Applications of Integration

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Area of Plane Region

If $f : [a, b] \longrightarrow \mathbb{R}^+$ be a non negative continuous function, then $\int_a^b f(x)dx$ is the area of the region R_x under the graph of f from a to b.

Theorem

If f and g are two continuous functions on $[a, b]$ and

$$f(x) \geq g(x), \ \forall x \in [a, b].$$

Then the area A of the region bounded by the graphs of f and g; $x = a$ and $x = b$ is

$$A = \int_a^b f(x) - g(x)dx.$$

Example 1 :
let $f(x) = x^2 + 1$ and $g(x) = x - 1$. Set up an integral that can be used to find the area of the shaded region.

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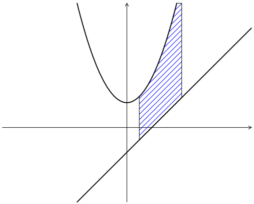


Solution: We have
The upper graph: $y = x^2 + 1$
The lower graph: $y = x - 1$
Then, the area A is given by

$$A = \int_a^b (x^2 + 1) - (x - 1)dx.$$

Remark

If f and g are two continuous functions on $[a, b]$. Then the area A of the region bounded by the graphs of f and g is

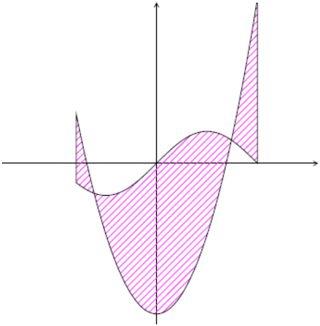
$$A = \int_a^b |f(x) - g(x)| dx.$$

For example if there is $c \in]a, b[$ such that $f(x) \geq g(x), \forall x \in [a, c], \forall x \in [a, c]$ and $f(x) \leq g(x), \forall x \in [c, b]$, then

$$A = \int_a^c f(x) - g(x) dx + \int_c^b g(x) - f(x) dx.$$

Example 2 :

$f(x) = x^2 - 4.7,\ g(x) = \sin x,\ [-3, \pi]$



$$A = \int_{-3}^{\pi} |x^2 - 4.7 - \sin x| dx.$$

Example 3 :
Find the area A of the region R bounded by the graphs of

$$y - x = 6,\ y = x^3 \text{ and } 2y + x = 0.$$

Solution.
Let $f(x) = x + 6$, $g(x) = x^3$ and $h(x) = -\frac{1}{2}x$. The points of intersection: $f(x) = h(x) \iff x = -4$, $g(x) = h(x) \iff x = 0$, $f(x) = g(x) \iff x^3 - x - 6 = 0$. We remak that $x = 2$ is the only solution of this equation and $f(x) = g(x) \iff x = 2$.

And we have $f(-4) = h(-4) = 2$, $g(0) = h(0) = 0$ and $f(2) = g(2) = 8$.
Clearly, if A_1 and A_2 are the area of the region R_1 and R_2 respectively, then

$$A = A_1 + A_2 = \int_{-4}^{0} f(x) - h(x)dx + \int_{0}^{2} f(x) - g(x)dx.$$

Therefore

$$
\begin{aligned}
A &= \int_{-4}^{0} (x+6) + \frac{1}{2} x dx + \int_{0}^{2} (x+6) - x^3 dx \\
&= \int_{-4}^{0} \frac{3}{2} x + 6 dx + \int_{0}^{2} x + 6 - x^3 dx \\
&= \left[\frac{3}{4} x^2 + 6x\right]_{-4}^{0} + \left[\frac{1}{2} x^2 + 6x - \frac{1}{4} x^4\right]_{0}^{2} \\
&= \left[\frac{3}{4} x^2 + 6x\right]_{-4}^{0} + \left[\frac{1}{2} x^2 + 6x - \frac{1}{4} x^4\right]_{0}^{2} \\
&= 22.
\end{aligned}
$$

Example 4 :
Find the area of the region between the graphs

$$f(x) = x^2 - 4; \; g(x) = x + 2$$

if x is restricted to the interval $[1, 4]$.
Solution: The points of intersection:

$$f(x) = g(x) \iff x^2 - 4 = x + 2 \iff x^2 - x - 6 = 0.$$

The only solution of this equation on the interval $[1, 4]$ is $x = 3$ and we have $f(3) = g(3) = 5$.

We have $f \leq g$ on the interval $[1, 3]$ and $g \leq f$ on the interval $[3, 4]$. Then

$$\begin{aligned} A &= \int_1^3 g(x) - f(x)dx + \int_3^4 f(x) - g(x)dx \\ &= \int_1^3 (x+2) - (x^2 - 4)dx + \int_3^4 (x^2 - 4) - (x+2)dx \\ &= \frac{17}{3}. \end{aligned}$$

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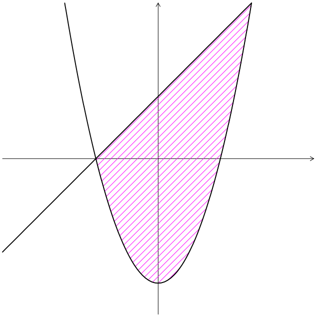

Solid of Revolution

The Disk Method

If a region R_x is revolved around the x-axis, the resulting solid is called: the solid of revolution generated by the region R_x.

Example 5 :
If $f : [a, b] \longrightarrow \mathbb{R}$ is a constant $f = c > 0$, then the region under the graph of f on the interval $[a, b]$ is rectangle.
The solid generated after revolving this region around the x-axis is a right cylinder.

Example 6 :
Consider the region under the graph of the function $f(x) = \sqrt{4 - x^2}$ for $x \in [-2, 2]$. If we revolve the region R_x around the x-axis, the solid generated is a ball of radius $r = 2$.

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuous function. The volume V of the solid of revolution generated by revolving the region bounded by the graphs of f, $y = 0$ $x = a$ and $x = b$ is given by

$$V = \int_a^b \pi f^2(x)dx.$$

Example 7 :

Let f the function defined on the interval $[-1, 2]$ by $f(x) = x^2 + 1$. Find the volume of the solid obtained by revolving the region under the graph of f around the x-axis

Solution.

The volume is equal to $\pi \int_{-1}^{2} (x^2 + 1)^2 dx = \pi \frac{78}{5}$.

Remark

If $x = g(y)$ where g is continuous and positive on $[c, d]$
If we revolve the region R_y around the y-axis, we obtain a solid of revolution which volume equal to

$$V = \pi \int_c^d g^2(y)dy.$$

Example 8 :
If $g(y) = y^2 - 4$ on the interval $[0, 2]$. The volume of the solid obtained by revolving the region under the graph of g around the y-axis.

$$V = \int_0^2 \pi(y^2 - 4)^2 dy.$$

The Washer Method

Let $f, g\colon [a, b] \longrightarrow \mathbb{R}^+$ be two continuous functions such that $f(x) \geq g(x) \geq 0$, $\forall x \in [a, b]$.
Let R is the region between the graph of f and the graph of g.
The volume of the solid obtained by revolving the region R around the x-axis is equal to

$$\pi \int_a^b f^2(x) - g^2(x) dx.$$

This formula can interpreted as

$$V = \pi \int_a^b (outer\ \ radius)^2 - (inner\ \ radius)^2 dx.$$

Example 9 :
If $f(x) = \cos(x)$ and $g(x) = \sin(x)$ on the interval $[0, \frac{\pi}{4}]$. Find the volume of the solid of revolution of the R betwen the graph of f and g around the x-axis.

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \cos^2(x) - \sin^2(x) dx \\ &= \pi \int_0^{\pi/4} \cos(2x) dx = \frac{\pi}{2}. \end{aligned}$$

Example 10 :
Let $f(x) = \sqrt{x}$ defined on the interval $[0, 4]$. If R is the region under the graph of f and S the solid of revolution of R around the axis $y = 2$. The volume of S is given by

$$V = \pi \int_0^4 2^2 - (2 - \sqrt{x})^2 dx = \frac{8\pi}{3}.$$

Here the outer radius is 2, the inner radius is $2 - y = 2 - \sqrt{x}$. .

Volume by Method of Cylindrical Shell

Theorem

Let $f : [a, b] \leq \mathbb{R}^+$ be a continuous function and R the region under the graph of f on the interval $[a, b]$.
The volume V of the solid of revolution generated by revolving the region R around the y-axis is given by

$$V = 2\pi \int_a^b xf(x)dx.$$

Example 11 :
Let $f : [2, 11] \longrightarrow \mathbb{R}^+$ the function defined by $\sqrt{x-2}$. The volume of the solid of revolution generated by revolving the region under the graph of f around the y-axis is

$$\begin{aligned} V = \int_2^{11} 2\pi x f(x) dx &= 2\pi \int_2^{11} x\sqrt{x-2} dx \\ &\overset{t=x-2}{=} \int_0^9 (t^{\frac{3}{2}} + 2t^{\frac{1}{2}}) \\ &= 4(\frac{243}{5} + 9). \end{aligned}$$

Arc Length and Surfaces of Revolution

Definition

Let $f : I \longrightarrow \mathbb{R}$ *be a function. We say that* f *is continuously differentiable if* $f'(x)$ *exists for all* $x \in I$ *and* f' *is itself continuous on* I*.*

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. Then the length of the curve of f denoted by L_a^b is given by:

$$L_a^b = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

Remark

Let $g\colon [c,d] \longrightarrow \mathbb{R}^+$, $y \mapsto g(y)$, be a continuously differentiable function.
Then the length of the curve of g from the point $(g(c), c)$ to the point $(g(d), d)$ is

$$L_c^d = \int_c^d \sqrt{1 + (g'(y))^2}\,dy.$$

Example 12 :
If $f(x) = \ln(\cos(x))$ defined on the interval on $[0, \frac{\pi}{4}]$. The length is given by

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}dx = \int_0^{\frac{\pi}{4}} \sec(x)dx = \ln(\sqrt{2} + 1).$$

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. Then the arc length function "s" for the graph of f on $[a, b]$ is defined by

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}dt, \ x \in [a, b].$$

We have

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (f'(x))^2}dx.$$

Area of Surface of Revolution

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. Then the area of the surface generated by revolving the curve $y = f(x)$ around the x-axis denoted SA is given by

$$SA = \int_a^b 2\pi |f(x)| \sqrt{1 + (f'(x))^2}\, dx.$$

Example 13 :

$f(x) = \dfrac{x^3}{3}$ defined on the interval $[0, 1]$. The surface of revolution of the graph of f around the $x-$axis is

$$\begin{aligned} S &= 2\pi \int_0^1 \frac{x^3}{3}\sqrt{1+x^2}dx \\ &\overset{t^2=1+x^2}{=} \frac{2\pi}{3}\int_1^{\sqrt{2}} t^4 - t^2 dt \\ &= \frac{2\pi}{3}\left(\frac{4\sqrt{2}-1}{5} - \frac{2\sqrt{2}-1}{3}\right). \end{aligned}$$

Remark

In the case $x = g(y)$, $y \in [c, d]$, the surface area generated by revolving the curve of g around the y-axis is given by

$$\begin{aligned} SA &= \int_c^d 2\pi|x|ds \\ &= \int_c^d 2\pi|g(y)|ds \\ &= \int_c^d 2\pi|g(y)|\sqrt{1+(g'(y))^2}dy. \end{aligned}$$