becomes

$$\begin{aligned} sX(s) - 1 &= -X(s) + 2Y(s), \\ sY(s) + 1 &= 2X(s) + Y(s), \end{aligned}$$

where $X(s) = \mathcal{L}(x(t))$, $Y(s) = \mathcal{L}(y(t))$.

Solving for $X(s)$, $Y(s)$. The above linear system of equations, we get

$$X(s) = \frac{s-3}{s^2-5} = \frac{a}{s-\sqrt{5}} + \frac{b}{s+\sqrt{5}}, \quad Y(s) = \frac{1-s}{s^2-5} == \frac{c}{s-\sqrt{5}} + \frac{d}{s+\sqrt{5}},$$

with $a = \dfrac{\sqrt{5}-3}{2\sqrt{5}},\ b = \dfrac{\sqrt{5}+3}{2\sqrt{5}},\ c = \dfrac{1-\sqrt{5}}{2\sqrt{5}},\ d = -\dfrac{1+\sqrt{5}}{2\sqrt{5}}$.

Taking the inverse Laplace transform, we obtain

$$x(t) = ae^{\sqrt{5}t} + be^{-\sqrt{5}t}, \quad y(t) = ce^{\sqrt{5}t} + de^{-\sqrt{5}t}.$$

Example 8 :
Consider the following system

$$\begin{cases} \dfrac{dx}{dt} &= -2x + y, \\ \dfrac{dy}{dt} &= x - 2y \end{cases}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$.

Taking Laplace transform, the system becomes

$$\begin{aligned} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{aligned}$$

where $X(s) = \mathcal{L}\,(x(t))$, $Y(s) = \mathcal{L}\,(y(t))$.

Solving for $X(s)$, $Y(s)$. The above linear system of equations, we get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

Since

$$\frac{s+4}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3},$$

Taking the inverse Laplace transform, we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

Example 9 :
$\mathcal{L}(x^3) = \frac{6}{s^4}$, then $\mathcal{L}(e^{2x})x^3) = \dfrac{6}{(s-2)^4}$.

$\mathcal{L}^{-1}(\dfrac{2s+5}{(s-3)^3}) = \mathcal{L}^{-1}(\dfrac{2(s-3)+11}{(s-3)^3}) = 2\mathcal{L}^{-1}(\dfrac{1}{(s-3)^2}) + 11\mathcal{L}^{-1}(\dfrac{1}{(s-3)^3}) = 2e^{3x} + 11xe^{3x}$.

Example 10 :
Consider the differential equation

$$y'' - 6y' + 9y = x^2e^{3x}, \quad y(0) = 2,\ y'(0) = 17.$$

We take the transform of each member of the differential equation:

$F(s) = \dfrac{2s+5}{(s-3)^2} + \dfrac{2}{(s-3)^5}$.

$$y = 2e^{3x} + 11xe^{3x}\frac{1}{12}x^4e^{3x}.$$

Example 11 :

Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Taking Laplace transforms of the differential equation, we get $(s^2+s+1)Y(s) - s = \frac{1}{s^2+1}$, and $Y(s) = \dfrac{s}{s^2+s+1} + \dfrac{1}{(s^2+s+1)(s^2+1)}$.

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\}.$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{1}{2}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2} - \frac{1}{\sqrt{3}}\frac{\frac{1}{2}\sqrt{3}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}\ x) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}\ x).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

We get $A = B = 1$, $C = -1$, $D = 0$, so that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}\ x), \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(x)$$

we obtain

$$y(x) = 2e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}\ x) - \cos(x).$$

Example 12 :

As we have known, a higher order differential equation can be reduced to a system of differential equations. Let us consider the system $\begin{cases} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= x - 2y \end{cases}$, with the initial conditions $x(0) = 1$, $y(0) = 2$.

Taking Laplace transforms the system becomes

$$\begin{cases} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{cases}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$. We get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

We have

$$\frac{s+4}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3}.$$

Using the inverse Laplace transform, we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

2.1 Translation on the $x-$axis

Definition 2.6

The unit step function is defined to be $U(x-a) = \begin{cases} 0 & 0 \le x \le a \\ 1 & x \ge a \end{cases}$.

For example, if $f(x) = \sin x$ for $x \ge \pi$ and 0 otherwise, then $f(x) = U(x-\pi)\sin x$. Also if f is the step function defined by $f(x) = e^x$ for $x \in [1,2)$, then $f(x) = e^x\left(U(x-1) - U(x-2)\right)$.
We have

$$\mathcal{L}\left(U(x-a)\right) = \frac{e^{-as}}{s}.$$

And if general

Theorem 2.7 *[Second Translation Theorem] If $F(s) = \mathcal{L}(f(x))$ and $a \ge 0$, then*

$$\mathcal{L}(f(x-a)U(x-a)) = e^{-as}F(s).$$

Proof .

$$\begin{aligned} \mathcal{L}(f(x-a)U(x-a)) &= \int_0^{+\infty} e^{-sx} f((x-a)U(x-a))dx \\ &= \int_a^{+\infty} e^{-sx} f(x-a)dx \\ &= e^{-as}\int_0^{+\infty} e^{-sx} f(x)dx = e^{-as}F(s). \end{aligned}$$

□

Example 13 :
$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s-3}\right) = e^{3(x-2)}U(x-3)$.

Example 14 :
Solve the following differential equation: $y' - 2y = f(x)$, with$y(0) = 3$, $f(x) = 3\cos x$ for $x \geq 1$ and $f(x) = 0$, for $0 \leq x \leq 1$.

$\mathcal{L}(f(x)) = -\frac{3s}{s^2+1}e^{-s}$. Then $sF(s) - 3 - 2F(s) = -\frac{3s}{s^2+1}e^{-s}$ and
$F(s) = \dfrac{1}{s-2}\left(3 - \dfrac{3s}{s^2+1}e^{-s}\right) = \dfrac{3}{s-2} + \dfrac{6}{5}\dfrac{s}{s^2+1}e^{-s} - \dfrac{3}{5}\dfrac{1}{s^2+1}e^{-s}$.

$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3e^{2x}$,
$\mathcal{L}^{-1}\left(\frac{6}{5}\frac{s}{s^2+1}e^{-s}\right) = \frac{6}{5}\cos(x-1)U(x-1)$, $\mathcal{L}^{-1}\left(\frac{3}{5}\frac{1}{s^2+1}e^{-s}\right) = \frac{3}{5}\sin(x-1)U(x-1)$.

$$y(x) = 3e^{2x} + \frac{6}{5}\cos(x-1)U(x-1) - \frac{3}{5}\sin(x-1)U(x-1).$$

2.2 Boundary-Value Problem

Example 15 : [Boundary-Value Problem]
Solve the boundary differential equation $y^{(4)}(x) = f(x)$, where $f(x) = b(1 - \dfrac{2}{a}x)$, for $0 \leq x \leq \dfrac{a}{2}$ and $f(x) = 0$, for $\dfrac{a}{2} \leq x \leq a$, $y(0) = y'(0) = 0$ and $y(a) = y'(a) = 0$.

Lemma 2.8

Let f be a piecewise continuous function and of exponential type, then $\displaystyle\int_0^x f(t)dt = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$, where $F = \mathcal{L}(f)$.

Proof .
In use of the Fubini Theorem, we get

$$\mathcal{L}\left(\int_0^x f(t)dt\right) = \int_0^{+\infty} e^{-xs}\left(\int_0^x f(t)dt\right) = \frac{1}{s}F(s).$$

Then $\displaystyle\int_0^x f(t)dt = \mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right)$.

□

Example 16 :
$\mathcal{L}^{-1}\left(\dfrac{1}{s(s^2+1)}\right) = \displaystyle\int_0^x \sin t dt = 1 - \cos x$.

Example 17 :
Consider the differential equation

$$y''(x) + y(x) = f(x), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f(x) = 1$ if $x \in [1, 2)$ and zero otherwise.
The function $f(x) = U(x-1) - U(x-2)$. Taking the Laplace transform, we get:

$$s^2Y(s) + Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Solving this equation, we obtain

$$Y(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-2s}}{s(s^2+1)}.$$

Then

$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos x$. Then using Lemma (**??**) and Theorem (2.7), we get $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(1-\cos x)\right) = (1-\cos(x-1))\,U(x-1)$.
Similarly $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(e^{-2s}\mathcal{L}(1-\cos x)\right) = \big(1-\cos(x-2)\big)U(x-2)$.
Hence, the solution is

$y(x) = \big(1-\cos(x-1)\big)\,U(t-1) - \big(1-\cos(x-2)\big)\,U(t-2)$.

Example 18 :
We wish to solve for $y(x)$ the following equation $x^2 = \int_0^x e^t y(t)dt$. We apply the Laplace transform, we get

$$\frac{2}{s^3} = \frac{1}{s}\mathcal{L}(e^t y(x)) = \frac{1}{s}Y(s-1),$$

where $Y(s) = \mathcal{L}(y(x))$. Thus $Y(s-1) = \frac{2}{s^2}$ or $Y(s) = \frac{2}{(s+1)^2}$. We use the shifting property again $y(x) = 2e^{-x}x$.

Example 19 :
Given $y'' + a^2y = 1$, with $y(0) = 0, y'(0) = 0,$.

We take the Laplace transform of the equation, we get $s^2Y(s) + a^2Y(s) = F(s) = \frac{1}{s}$. Then $Y(s) = \frac{1}{s^2+a^2}\frac{1}{s}$. Taking the inverse Laplace transform of $Y(s)$ we obtain

$$y(x) = \frac{1-\cos(ax)}{a^2}.$$

2.3 Exercises

Exercise 1 :
Using the Heaviside function write down the piecewise function that is 0 for $x < 0$, x^2 for x in $[0,1]$ and x for $x > 1$.

Solution of Exercise 1:
$f(x) = xU(x-1) + x^2(U(x) - U(x-1))$.

Exercise 2 :
Using the Laplace transform solve

$$y'' + 3y' + 2y = 0, \quad y(0) = a, \quad y'(0) = b,$$

Solution of Exercise 2:
Using the Laplace transform, $\mathcal{L}(y^{'}) = sY - a$, $\mathcal{L}(y^{''}) = s^2Y - as - b$.Then $(s^2 + 3s + 2)Y = as + (b + 3a) \iff Y = \dfrac{b+2a}{s+1} - \dfrac{b+a}{s+2}$. Then $y(x) = (b+2a)e^{-x} - (b+a)e^{-2x}$.

Exercise 3 :
Using the Laplace transform solve

$$y^{''} + y^{'} + y = 0, \quad y(0) = a, \quad y^{'}(0) = b,$$

Solution of Exercise 3:
Using the Laplace transform,
$(s^2 + s + 1)Y = as + (b + a) \iff Y = \dfrac{as + (b+a)}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$. Then

$$y(x) = ae^{-\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x) + \frac{2}{\sqrt{3}}(b + \frac{a}{2})e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x)$$

Exercise 4 :
Using the Laplace transform solve

$$y^{''} + 2y^{'} + y = 0, \quad y(0) = a, \quad y^{'}(0) = b,$$

Solution of Exercise 4:
Using the Laplace transform,

$$y^{''} + 2y^{'} + y = 0, \quad y(0) = a, \quad y^{'}(0) = b,$$

$(s^2 + 2s + 1)Y = as + (b + a) \iff Y = \dfrac{as + (b+2a)}{(s+1)^2} = \dfrac{a}{s+1} + \dfrac{b+a}{(s+1)^2}$. Then

$$y(x) = ae^{-x} + (a+b)xe^{-x}.$$

Exercise 5 :
Solve $y^{''} + y = U(x-1)$ for initial conditions $y(0) = 0$ and $y^{'}(0) = 0$.

Solution of Exercise 5:
Using the Laplace transform,
$(s^2 + 1)Y = \frac{1}{s}e^{-s}$, then $Y = \dfrac{1}{s}\dfrac{e^{-s}}{s^2+1}$. Then

$$y(x) = \int_0^x \sin tU(t-1)dt = (1 - \cos x)U(x-1).$$

Exercise 6 :
Solve $y^{(3)} + y = x^3U(x-1)$ for initial conditions $y(0) = 1$ and $y^{'}(0) = 0$, $y^{''}(0) = 0$.

Solution of Exercise 6:
Using the Laplace transform,
$s^3Y - s^2 + Y = e^{-s}\left(\frac{1}{s} + \frac{3}{s^2} + \frac{6}{s^3} + \frac{6}{s^4}\right)$. Then

$$Y = e^{-s}\left(\frac{1}{s^3(s-1)} + \frac{3}{s^4(s-1)} + \frac{6}{s^5(s-1)} + \frac{6}{s^6(s-1)}\right).$$

Exercise 7 :
Solve $y'' - y = (x^2-1)U(x-1)$ for initial conditions $y(0) = 1$, $y'(0) = 2$ using the Laplace transform.

Solution of Exercise 7:
$y(x) = (2e^{x-1} - x^2 - 1)U(x-1) - \frac{e^{-x}}{2} + \frac{3e^x}{2}$.